Nilpotent products of groups with amalgamations.

By JAMES WIEGOLD (Keele, Staffordshire).

§ 1. Introduction.

The contents of the following pages represent an attempt to combine something of the ideas of the theory of generalised free and generalised direct products of groups with amalgamations (SCHREIER [12], HANNA NEUMANN [10], [11], B. H. NEUMANN [9]) with those of GOLOVIN ([2], [3]) and MORAN ([6]) on regular products. One is naturally led to think about "regular products of groups with amalgamations", or more accurately to try to generalise the definitions of some subclass of the class of regular products so that the constituent groups of a product from such a subclass intersect in possibly non-trivial subgroups. This I have done for the nilpotent products of GOLOVIN and the verbal products of MORAN, and so the generalised free n-th nilpotent Golovin product of groups with amalgamations and the generalised free V-verbal product of groups with amalgamations are defined. It is inconvenient to give the precise definitions at this point, and the reader is referred for them to § 4 of the present work.

Sections 2 and 3 are of an elementary and preliminary nature, containing as they do most of the definitions and simple or known results that are necessary for the subsequent investigation. In § 4 we define the generalised n-th nilpotent and the generalised V-verbal products, and explain what is meant by free products of this kind. This section also contains some results which are fundamental for the rest of the paper. In § 5 we enlarge upon the contents of § 4 and state which of the very large number of problems arising from the definitions we tackle in this paper. It is namely a discussion of conditions necessary and sufficient for the existence of the generalised free second nilpotent products (one arising from the work of GOLOVIN and one from that of MORAN) of two groups with an amalgamation. This will be the central theme of the paper after § 6, though incidental results are derived throughout. Section 6 itself is devoted to a comparison of the "freeness" of certain pairs of the defined multiplications, where in fact we find a very surprising lack of such comparison. Also in this section the connection between the generalised free n-th nilpotent products *à la* GOLOVIN and *à la* MORAN for the case of two groups, for each $n \geqq 1$, is worked out; we find that the former may exist while the latter does not, but that if the latter exists so does the former, and they coincide. Section 7 is devoted to a discussion of the sort of conditions that are likely to succeed for the situation here presented. In fact one finds that most of the things one feels should happen do not, and conversely. The examples given in this section will clarify this remark, and lead the way to a conjecture that the sort of conditions one would most naturally think about are in fact inadequate for the situation. In § 8 we give necessary and sufficient conditions that the generalised free second nilpotent Golovin product of two groups with amalgamation exist. The conditions are unfortunately not very manageable, but this is felt to be in the nature of the problem. Moreover even when we know that a product exists, it is not always easy to obtain all its more important properties. In § 9 we give some normal forms for the elements of a generalised free second nilpotent product. It is unusual (or at least unexpected) that a normal form should be constructed as a result of successful investigations as to the existence of necessary and sufficient conditions of the sort here mentioned; one would expect to use the normal form to actually construct the product. It is true that this can be done, but the construction is so complicated as to be unworthy of serious attention. Also in § 9 we give (once more unmanageable) necessary and sufficient conditions that the generalised free second nilpotent Moran product exist. Finally in the appendix we take a cursory glance at the very hard problem of the third nilpotent products.

The greater part of this paper was presented as part of a thesis for the degree of Doctor of Philosophy to the University of Manchester. It is my very great pleasure to record my deep indebtedness to DR. B. H. NEUMANN, in particular for suggesting this problem to me and for his ever-ready advice and encouragement throughout its solution. I would also like to thank my Parents for making possible this opportunity of doing mathematical research.

§ 2. Notation and preliminary remarks.

In this section we assemble most of the notation, definitions, basic concepts and elementary results which are necessary for the investigation.

Groups will always be written multiplicatively, 1 will stand for the unit element of all groups occurring, and E for the subgroup consisting of 1 only. If g, h are elements of the group G, by g^h we mean the *transform* $h^{-1}gh$ of g by h, and by $[g, h]$ the *commutator* $g^{-1}h^{-1}gh$. The following easy results will be used throughout this paper:

$$(g_1 g_2 \cdots g_r)^a = g_1^a g_2^a \cdots g_r^a,$$
$$(g^n)^a = (g^a)^n,$$
$$[ab, c] = [a, c]^b [b, c],$$
$$[a^{-1}, b] = [b, a]^{a^{-1}}.$$

In these, r is any integer $\geqq 1$, n is any integer, and the other symbols denote group elements.

The order of the group G is denoted by $|G|$. We express the fact that G is generated by the set X of generators with the set R of defining relations in these generators by the equality

$$G = \mathrm{Gp}(X; R),$$

or, if the relations are understood or unimportant, by $G = \mathrm{Gp}(X)$.

The *normal closure* of the subgroup Y of G is the least normal subgroup of G containing Y, and is denoted by Y^G, by obvious analogy with the above notation for transforms. If Y is a normal subgroup, so that $Y = Y^G$, we write $Y \trianglelefteq G$.

If A and B are subgroups of G, the symbol $[A, B]$ stands for the subgroup of G generated by all commutators of the form $[a, b]$, where $a \in A$, $b \in B$. The *lower central series* of G is

$$G = {}^0G \supseteqq {}^1G \supseteqq \cdots \supseteqq {}^nG \supseteqq \cdots,$$

where ${}^{n+1}G = [{}^nG, G]$ for $n \geqq 0$; G is *nilpotent of class* n if its lower central series terminates in E after a finite number of steps, and n is the first integer for which ${}^nG = E$. The *upper central series* of G is

$$E = Z_0(G) \subseteqq Z_1(G) \subseteqq \cdots \subseteqq Z_n(G) \subseteqq \cdots,$$

where $Z_{n+1}(G)/Z_n(G)$ is the centre of $G/Z_n(G)$, for $n \geqq 0$. Then as is well known G is nilpotent of class n if and only if the upper central series terminates in G after a finite number of steps, and n is the first integer for which $Z_n(G) = G$. Lastly the *derived series* of G is

$$G = G^{(0)} \supseteqq G^{(1)} \supseteqq \cdots \supseteqq G^{(n)} \supseteqq \cdots,$$

where $G^{(n+1)} = [G^{(n)}, G^{(n)}]$ for $n \geqq 0$; G is *soluble of length* n if this series terminates in E after a finite number of steps, and n is the first integer for which $G^{(n)} = E$.

The members of the lower central series and of the derived series are particular instances of the more general phenomenon of *verbal subgroups*, which are defined in the following way (B. H. NEUMANN [8]).

Definition 2. 1. *Let* $v_\alpha(x_1^{(\alpha)}, x_2^{(\alpha)}, \ldots, x_{n(\alpha)}^{(\alpha)})$, *where* α *ranges over some index set, be a set* V *of words in the variables* $x_i^{(\alpha)}$, *where by "word" we mean any finite power-product of the* $x_i^{(\alpha)}$ *and their inverses. Let* G *be an arbitrary group, and let the* $x_i^{(\alpha)}$ *range over all the elements of* G. *Then the subgroup* $V(G)$ *generated by all the "values" of the words thus obtained is the verbal subgroup of* G *corresponding to the given set of words.*

Thus ${}^nG$ is the verbal subgroup of G corresponding to the word

$$[\ldots[[x_1, x_2], x_3], \ldots];$$

$G^{(n)}$ is that corresponding to the word

$$[\ldots[[[x_1, x_2], [x_3, x_4]], [[x_5, x_6], [x_7, x_8]]], \ldots],$$

where this commutator involves 2^n variables.

Suppose that G is generated by its subgroups G_α, where α ranges over some index set $\mathfrak{M}$. Then $[G_\alpha]$ denotes the subgroup of G generated by all commutators of the form $[g_\alpha, g_\beta]$, where $g_\alpha \in G_\alpha, g_\beta \in G_\beta$, and $\alpha \neq \beta$. (Note that $G_\alpha \cap G_\beta$ may well be more than E.) The normal closure $[G_\alpha]^G$ of $[G_\alpha]$ will be called the *cartesian* of the G_α in G. We shall use throughout the following results on cartesians and verbal subgroups.

(2. 2) If G is generated by its subgroups A and B, then the subgroup $[A, B]$ is normal in G. (GOLOVIN [2]).

(2. 3) If G is generated by its subgroups G_α $(\alpha \in \mathfrak{M})$, and θ is any homomorphism of G, $[G_\alpha]^G\theta = [G_\alpha\theta]^{G\theta}$. (GOLOVIN [2]).

(2. 4) If $V(G)$ is any verbal subgroup of G, and θ is any homomorphism of G, $V(G)\theta = V(G\theta)$. (MORAN [6]).

(2. 5) Verbal subgroups are fully invariant.

Let G once more be generated by its subgroups G_α $(\alpha \in \mathfrak{M})$. The following sequence of normal subgroups of G play a fundamental role in the present work:

$$[G_\alpha]^G = O_1(G) \supseteqq O_2(G) \supseteqq \cdots \supseteqq O_n(G) \supseteqq \cdots$$

where $O_{n+1}(G) = [G, O_n(G)]$ for $n \geqq 1$. The fact that this sequence decreases follows from (2. 2), which shows that the members are normal in G. The letter "O" has the advantage that its use is rare in group theoretical writings, and that it is the first letter of GOLOVIN's first name. One should perhaps write $O_n(G; \{G_\alpha\})$ for $O_n(G)$, to be quite accurate; and where confusion may arise, we shall do this.

By a *right transversal* T of the group G modulo its subgroup H we shall understand a set of right coset representatives of G modulo H, in which the unit element of G represents the coset H. Thus every element of G has a unique expression of the form ht, where $h \in H$ and $t \in T$. *Left transversals* are defined in the obviously analogous fashion.

Of great importance for the later sections is the *tensor product* of two groups A and B. This is denoted by $A \otimes B$, and is generated by all elements of the form $a \otimes b$, subject to the relations

$$a_1 a_2 \otimes b = (a_1 \otimes b)(a_2 \otimes b)$$
$$a \otimes b_1 b_2 = (a \otimes b_1)(a \otimes b_2).$$

It is well-known that $A \otimes B$ is always abelian (this will in any case follow from considerations in the next section) and that if m is any integer,

$$a^m \otimes b = a \otimes b^m = (a \otimes b)^m.$$

To end this section, we give the definition of the useful and powerful generalised free product of groups with amalgamations (HANNA NEUMANN [10]). Of the several possible definitions, it is most convenient to give that of B. H. NEUMANN in [9].

Definition 2.6. *Let G be a group and S a set of generators of G. Let S be the union of not necessarily disjoint subsets S_α, where α ranges over the index set $\mathfrak{M}$:*

$$S = \bigcup_{\alpha \in \mathfrak{M}} S_\alpha.$$

Let $G_\alpha = \mathrm{Gp}(S_\alpha)$, so that $G = \mathrm{Gp}(G_\alpha, \alpha \in \mathfrak{M})$. Let R_α denote a system of defining relations for G_α; if now all these relations together,

$$R = \bigcup_{\alpha \in \mathfrak{M}} R_\alpha,$$

form a set of defining relations for G, then G is called the generalised free product of the subgroups G_α, with amalgamations $G_\alpha \cap G_\beta = H_{\alpha\beta}$ $(\alpha \neq \beta)$.

§ 3. Regular products of groups.

This section is also largely devoted to the assembling of necessary material, but it is of a less elementary and well-known character.

In answer to a famous problem of KUROSCH, which asks about the existence of multiplications on a set of groups which enjoy certain properties possessed by the free and the direct multiplications, O. N. GOLOVIN wrote his paper [2], in which he introduced the concept of *regular product*. The following is a paraphrase of his definition.

Definition 3.1. *If the group G is generated by its subgroups G_α, where α ranges over the index set $\mathfrak{M}$, in such a way that for each $\alpha \in \mathfrak{M}$, $G_\alpha \cap B_\alpha^G = E$, where $B_\alpha = \mathrm{Gp}(G_\beta, \beta \in \mathfrak{M} - \{\alpha\})$, then G is said to be a regular product of the subgroups G_α.*

Golovin showed that G is a regular product of groups G_α if and only if $G \cong F/N$, where F is the free product of the G_α and N is a normal subgroup of F contained in the free cartesian $[G_\alpha]^F$. The extreme cases $N = E$ and $N = [G_\alpha]^F$ correspond to the free and direct multiplications respectively. By a suitable choice of kernels N, Golovin produced countably many associative regular multiplications, in the following way:

Definition 3.2. *The n-th nilpotent product of the groups G_α is $F/O_n(F)$, where F is their free product.*

Later S. Moran, in [6], defined a more extensive class, which he called the verbal products.

Definition 3.3. *The V-verbal product of the groups G_α is*

$$F/V(F) \cap [G_\alpha]^F \quad (\alpha \in \mathfrak{M}),$$

where F is their free product, and $V(F)$ is a verbal subgroup of F.

Definition 3.4. *With $V(F) = {}^nF$ we get the n-th nilpotent product in the sense of* Moran.

However it follows from results of Golovin in [2] that $O_n(F) = {}^nF \cap [G_\alpha]^F$, which means that the n-th nilpotent multiplication of Moran coincides with that of Golovin, for each $n = 1, 2, \ldots$.

The following results will be needed.

(3.5) The k-th nilpotent product of groups which are nilpotent of class at most m is nilpotent of class at most $\max(k, m)$. (Golovin [2]).

(3.6) The V-verbal product G of groups G_α such that $V(G_\alpha) = E$ is also such that $V(G) = E$. Further, any group H generated by isomorphic copies of the G_α in such a way that $V(H) = E$ is a factor group of G. (Moran [6]).

(3.7) The V-verbal product G of groups G_α is maximal in the sense that if H is any group generated by isomorphic copies of the G_α in such a way that $V(H) \cap [G_\alpha]^H = E$, then H is a factor group of G. (Moran [6]).

(3.8) The n-th nilpotent product G of groups G_α is maximal in the sense that if H is any group generated by isomorphic copies of the G_α in such a way that $O_n(H) = E$, then H is a factor group of G.

For, let H be generated by isomorphic copies $\hat{G}_\alpha$ of the G_α, where to each α there is an isomorphism φ_α of G_α onto $\hat{G}_\alpha$. Then by the characteristic property of the ordinary free product F of the G_α, these isomorphisms φ_α can be simultaneously extended to a homomorphism φ of F onto H. Let N be the kernel of φ, so that by Remark (2.3),

$$O_n(F; \{G_\alpha\})\varphi = O_n(F\varphi; \{G_\alpha\varphi\}) = O_n(H; \{\hat{G}_\alpha\}) = E.$$

Thus $N \supseteq O_n(F)$ and we see

$$H \simeq F/N \simeq F/O_n(F) \Big/ N/O_n(F).$$

But $F/O_n(F) \simeq G$, and we are through.

Consider next the second nilpotent product G of the groups A and B. By definition it is $F/[F,[A,B]]$, so that the cartesian of A and B is $[A,B]^F/[F,[A,B]]$. This means that the cartesian is central in G, and it now follows quickly that

$$[a_1 a_2, b] = [a_1, b][a_2, b]$$
$$[a, b_1 b_2] = [a, b_1][a, b_2],$$

in G. Thus $[A,B]^G$ is seen to be a factor group of the tensor product $A \otimes B$, by VON DYCK's theorem. We shall now show that in fact $A \otimes B$ and $[A,B]^G$ are isomorphic. This was conjectured by B. H. NEUMANN, and first proved by T. S. MAC HENRY in [5]. The following proof, due to the author, is shorter than MAC HENRY's. We first need a preparatory lemma.

Lemma 3.9. *Let F be the free product of two groups A and B. Then the subgroup $[F,[A,B]]$ is generated by all transforms by elements of $[A,B]$ of all elements of the form $[[a_1,b_1],a]$, $[[a_1,b_1],b]$, where a, a_1 range over A and b, b_1 range over B.*

PROOF. Since F is generated by A and B, and $[A,B]$ is normal in F, it follows from results in [2] that $[F,[A,B]]$ is generated by all elements $[u,a]$, $[u,b]$, where u ranges over $[A,B]$, a over A and b over B.

Consider $[u,a]$. Firstly since $[A,B]$ is generated by all commutators $[a,b]$, u is expressible in the form

$$u = [a_1, b_1]^{\varepsilon_1}[a_2, b_2]^{\varepsilon_2} \cdots [a_s, b_s]^{\varepsilon_s},$$

where $\varepsilon_i = \pm 1$ for each i. Thus $[u,a]$ is a product of transforms by elements of $[A,B]$ of commutators of the form $[[a_i, b_i]^{\varepsilon_i}, a]$. Now if $\varepsilon_i = -1$, we use the equation

$$[[a_i, b_i]^{-1}, a] = [a, [a_i, b_i]]^{[a_i, b_i]}$$

to see that $[u,a]$ is a product of transforms by elements of $[A,B]$ of elements of the form $[[a_i, b_i], a]$ and their inverses. A similar argument applies for $[u,b]$, and the lemma follows.

Theorem 3.10. *The tensor product of two groups A and B is isomorphic with the cartesian of A and B in the second nilpotent product G of A and B, and the correspondence*

$$[a, b] \leftrightarrow a \otimes b$$

generates an isomorphism.

PROOF. Since $G = F/[F, [A, B]]$, it follows that the cartesian in question is the factor group $[A, B]^F/[F, [A, B]]$. Now by Lemma 3. 9, $[F, [A, B]]$ is generated by all transforms by elements of $[A, B]^F$ of all triple commutators $[[a_1, b_1], a]$, $[[a_1, b_1], b]$. In other words, $[F, [A, B]]$ is the normal closure in $[A, B]^F$ of these triple commutators. But

$$[[a_1, b_1], a] = [a_1, b_1]^{-1}[a_1 a, b_1][a, b_1]^{-1},$$
$$[[a_1, b_1], b] = [a_1, b_1]^{-1}[a_1, b]^{-1}[a_1, b_1 b].$$

Moreover, by results in [2], $[A, B]^F$ is free on the commutators $[a, b]$ where $a \neq 1$, $b \neq 1$. This means that $[A, B]^G$ may be generated by formal symbols $a \otimes b$ subject to relations

$$a_1 a \otimes b_1 = (a_1 \otimes b_1)(a \otimes b_1),$$
$$a_1 \otimes b b_1 = (a_1 \otimes b_1)(a \otimes b).$$

But this group is clearly also abelian, and so it is isomorphic with the tensor product $A \otimes B$, as required.

§ 4. Generalised products of groups with amalgamations.

Observing that the extreme cases of regular multiplication have been generalised to include the contingency that the constituent subgroups do not intersect in the unit element (O. SCHREIER [12], H. NEUMANN [10], [11], B. H. NEUMANN [8]) we are naturally led to attempt a similar extension of the concepts of GOLOVIN and MORAN quoted in § 2. What this would mean in the utmost generality is not clear (at least not to the author), but it is relatively simple to extend the definitions of Golovin nilpotent products and Moran verbal products, in the following ways.

Definition 4. 1. *The group G is a generalised Golovin n-th nilpotent product of its subgroups G_α (where α ranges over some index set $\mathfrak{M}$) with amalgamations $G_\alpha \cap G_\beta = H_{\alpha\beta}$ $(\alpha \neq \beta)$,* if

$$\text{(i)} \qquad G = \mathrm{Gp}(G_\alpha, \alpha \in \mathfrak{M}),$$
$$\text{(ii)} \qquad O_n(G; \{G_\alpha\}) = E.$$

For brevity we say G is a generalised GN_n product of the G_α with the given amalgamations.

Definition 4. 2. *If V is a set of words, the group G is a generalised Moran V-verbal product (for short, generalised MV-product) of its subgroups G_α (where α ranges over some index set $\mathfrak{M}$) with amalgamations $G_\alpha \cap G_\beta = H_{\alpha\beta}$ $(\alpha \neq \beta)$, if*

$$\text{(i)} \qquad G = \mathrm{Gp}(G_\alpha, \alpha \in \mathfrak{M}),$$
$$\text{(ii)} \qquad V(G) \cap [G_\alpha]^G = E.$$

With $V(G)={}^{n}G$ *we have the generalised* MN_n *product, and with* $V(G)=G^{(n)}$ *we have the generalised* MS_n *product.*

In these abbreviations, the N stands for nilpotent and the S for soluble.

Examples of generalised products of these kinds readily come to mind. For instance the regular nilpotent and verbal products are such; furthermore any group which is nilpotent of class at most n is a generalised GN_n product of any generating set of subgroups, and any group G such that $V(G)=E$ is a generalised MV product of any generating set of subgroups.

If will be noted that Definitions 4.1 and 4.2 carry no mention of the "freeness" of the product in question. This is because it is found more convenient to prove, from a knowledge of the existence of a product, the existence of a "free" one. To make this idea precise, we let X stand for a relevant property, and employ the following definition of freeness. (Cf. GRACE E. BATES [1]).

Definition 4.3. *The group* G *is a generalised free* X *product of its subgroups* G_α $(\alpha\in\mathfrak{M})$ *with amalgamations* $H_{\alpha\beta}=G_\alpha\cap G_\beta$ $(\alpha\neq\beta)$, *if the following two conditions hold.*

(i) *G is a generalised X product of the G_α with the given amalgamations.*

(ii) *If $\hat{G}$ is a generalised X product of its subgroups $\hat{G}_\alpha$ $(\alpha\in\mathfrak{M})$ with amalgamations $\hat{G}_\alpha\cap\hat{G}_\beta=\hat{H}_{\alpha\beta}$ $(\alpha\neq\beta)$, where for each $\alpha\in\mathfrak{M}$ there exists an isomorphism φ_α of G_α onto $\hat{G}_\alpha$ such that*

(a) *for every* $\beta\neq\alpha$, $H_{\alpha\beta}\varphi_\alpha=\hat{H}_{\alpha\beta}$,

(b) *if* $h\in H_{\alpha\beta}$ *then* $h\varphi_\alpha=h\varphi_\beta$,

then the φ_α can be simultaneously extended to a homomorphism of G onto $\hat{G}$.

According to this definition, G is the 'largest' group of its kind.

Example 4.4. The elementary abelian group of order 8 is a GN_2 product of two of its four-group subgroups amalgamating a cycle of order 2. The free one of this type is however

$$G=\mathrm{Gp}\,(a,b,c;\ a^2=b^2=c^2=[a,b]=[a,c]=[[x,y],z]=1),$$

where x,y,z range over the elements a,b,c. It is, in fact, the generalised free product "made second nilpotent". It is readily verified that G is of order 16 and nilpotent of class 2.

In this example we have implied the uniqueness of the generalised free X product — cf. the phrase "the free one". This uniqueness is demonstrated in the following theorem, which may be proved by an almost word-for-word repetition of the proof of the uniqueness of the generalised free product, in [9], pp. 505—506.

Theorem 4.5. *Let G, $\hat{G}$ be generalised free X products of their subgroups G_α $(\alpha \in \mathfrak{M})$ and $\hat{G}_\alpha$ $(\alpha \in \mathfrak{M})$ respectively with respective amalgamations $H_{\alpha\beta} = G_\alpha \cap G_\beta$ and $\hat{H}_{\alpha\beta} = \hat{G}_\alpha \cap \hat{G}_\beta$ $(\alpha \neq \beta)$. Let there be, further, to each $\alpha \in \mathfrak{M}$ an isomorphism φ_α of G_α onto $\hat{G}_\alpha$ which maps the intersections $H_{\alpha\beta}$ onto corresponding intersections $\hat{H}_{\alpha\beta}$. Specifically if $h \in H_{\alpha\beta}$ we assume*

$$h\varphi_\alpha = h\varphi_\beta,$$

and if $\hat{h} \in \hat{H}_{\alpha\beta}$,

$$\hat{h}\varphi_\alpha^{-1} = \hat{h}\varphi_\beta^{-1}.$$

Then G is isomorphic with $\hat{G}$.

This theorem says nothing about the existence of generalised free X products with amalgamations, it merely states that whenever such a product exists, it is in a very strong sense unique. However, starting with the knowledge of the existence of a generalised GN_n or MV product of a system of groups, we now give an explicit construction for free products of this kind.

Theorem 4.6. *If G is a generalized GN_n (MV) product of its subgroups G_α $(\alpha \in \mathfrak{M})$ with amalgamations $G_\alpha \cap G_\beta = H_{\alpha\beta}$ $(\alpha \neq \beta)$, then*

(i) *the generalised free product F of the G_α with amalgamated $H_{\alpha\beta}$ exists,*

(ii) *the generalised free GN_n (MV) product of the G_α with amalgamated $H_{\alpha\beta}$ exists, and it is $F/O_n(F)$ $(F/(V(F) \cap [G_\alpha]^F))$.*

PROOF. The existence of the generalised free product F follows for example from [9], pp. 518—520. For clarity of exposition we assume that F is the generalised free product of the subgroups $\hat{G}_\alpha$ with intersections $\hat{H}_{\alpha\beta} = \hat{G}_\alpha \cap \hat{G}_\beta$ $(\alpha \neq \beta)$, where to each $\alpha \in \mathfrak{M}$ there is an isomorphism φ_α of $\hat{G}_\alpha$ onto G_α such that

$$\hat{H}_{\alpha\beta}\varphi_\alpha = H_{\alpha\beta},$$

$$\hat{h} \in \hat{H}_{\alpha\beta} \Rightarrow \hat{h}\varphi_\alpha = \hat{h}\varphi_\beta,$$

$$h \in H_{\alpha\beta} \Rightarrow h\varphi_\alpha^{-1} = h\varphi_\beta^{-1}.$$

Then by the characteristic property of the generalised free product ([9], p. 505.), these isomorphisms φ_α can be simultaneously extended to a homomorphism φ of F onto G. Suppose then that the kernel of φ is N, and let us treat the two parts of the theorem separately.

(a) Suppose G is a GN_n product of the G_α with amalgamations $H_{\alpha\beta}$, so that $O_n(G; \{G_\alpha\}) = E$. However by (2. 3),

$$O_n(F; \{\hat{G}_\alpha\})\varphi = O_n(F\varphi; \{\hat{G}_\alpha\varphi\}) = O_n(G; \{G_\alpha\}) = E.$$

In other words, $N \supseteq O_n(F; \{\hat{G}_\alpha\}) = O_n(F)$. Next put $Q = F/O_n(F)$, and let θ

be the canonic homomorphism of F onto Q. We shall show that Q is the required generalised free GN_n product. This is done in four stages.

(i) *Q is generated by isomorphic copies of the G_α.* This is because it is generated by the $\hat{G}_\alpha\theta$, and

$$\hat{G}_\alpha\theta \cong \hat{G}_\alpha/(\hat{G} \cap O_n(F)) \cong \hat{G}_n,$$

since

$$\hat{G}_\alpha \cap O_n(F) \subseteq \hat{G}_\alpha \cap N,$$

and φ is isomorphic on the constituents $\hat{G}_\alpha$.

(ii) $\hat{G}_\alpha\theta \cap \hat{G}_\beta\theta = \hat{H}_{\alpha\beta}\theta$ $(\alpha, \beta \in \mathfrak{M})$. For

$$x \in \hat{G}_\alpha\theta \cap \hat{G}_\beta\theta \Longrightarrow x = \hat{g}_\alpha\theta = \hat{g}_\beta\theta \Longrightarrow \hat{g}_\beta^{-1}\hat{g}_\alpha \in O_n(F),$$

for some $\hat{g}_\alpha \in \hat{G}_\alpha, \hat{g}_\beta \in \hat{G}_\beta$. But since $O_n(F) \subseteq N$,

$$1 = (\hat{g}_\beta^{-1}\hat{g}_\alpha)\varphi = \hat{g}_\beta^{-1}\varphi_\beta\hat{g}_\alpha\varphi_\alpha,$$

so that

$$\hat{g}_\alpha\varphi_\alpha = \hat{g}_\beta\varphi_\beta \in H_{\alpha\beta}.$$

Hence $\hat{g}_\beta = \hat{g}_\alpha \in \hat{H}_{\alpha\beta}$, and therefore

$$\hat{G}_\alpha\theta \cap \hat{G}_\beta\theta \subseteq \hat{H}_{\alpha\beta}\theta.$$

This together with the obvious reverse inclusion gives the assertion.

(iii) *Q is a generalised GN_n product of its subgroups $\hat{G}_\alpha\theta$ with amalgamations $\hat{H}_{\alpha\beta}\theta$, where of course $\hat{H}_{\alpha\beta} \cong \hat{H}_{\alpha\beta}\theta$.* This is because

$$O_n(Q; \{\hat{G}_\alpha\theta\}) = O_n(F\theta; \{\hat{G}_\alpha\theta\}) = O_n(F; \{\hat{G}_\alpha\})\theta = E.$$

(iv) *In fact Q is the free product of this kind.* This follows from the isomorphisms

$$G \cong F/N \cong F/O_n(F)\big/N/O_n(F),$$

which show that G is a homomorphic image of Q.

(b) If, on the other hand, G is a generalised MV product, it follows that $N \supseteq V(F) \cap [G_\alpha]^F$. For $V(F)\varphi = V(F\varphi)$,

$$[\hat{G}_\alpha]^F\varphi = [\hat{G}_\alpha\varphi]^{F\varphi} = [G_\alpha]^G.$$

Hence

$$(V(F) \cap [\hat{G}_\alpha]^F)\varphi \subseteq V(G) \cap [G_\alpha]^G = E,$$

and we get the answer. The proof that $F/(V(F) \cap [G_\alpha]^F)$ is the required generalised free MV product follows in a manner exactly analagous to that of part (a) of this theorem.

The following two properties of the generalised multiplications will be found useful later.

(4. 7) By (3. 7), the generalised free MV product of groups G_α with amalgamations is a factor group of the (free) regular V-verbal product of the G_α; so that in particular the generalised free MV product G of groups G_α such that $V(G_\alpha)=E$ also has this property, $V(G)=E$.

(4. 8) Further the generalised free GN_n product of groups which are nilpotent of class at most m is nilpotent of class at most $\max(n, m)$.

This follows quickly from (3. 5) and (3. 8).

§ 5. Statement of the problem.

Up to now we have dealt only with groups which we already know to be generalised GN_n or MV products of certain subgroups. The chief aim of this paper is to consider something of the converse and more difficult situation, which is perhaps best stated as follows.

Let a set G_α of groups be given, where α ranges over some index set $\mathfrak{M}$. In every G_α and to every $\beta \in \mathfrak{M}$ let a subgroup $H_{\alpha\beta}$ be distinguished where (to avoid special consideration of the case $\alpha=\beta$) we insist that $H_{\alpha\alpha}=G_\alpha$, for each $\alpha \in \mathfrak{M}$. Does there exist a group G which is in some sense the generalised free X product (X being a GN_n or an MV) of the G_α with amalgamations $H_{\alpha\beta}$? More precisely, we ask about the existence of a group G which is the generalised free X product of its subgroups $\hat{G}_\alpha$, which intersect pairwise thus:

$$\hat{H}_{\alpha\beta}=\hat{G}_\alpha \cap \hat{G}_\beta=\hat{H}_{\beta\alpha},$$

and moreover there is to exist, to each $\alpha \in \mathfrak{M}$, an isomorphism φ_α of G_α onto $\hat{G}_\alpha$,

$$G_\alpha \varphi_\alpha=\hat{G}_\alpha,$$

such that

$$H_{\alpha\beta}\varphi_\alpha=\hat{H}_{\alpha\beta}.$$

If this is so, we say that the generalised free X product of the G_α with amalgamations $H_{\alpha\beta}$ exists.

We already know (Theorem 4. 6) that to guarantee the existence of the generalised free X product of the G_α, we need only show the existence of a G which is a generalised X product of its subgroups G_α (not necessarily free).

On the other hand, certain conditions necessary for the existence of the generalised free X product are evident. Firstly $H_{\alpha\beta}$ and $H_{\beta\alpha}$ have to be isomorphically mapped onto the same group,

$$\hat{G}_\alpha \cap \hat{G}_\beta=\hat{H}_{\alpha\beta}=\hat{H}_{\beta\alpha},$$

which means that they themselves have to be isomorphic. Moreover the mapping

$$\iota_{\alpha\beta}=\varphi_\alpha \varphi_\beta^{-1}$$

must be an isomorphism of $H_{\alpha\beta}$ onto $H_{\beta\alpha}$, and

$$\iota_{\beta\alpha} = \varphi_\beta \varphi_\alpha^{-1}$$

is inverse to $\iota_{\alpha\beta}$. These $\iota_{\alpha\beta}$ are called the amalgamating isomorphisms. It will be noted that they play an essential role in the construction of the generalised free X product; therefore to ensure strict accuracy the notation should bear some reference to the $\iota_{\alpha\beta}$. However, though they may not be explicitly mentioned, they are always implicit in any example, so that in fact the notation is unambiguous.

Even further, it follows from results in [9] that it is necessary for the existence of the generalised free X product that the generalised free product of the G_α with the given amalgamations exist. We shall consequently only consider systems of groups for which this condition holds. Certain sets of criteria for this to be so have been derived, and may conveniently be found in [9]; however they do not concern us here, for our attention will be chiefly centred on the case where our system consists of two groups with an amalgamation — and in this case the famous theorem of SCHREIER tells us that the generalised free product exists.

The problem we set ourselves is to find necessary and sufficient conditions that the generalised free GN_2 and MN_2 product of two groups with an amalgamation exists. This we do in § 9, after some preliminary investigation of the situation, in § 6 and § 7.

We next record some useful facts.

(5. 1) If the generalised free MV product of a set of groups exists, and V' is a set of words such that

$$V'(L) \subseteqq V(L)$$

for all groups L, then the generalised free MV' product also exists. For if G is the generalised free MV product of groups G_α, $V(G) \cap [G_\alpha]^G = E$, which gives $V'(G) \cap [G_\alpha]^G \subseteqq V(G) \cap [G_\alpha]^G = E$. This means that G is an MV' product of the G_α, though of course not necessarily the free one. (Example 4. 4.)

(5. 2) Suppose that G is the generalised direct product of its subgroups G_α, where by this we of course mean that $G = \mathrm{Gp}(\{G_\alpha\})$ and $[G_\alpha] = E$. Then since $O_n(G; \{G_\alpha\}) \subseteqq [G_\alpha]^G$, $V(G) \cap [G_\alpha]^G \subseteqq [G_\alpha]^G$, G is both a generalised MV product and a generalised GN_n product of the G_α, for any set V of words and any positive integer n.

(5. 1), (5. 2) and Theorem 4. 6 now enable us to construct many examples of generalised free X products, for various properties X; for we need only take all the groups to be abelian, and the amalgamated subgroup to be one and the same group.

(5. 3) We now give an example of a system of two groups whose generalised free GN_2 product does not exist. Both groups are nilpotent of class two, being dihedral groups of order 8:

$$G_1 = \mathrm{Gp}\,(a, b; a^4 = b^2 = (ab)^2 = 1),$$
$$G_2 = \mathrm{Gp}\,(c, d; c^4 = d^2 = (cd)^2 = 1).$$

Then put $H_1 = \mathrm{Gp}\,(a^2)$, $H_2 = \mathrm{Gp}\,(d)$, and define an isomorphism φ of H_1 onto H_2 by the rule $a^2\varphi = d$. If we now try to form the generalised free GN_2 product of G_1 and G_2 amalgamating H_1 and H_2 according to φ, it has to be

$$G = \mathrm{Gp}\,(a, b, c, d; a^4 = b^2 = c^4 = d^2 = (ab)^2 = (cd)^2 = 1, a^2 = d, [[x, y], z] = 1),$$

where x, y, z can each assume the four values a, b, c, d. This fact follows from Theorem 4. 6 and Remark 4. 8. Turning our attention to G, we see that $[a^2, c] = [d, c] = c^2$, and $[a^2, c] = [[a, b], c] = 1$. This means that in G, the group G_2 is not isomorphically represented in the way that it should be; which means that the required generalised free GN_2 product does not exist.

Examples of this nature may be endlessly multiplied with only a small amount of difficulty.

(5. 4) It is worthy of attention that the generalised free X product of certain groups may coincide with the generalised free X' product (X, X' denoting different properties), even though this is not the case for the associated regular products. For instance let $A = \mathrm{Gp}\,(a)$ and $B = \mathrm{Gp}\,(b)$ be infinite cycles; then the generalised free GN_2 product of A and B amalgamating any subgroup exists. We form that amalgamating the cycle $\mathrm{Gp}\,(a^2)$ in A with $\mathrm{Gp}\,(b^3)$ in B. It is

$$G = \mathrm{Gp}\,(a, b; a^2 = b^3 = [[a, b], a] = [[a, b], b] = 1).$$

But then $1 = [b^3, b] = [a^2, b] = [a, b]^a [a, b] = [a, b]^2$, and similarly $1 = [a, a^2] = [a, b^3] = [a, b]^3$. It thus follows that $[a, b] = 1$, and therefore that G is the generalised direct product of A and B with the given amalgamation. However the free second nilpotent group on two generators differs from the free abelian group of rank two.

Theorem 5. 5. *The generalised free GN_n (MV) product of groups G_α ($\alpha \in \mathfrak{M}$) with amalgamations $H_{\alpha\beta}$ exists if and only if*

(i) *the generalised free product F exists,*

(ii) $g_\alpha g_\beta \in O_n(F)$ ($\in V(F) \cap [G_\alpha]^F$) *and* $g_\alpha \in G_\alpha, g_\beta \in G_\beta$ ($\alpha \neq \beta$) implies that $g_\alpha g_\beta = 1$.

PROOF. We observe first of all that no confusion arises from using the G_α themselves and not isomorphic copies of them in the generalised free product. Secondly that the necessity of conditions (i) and (ii) follows immediately from the proof of Theorem 4.6.

Suppose on the other hand that conditions (i) and (ii) are satisfied; once again we prove their sufficiency for the GN_n case, the MV following by a very similar argument.

We show (as would be expected) that $Q = F/O_n(F)$ is a generalised GN_n product of the G_α with amalgamations $H_{\alpha\beta}$. To this end we let θ be the canonic homomorphism of F onto Q.

(a) *Q is generated by isomorphic copies of the G_α.* This is because it is generated by the $G_\alpha\theta$, and

$$G_\alpha\theta \cong G_\alpha/(G_\alpha \cap O_n(F)).$$

However condition (ii) with $g_\beta = 1$ implies that $G_\alpha \cap O_n(F) = E$, so that $G_\alpha\theta \cong G_\alpha$. Moreover $H_{\alpha\beta}\theta \cong H_{\alpha\beta}$.

(b) $G_\alpha\theta \cap G_\beta\theta = H_{\alpha\beta}\theta$ $(\alpha \neq \beta)$. Firstly suppose that $x \in G_\alpha\theta \cap G_\beta\theta$. Then

$$x = g_\alpha\theta = g_\beta\theta,$$

where $g_\alpha \in G_\alpha, g_\beta \in G_\beta$, so that

$$g_\beta^{-1} g_\alpha \in O_n(F).$$

Condition (ii) now gives $g_\beta = g_\alpha \in (G_\alpha \cap G_\beta) = H_{\alpha\beta}$, which means that

$$G_\alpha\theta \cap G_\beta\theta \subseteq H_{\alpha\beta}\theta.$$

The reverse inclusion is obvious, and the assertion follows.

(c) *Q is a generalised GN_n product.* For $O_n(Q) = O_n(F\theta) = O_n(F)\theta = E$.

Thus Q has all the desired properties, and the theorem is proved.

This is a convenient stage at which to discuss the interaction of the various generalised multiplications, before proceeding to a deeper investigation of the generalised free second nilpotent products.

§ 6. Comparison of the generalised multiplications.

This section is devoted to an investigation of the connection between the GN_n and the MN_n products, and a demonstration of the lack of connection between the GN_n and the MS_m products. The meaning of this will become clear as the section proceeds.

We first observe that the free regular nilpotent multiplications of class n are identical (§ 3). Moreover, every generalised MN_n product of certain groups is simultaneously a GN_n product of these groups; for if G is a generalised MN_n product of its subgroups G_α,

$$O_n(G; \{G_\alpha\}) \subseteq {}^nG \cap [G_\alpha]^G = E.$$

by (6. 2) and (3. 8), we conclude that every element of ${}^nQ$ has an expression of the form $\bar{a}\bar{b}$, for some $\bar{a}\in{}^n(A\theta)$, $\bar{b}\in{}^n(B\theta)$. But ${}^nQ=({}^nG\theta)=({}^nG)\theta$, so that if $g\theta\in{}^nG$, $g=a\theta b\theta=(ab)\theta$ for suitable $a\in{}^nA$, $b\in{}^nB$. Thus $g=abu$, where $u\in O_n(G)$, as required.

We can now prove the following theorem.

Theorem 6. 4. *If the generalised free MN_n and GN_n products of two groups A and B exist, these products are identical (in the sense that they are factor groups of the generalised free product F of A and B by one and the same normal subgroup).*

PROOF. By Theorem 4. 6, the generalised free MN_n product is $F/{}^nF\cap[A,B]^F$ and the generalised fre GN_n product is $F/O_n(F)$.

Now since $[A,B]\trianglelefteq F$,

$${}^nF\cap[A,B]\supseteqq O_n(F).$$

On the other hand let $x\in{}^nF\cap[A,B]$. Then since $x\in{}^nF$, by Lemma 6. 3 we have $x=abu$ for some $a\in{}^nA$, $b\in{}^nB$, $u\in O_n(F)$. Thus, as $x\in[A,B]$ it follows that $ab\in[A,B]$; but $ab\in{}^nF$ since both a and b do, so that in fact $ab\in{}^nF\cap[A,B]$. Theorem 5. 5 now gives us that $ab=1$. Hence $x=u\in O_n(F)$, and

$${}^nF\cap[A,B]\subseteqq O_n(F).$$

The two inclusions now established prove the theorem.

I do not know if a result analogous to Theorem 6. 4 is true if one considers systems of more than two groups.

It might be conjectured that if the generalised free GN_2 product of two groups exists then it must be true that the generalised free MN_3 or MN_4 product exists. This is not in fact true, far from it; we next give, for each n, an example of two groups whose generalised free GN_2 product exists, but for which not even the generalised free MS_n product exists.

Example 6. 5. In [4], PHILLIP HALL gives p-groups of arbitrary high nilpotency class such that

$$G^{(i)}={}^{2^i-1}G$$

for each $i=0,1,2,\ldots$. Let G be one of these groups which is soluble of length $n+2$:

$$G=G^{(0)}\supset G^{(1)}\supset\cdots\supset G^{(n+1)}\supset G^{(n+2)}=E,$$
$$G={}^0G\supset{}^1G\supset\cdots\supset{}^{m-1}G\supset{}^mG=E,$$

where the inclusions are strict. Then we assert that $m+1>2^{n+1}$. For if not $m+1\leqq 2^{n+1}$ and

$$G^{(n+1)}={}^{2^{n+1}-1}G\subseteqq{}^mG=E,$$

which is clearly not the case.

Let now Z be the centre of G and let h be an element of ${}^{m-2}G$ lying outside Z. Such an element must exist, else G would have nilpotency class less than m. Put

$$H = \mathrm{Gp}\,(h);$$

our system of groups is now to consist of H and G itself.

(a) The generalised free GN_2 product of G and H amalgamating H exists. Indeed, G is that very product; for it is the generalised free product, and moreover

$$[H, G] \subseteq {}^{m-1}G,$$

so that

$$[[G, H], G] = E.$$

Thus by Theorem 5. 5, G is the required generalised free GN_2 product.

(b) However the generalised free MS_n product does not exist. For in G, which is (as above) the generalised free product of G and H, we give a non-trivial element of $[G, H] \cap G^{(n)}$. To do this, let g be an element of G not commuting with h; then $1 \neq [g, h] \in [G, H] \cap {}^{m-1}G$. Now as $m+1 > 2^{n+1}$, we have $m \geqq 2^n$; therefore

$${}^{m-1}G \subseteqq {}^{2^n-1}G = G^{(n)},$$

and further $[g, h] \in [G, H] \cap G^{(n)}$. This completes the proof of (b).

Example 6. 6. The phenomenon which is converse to that of Example 6. 5 is much more readily demonstrated. Let G be the free regular MS_n product of the infinite cycles $\mathrm{Gp}\,(a)$ and $\mathrm{Gp}\,(b)$, and put $H = \mathrm{Gp}\,(a)$. Our system of groups is to consist of G and H.

(a) Obviously $[[H, G], G] \neq E$, since $[H, G]$ contains the commutator $[a, b]$, which is not central (unless $n = 1$, which trivial case we exclude). Thus by the now familiar argument, the generalised free GN_2 product of G and H does not exist.

(b) However G is the generalised free MS_n product of G and H. This follows because $G^{(n)} = E$ and therefore $G^{(n)} \cap [A, B] = E$.

Example 6. 7. If A and B are both abelian, then both the generalised free GN_n and the generalised free MS_n product of A and B amalgamating any isomorphic subgroups exist.

Example 6. 8. To complete the picture, that is to give an example of a pair of groups whose generalised free GN_n and MS_n products do not exist for any n, let G be any non-abelian simple group, and H any proper subgroup of G. Then $[G, H] = {}^nG = G^{(n)} = O_n(G) = G$, so that none of the above products of G and H amalgamating H exists.

It is found, therefore, that there is in fact no comparison in freeness between the generalised GN_n and MS_n products. This somewhat pathological situation can be perhaps explained by the bad behaviour of intersections under homomorphisms. Namely if N and M are subgroups of G and φ is a homomorphism of G then it is not necessarily true that

$$(N \cap M)\varphi = N\varphi \cap M\varphi,$$

though we always have

$$(N \cap M)\varphi \subseteq N\varphi \cap M\varphi.$$

Thus we cannot always assert that homomorphic images of generalised MV products are themselves MV products; and in fact Example 6.1 gives examples of homomorphic images of free regular V-verbal products which are not generalised MV products of the homomorphic images of the constituents. For in that example G is a homomorphic image of the free regular GN_n product of an isomorphic copy of G and an isomorphic copy of Z_{n-1} (Remark 3.8). This answers a question of MORAN [6].

§ 7. The generalised GN_2 and MN_2 products; preliminary investigation.

We shall now turn our attention more exclusively to a consideration of the generalised free GN_2 and MN_2 products of a pair of groups with amalgamation. Our aim is to find necessary and sufficient conditions for the existence of such products; the present section is devoted to finding out something of the sort of conditions that can be expected to work.

It is found useful in this section to use the amalgam $\mathfrak{A}$ of the groups A and B with the subgroup H amalgamated; in symbols

$$\mathfrak{A} = \{A, B;\ A \cap B = H\}.$$

That is, in the system of groups A, B where $A \supseteqq H$, $B \supseteqq H\varphi$, φ being an isomorphism of H onto $H\varphi$, we regard each element h of H as identified with its image $h\varphi$ in $H\varphi$. The amalgam $\mathfrak{A}$ is then an 'incomplete group' (see [9]). We look for conditions that $\mathfrak{A}$ generate a group which is a GN_2 or MN_2 product of A and B intersecting in H, as it were keeping the amalgam intact. To abbreviate certain frequently-occurring phrases we call conditions that the generalised free GN_2 (MN_2) product of $\mathfrak{A}$ exist GN_2 (MN_2) conditions for $\mathfrak{A}$.

The following lemmas give some necessary GN_2 conditions for $\mathfrak{A}$ and therefore necessary MN_2 conditions, as every condition which is necessary GN_2 for $\mathfrak{A}$ is automatically necessary MN_2 for $\mathfrak{A}$.

Lemma 7.1. *If the group G is a generalised GN_n product of its subgroups G_α $(\alpha \in \mathfrak{M})$ with amalgamations $H_{\alpha\beta} = G_\alpha \cap G_\beta$ $(\alpha \neq \beta)$ then*

$$[\ldots[[H_{\alpha\beta}, G_\alpha], G_\alpha], \ldots, G_\alpha] = E,$$

the number of commutations being n. That is, for each α *and each* $\beta \neq \alpha$, $H_{\alpha\beta}$ *is a subgroup of* $Z_n(G_\alpha)$.

PROOF. By definition

$$O_n(G) = [\ldots[[G_\alpha]^G, G], \ldots, G] = E.$$

But if $h \in H_{\alpha\beta}$ then $h \in G_\beta$; and it therefore follows from the above that if $g_1, g_2, \ldots, g_n$ are arbitrary elements of G_α, the commutator

$$[\ldots[[h, g_1], g_2], \ldots, g_n]$$

must equal the unit element. But this is exactly the requirement that $h \in Z_n(G_\alpha)$.

COROLLARY 7.2. *For each pair* α, β *of indices with* $\alpha \neq \beta$, *the group* $H_{\alpha\beta}$ *is nilpotent of class at most n.*

Lemma 7.3. *If the group G is a generalised* GN_n *product of its subgroups A and B,*

$$[{}^{n-1}A, B] = [{}^{n-1}B, A] = E.$$

PROOF. It suffices to remark that the result is true in the free regular n-th nilpotent product K of A and B (see [2]) and that by Remark (3.8) of this paper, G is a factor group of K.

Lemma 7.4. *If the group G is a generalised* MN_n *product of its subgroups* G_α $(\alpha \in \mathfrak{M})$ *with amalgamations* $H_{\alpha\beta}$, *then for each* $\beta \neq \alpha$,

$$^{n}G_\alpha \cap [G_\alpha, H_{\alpha\beta}] = E.$$

The proof is obvious, and omitted.

We can now write down the following set of necessary GN_2 conditions for the amalgam $\mathfrak{A} = \{A, B; A \cap B = H\}$:

$$[[A, H], A] = [[B, H], B] = [A' \cap H, B] = [B' \cap H, A] = E. \tag{7.5}$$

Observe that all commutations are carried out completely inside one or other of the factors. Further (7.5) together with the following set of conditions are MN_2 necessary for $\mathfrak{A}$:

$$\begin{aligned}[A, A'] \cap [H, A] = [B, B'] \cap [B, H] = [A, A'] \cap H \cap [B,H] = \\ = [B, B'] \cap H \cap [A, H] = E.\end{aligned} \tag{7.6}$$

For some time it was thought that conditions (7.5) were also GN_2 sufficient for $\mathfrak{A}$. Such is very far from the truth, as we shall now see.

Lemma 7.7. *Let* $\mathfrak{A} = \{A, B; A \cap B = H\}$ *be an amalgam such that there exist* $a \in A, b \in B$ *where for some positive integer n,*

(i) $a^n \in H, b^n = 1,$

(ii) $[b, a^n] \neq 1.$

Then the generalised free GN_2 *product of* $\mathfrak{A}$ *does not exist.*

PROOF. Suppose that it does, and let it be G. Then in G, the commutator $[a, b]$ is central so that

$$[a^n, b] = [a, b]^n = [a, b^n] = 1,$$

which is a contradiction.

This lemma (of which it is easy to imagine variants) is a fruitful source of counterexamples. For instance the following example immediately shows that (7. 5) are not GN_2 sufficient.

Example 7. 8. Our amalgam consists of a dihedral group of order 8 and a cyclic group of order 8 intersecting in a cyclic group of order 4. Namely

$$A = \mathrm{Gp}(f; f^8 = 1),$$

$$B = \mathrm{Gp}(c, d; c^4 = d^2 = (cd)^2 = 1),$$

where we put $f^2 = c$ and $H = \mathrm{Gp}(c)$. This amalgam satisfies conditions (7. 5) — indeed any amalgam one group of which is abelian and one nilpotent of class 2 will do so — and the requirements of Lemma 7. 7, with $n = 2, a = f$, $b = d$.

Example 7. 8 is an example of an amalgam of one abelian group and one nilpotent group of class two which cannot generate a nilpotent group of class two. Actually, something much worse is the case, namely we shall now give such an amalgam which can generate no nilpotent group, of finite or transfinite class. This means that conditions (7. 5) are not even GN_n sufficient, for any $n > 0$. The example depends on the following lemma, which is perhaps interesting in its own right.

Lemma 7.9. *Let G be an arbitrary group, g an element of G, and d an element of G whose square commutes with g. Then for each $r \geqq 0$, ${}^{r+1}G$ contains the commutator $[g^{2^r}, d]$.*

PROOF. This proceeds by induction. With $r = 0$, the result is obvious, since $[g, d]$ is clearly contained in ${}^1G$.

Suppose then that $r \geqq 0$ and that we have shown that ${}^{r+1}G$ contains $[g^{2^r}, d]$. Then ${}^{r+2}G$ contains the commutator

$$k = [g^{2^r} d, [g^{2^r}, d]] = d^{-1} g^{-2^r} d^{-1} g^{-2^r} d g^{2^r} g^{2^r} d g^{-2^r} d^{-1} g^{2^r} d.$$

Putting $d = d^{-1} z$ we see that z commutes with d and g; and as ${}^{r+2}G \trianglelefteq G$, it contains the transform k' of k by the element $d^{-1} g^{-2^r} d$. A little computation now gives

$$k' = z^{-1} g^{-2^r} d g^{2^{r+1}} d g^{-2^r} = g^{2^r} g^{-2^{r+1}} d^{-1} g^{2^{r+1}} d g^{-2^r}.$$

Further, ${}^{r+2}G$ contains the transform k'' of k' by g^{2^r},

$$k'' = g^{-2^{r+1}} d^{-1} g^{2^{r+1}} d$$
$$= [g^{2^{r+1}}, d].$$

This completes the induction, and proves the lemma.

Example 7.10. Let A be the dihedral group of order 8:

$$A = \mathrm{Gp}\,(c, d;\, c^4 = d^2 = (cd)^2 = 1).$$

B is a Prüfer group of type 2^∞:

$$B = \mathrm{Gp}\,(a_1, a_2, \ldots, a_n, \ldots;\, a_1^2 = 1, a_{n+1}^2 = a_n,\ n = 1, 2, \ldots).$$

We form an amalgam $\mathfrak{A}$ of A and B by putting $a_2 = c$ and $H = \mathrm{Gp}\,(c)$. Let then G be any group generated by the amalgam — such groups exist since the generalised free product does. By Lemma 7.9, ${}^{r+1}G$ contains $[g^{2^r}, d]$ for any $g \in G$. Observing that

$$a_{n+2}^{2^n} = a_2$$

for each $n = 0, 1, 2, \ldots$, we see that ${}^{r+1}G$ contains the commutator

$$[a_{r+2}^{2^r}, d] = [a_2, d] = [c, d] = c^2.$$

But $c^2 \neq 1$, so that G is not nilpotent, not even of transfinite class.

This example amply demonstrates the considerable amount of deviation from the expected that one encounters in this investigation. It leads one to conjecture that necessary and sufficient GN_2 conditions will have to be rather different from those we have so far examined. Precisely, we take our general amalgam $\mathfrak{A} = \{A, B;\, A \cap B = H\}$ and allow ourselves three methods of forming groups from the constituents of $\mathfrak{A}$, namely

(i) commutation;

(ii) intersection;

(iii) multiplication of normal subgroups.

We form all possible groups from $\mathfrak{A}$ using these operations; if G_1 and G_2 are such groups the equation

$$G_1 = G_2$$

is termed a *CIM* condition for $\mathfrak{A}$.

CONJECTURE 7.11. *No set of CIM conditions can be GN_2 necessary and sufficient for $\mathfrak{A}$.*

In fact it seems likely that the amalgam in Example 7.7 satisfies all possible necessary conditions of this sort.

In our search for GN_2 conditions for an amalgam of two groups, we have always finally investigated properties of groups not arising from the amalgam by the use of operations (i), (ii) and (iii) above; up to now it has been the generalised free product. A similar situation is present in the next section, when we investigate properties of the free regular second nilpotent product and of the tensor product. It thus appears that the difficulties here arising are very different in nature from those one finds in a study of the generalised free and generalised direct products. In the former case there is no restriction on the subgroup to be amalgamated; in the latter it has to be central in both constituents. This is a *CIM* condition, namely

$$[A, H] = [B, H] = E.$$

§ 8. Necessary and sufficient GN_2 conditions.

Instead of using the amalgam of groups we find it easier from now on to equip ourselves with a pair of groups A and B, where A contains a subgroup H and B a subgroup $H\varphi$ isomorphic to H under the isomorphism φ. In the present section we derive necessary and sufficient conditions that the generalised free GN_2 product of A and B amalgamating H and $H\varphi$ according to φ exist. The conditions achieved are not very manageable, or easily applied; but this is felt to be inherent in the nature of the problem.

We first need the following lemmas.

Lemma 8.1. *If the generalised free GN_2 product G of A and B amalgamating H and $H\varphi$ according to φ exists it is G_0/N, where*

(i) *G_0 is the free regular GN_2 product of A and B,*

(ii) *N is the normal closure in G_0 of all elements of the form $h^{-1}h\varphi$, where h ranges over H.*

PROOF. Let F_0 be the ordinary free product of A and B, and let F be the generalised free product amalgamating H and $H\varphi$. Then $F \cong F_0/M$, where M is the normal closure in F_0 of all elements $h^{-1}h\varphi$. Further by definition $G_0 = F_0/[F_0, [A, B]]$, and by Theorem 4.6, $G = F/[F, [A, B]]$. Thus

$$G \cong F_0/M \Big/ M[F_0, [A, B]]/M \cong F_0/M[F_0, [A, B]].$$

In other words G is obtained from F_0 first by making the cartesian central, and then by applying all relations $h = h\varphi$, which means by VON DYCK's theorem that $G \cong G_0/N$.

Lemma 8.2. *With the same notation as in Lemma 8.1, the generalised free GN_2 product of A and B amalgamating H and $H\varphi$ exists if and only if the implication*

$$a \in A, b \in B, ab \in N \Longrightarrow a \in H, b = a^{-1}\varphi$$

always holds.

PROOF. (i) Suppose that N has the properties mentioned in the data, and let θ be the canonic homomorphism of G_0 onto G_0/N.

(a) $G_0\theta$ is generated by isomorphic copies of A and B. For let $a \in A \cap N$. Then $a = a \cdot 1$, so that by the data $a = 1$. This means that $A \cap N = E$, and similarly $B \cap N = E$. But $G_0\theta$ is generated by $A\theta$ and $B\theta$, and

$$A\theta \cong A/(A \cap N) \cong A$$
$$B\theta \cong B/(B \cap N) \cong B,$$

In this way we show also that $H\theta \cong H$.

(b) $A\theta \cap B\theta = H\theta = (H\varphi)\theta$. Firstly $H\theta = (H\varphi)\theta$, because if h is any element of H, $h^{-1}h\varphi \in N$ so that $(h^{-1}h\varphi)\theta = 1$, $h\theta = (h\varphi)\theta$.

Next, $x \in A\theta \cap B\theta$ means that $x = aN = bN$ for some $a \in A$, $b \in B$. Thus $ab^{-1} \in N$, and therefore $a = h \in H$, $b^{-1} = h^{-1}\varphi$, which means that $A\theta \cap B\theta \subseteq H\theta$. With the obvious reverse inclusion this now gives $A\theta \cap B\theta = H\theta$, so that in G_0, $A\theta$ and $B\theta$ intersect in $H\theta$, and moreover H and $H\varphi$ are amalgamated according to φ.

(c) By the well-known property of commutator subgroups under homomorphisms, $[G_0\theta, [A\theta, B\theta]] = [G_0, [A, B]]\theta = E\theta = E$. Thus $G_0\theta$ is a generalised GN_2 product of A and B amalgamating H with $H\varphi$, and therefore the required generalised free GN_2 product exists.

(ii) Conversely, suppose the generalised free GN_2 product G of A and B amalgamating H with $H\varphi$ exists. Then G_0 can be mapped by a homomorphism ψ onto G in such a way that ψ is isomorphic on A and on B; and by Lemma 8.1 the kernel of ψ is N. Suppose then that $a \in A$, $b \in B$, and the product $ab \in N$; then $(ab)\psi = 1$, and $a\psi = b^{-1}\psi$. But then it quickly follows that $a \in H$ and $b = a^{-1}\varphi$, since $A\psi$ and $B\psi$ intersect in the subgroup $H\psi = (H\varphi)\psi$, in $G_0\psi$. This completes the proof of the lemma.

Following HANNA NEUMANN, we shall say that N is "tidy" in G_0 with respect to A and B if it has the property formulated in Lemma 8. 2. In order to find necessary and sufficient conditions for this to be so, we first remark that by Theorem 1. 1 in Chapter II of [2] and Theorem 3. 10 of this paper, we can express G_0 as the set of all triples

$$(a, b, c),$$

where a ranges over A, b ranges over B, and c over the tensor product $C = A \otimes B$. The multiplication in G_0 is given by

$$(a_1, b_1, c_1)(a_2, b_2, c_2) = (a_1a_2, b_1b_2, c_1c_2\{a_2^{-1} \otimes b_1\});$$

furthermore the unit element is $(1, 1, 1)$, and

$$(a, b, c)^{-1} = (a^{-1}, b^{-1}, c^{-1}\{a^{-1} \otimes b\}).$$

These all follow from the fact that the cartesian is central in G_0. The set of all triples $(a, 1, 1)$ forms a subgroup A_0 isomorphic with A, and the set of all triples $(1, b, 1)$ a subgroup B_0 isomorphic with B. We may without confusion identify $(a, 1, 1)$ with a, A_0 with A, and so on.

Lemma 8.3. *Let (a_i, b_i, c_i), $(i=1,2,\dots,r)$ be r elements of G_0, where $r \geqq 2$. Then*

$$\prod_{i=1}^{r}(a_i, b_i, c_i)=\left(\prod_{i=1}^{r} a_i, \quad \prod_{i=1}^{r} b_i, \quad \prod_{i=1}^{r} c_i \prod_{1\leqq k<j\leqq r} a_j^{-1}\otimes b_k\right).$$

PROOF. This is by induction on r. For $r=2$ the result is merely a statement of the multiplication in G_0. Suppose then that for arbitrary $(a_i, b_i, c_i)\in G$ we have shown that

$$\prod_{i=1}^{r}(a_i, b_i, c_i)=\left(\prod_{i=1}^{r} a_i, \quad \prod_{i=1}^{r} b_i, \quad \prod_{i=1}^{r} c_i \prod_{1\leqq k<j\leqq r} a_j^{-1}\otimes b_k\right),$$

and let $(a_{r+1}, b_{r+1}, c_{r+1})$ be a further element of G_0. Then using the commutativity of the tensor product,

$$\prod_{i=1}^{r+1}(a_i, b_i, c_i)=\left(\prod_{i=1}^{r+1} a_i, \quad \prod_{i=1}^{r+1} b_i, \quad \prod_{i=1}^{r+1} c_i \left\{\prod_{1\leqq k<j\leqq r} a_j^{-1}\otimes b_k\right\}\left\{a_{r+1}^{-1}\otimes \prod_{i=1}^{r} b_i\right\}\right)=$$

$$=\left(\prod_{i=1}^{r+1} a_i, \quad \prod_{i=1}^{r+1} b_i, \quad \prod_{i=1}^{r+1} c_i \left\{\prod_{1\leqq k<j\leqq r} a_j^{-1}\otimes b_k\right\}\left\{\prod_{i=1}^{r} a_{r+1}^{-1}\otimes b_i\right\}\right)=$$

$$=\left(\prod_{i=1}^{r+1} a_i, \quad \prod_{i=1}^{r+1} b_i, \quad \prod_{i=1}^{r+1} c_i \prod_{1\leqq k<j\leqq r+1} a_j^{-1}\otimes b_k\right).$$

This completes the induction, and the lemma is proved.

In the succeeding work we use the following notation.

G_0 is the free regular second nilpotent product of A and B;

N is the normal closure in G of all elements $(h^{-1}, h\varphi, 1)$, $h\in H$;

C is the tensor product $A\otimes B$;

D is the subgroup of C generated by all elements $a\otimes h\varphi$, where $a\in A$, $h\in H$;

D° is the subgroup of C generated by all elements $h\otimes b$, where $h\in H$, $b\in B$.

Lemma 8.4. *If N is tidy in G_0 with respect to A and B, there is a homomorphism ϑ of D onto $[A, H]$ defined by*

$$(a\otimes h\varphi)\vartheta=[a, h^{-1}].$$

PROOF. Put $n=(h^{-1}, h\varphi, 1)$ and let $a\in A$. Then using Lemma 8.3,

$$\begin{aligned}[a, n] &= (a^{-1}, 1, 1)(h, h^{-1}\varphi, h\otimes h\varphi)(a, 1, 1)(h^{-1}, h\varphi, 1)=\\ &=(a^{-1}hah^{-1}, 1, \{a\otimes h\varphi\}\{h\otimes h^{-1}\varphi\}\{h\otimes h\varphi\})=\\ &=([a, h^{-1}], 1, a\otimes h\varphi).\end{aligned}$$

Also $[a, n] \in N$ since $N \trianglelefteq G_0$. Now put

$$n_i = (h_i^{-1}, h_i\varphi, 1) \qquad (i = 1, 2, \ldots, r)$$

and let $a_1, a_2, \ldots, a_r$ be r elements of A. Then by the foregoing,

$$[a_i, n_i] = ([a_i, h_i^{-1}], 1, a_i \otimes h_i\varphi),$$
$$[a_i, n_i]^{-1} = ([a_i, h_i^{-1}]^{-1}, 1, \{a_i \otimes h_i\varphi\}^{-1}).$$

Thus since the middle component of each of these is 1, if $\varepsilon_i = \pm 1$ for each $i = 1, 2, \ldots, r$ we have

$$n^* = \prod_{i=1}^{r} [a_i, n_i]^{\varepsilon_i} = \left(\prod_{i=1}^{r} [a_i, h_i^{-1}]^{\varepsilon_i}, \quad 1, \quad \prod_{i=1}^{r} (a_i \otimes h_i\varphi)^{\varepsilon_i} \right).$$

Suppose now that $\prod_{i=1}^{r} (a_i \otimes h_i\varphi)^{\varepsilon_i} = 1$. Then n^* lies in A, so that since it also lies in N, we deduce from the tidiness of N that

$$\prod_{i=1}^{r} [a_i, h_i^{-1}]^{\varepsilon_i} = 1.$$

This clearly proves the lemma.

Proved by a method very similar is the following lemma.

Lemma 8.5. *If N is tidy in G_0 with respect to A and B, there is a homomorphism δ° of D° onto $[B, H\varphi]$ defined by*

$$(h \otimes b)\delta^\circ = [b, h\varphi].$$

So far we have only used the condition that the constituents A and B have trivial intersection with N. The stronger condition that no product ab may lie in N is used in the proof of the following lemma, which brings out the consonance between the homomorphisms δ and δ°.

Lemma 8.6. *If N is tidy in G_0 with respect to A and B, and d is any element of $D \cap D^\circ$, then*

$$d\delta \in H,$$
$$d\delta\varphi = d\delta^\circ.$$

PROOF. Let $d \in D \cap D^\circ$. Then by the preceding two lemmas,

$$n_1 = (d\delta, 1, d) \in N,$$
$$n_2 = (1, d\delta^\circ, d) \in N.$$

Further

$$n_1^{-1} n_2 = (d^{-1}\delta, 1, d^{-1})(1, d\delta^\circ, d) = (d^{-1}\delta, d\delta^\circ, 1).$$

This also lies in N, so that by the tidiness of this subgroup we conclude that $d\delta \in H$ and $d\delta\varphi = d\delta^\circ$, as required.

Before carrying this investigation further, we show that the necessary conditions derived in Lemmas 8.4 and 8.5 contain the conditions (7.5). In fact we show a little more. For any pair X, Y of groups we define $J_X(Y)$ to be the subgroup of Y which *"centralises X tensorwise"*, that is, $y \in J_X(Y)$ if and only if $x \otimes y = 1$ in $X \otimes Y$, for all $x \in X$.

Theorem 8. 7. *The existence of the homomorphisms δ and δ° of Lemmas* 8. 4 *and* 8. 5 *implies the following conditions:*

$$[[A, H], A] = [[B, H], B] = E,$$
$$[(J_B(A) \cap H)\varphi, B] = [(J_A(B) \cap H\varphi)\varphi^{-1}, A] = E.$$

PROOF. (i) Consider the subgroup $[[A, H], A]$ of A. It is generated by the conjugates in A of all commutators $[[a, h], a_1]$, where a, a_1 range over A and h ranges over H. Now

$$[aa_1, h] = [a, h]\,[[a, h], a_1]\,[a_1, h],$$

so that

$$[[a, h], a_1] = [a, h]^{-1}[aa_1, h]\,[a_1, h]^{-1}.$$

Consider then the element

$$(a \otimes h^{-1}\varphi)^{-1}(aa_1 \otimes h^{-1}\varphi)\,(a_1 \otimes h^{-1}\varphi)^{-1}$$

of D. From the relations of the tensor product, it must be the unit element; moreover its image under δ is

$$[a, h]^{-1}[a\,a_1, h]\,[a_1, h]^{-1} = [[a, h], a_1].$$

It thus follows that $[[a, h], a_1] = 1$, and hence that $[[A, H], A] = E$. The proof that $[[B, H\varphi], B] = E$ follows along closely similar lines.

(ii) Consider the subgroup $[(J_B(A) \cap H)\varphi, B]$ of B, and let h be any element of $J_B(A) \cap H$. Since $h \in J_B(A)$, by definition $h \otimes b = 1$, for all $b \in B$. Hence if b is any element of B,

$$1 = 1\delta^\circ = (h \otimes b)\,\delta^\circ = [b, h\varphi].$$

But this means that $(J_B(A) \cap H)\varphi$ centralises B, as we wanted to show.

The final part of the theorem is proved in a similar manner.

Conditions (7. 5) now follow from the obvious remark that $J_X(Y)$ always contains Y', for all groups X.

Most of the remainder of this section is devoted to a proof that the necessary conditions found in Lemmas 8. 4, 8. 5, 8. 6 for the tidiness of N are also sufficient. The proof is long, and is split up into the proof of several lemmas.

Lemma 8. 8. *The subgroup N of G_0 is generated by all elements of the form*

$$(a^{-1}h^{-1}a,\, b^{-1}h\varphi b,\, \{h^{-1} \otimes b\}\,\{a \otimes h^{-1}\varphi\}),$$

where a ranges over A, b over B, h over H.

PROOF. Since N is the normal closure in G_0 of all elements $(h^{-1}, h\varphi, 1)$, it is generated by all conjugates in G_0 of all these elements. Thus to prove the lemma all we have to do is to show that such a conjugate has the form required in the lemma. A typical such element is, for some $a\in A, b\in B, h\in H, c\in C$,

$$\begin{aligned} n &= (a, b, c)^{-1}(h^{-1}, h\varphi, 1)(a, b, c) = \\ &= (a^{-1}, b^{-1}, c^{-1}\{a^{-1}\otimes b\})(h^{-1}, h\varphi, 1)(a, b, c) = \\ &= (a^{-1}h^{-1}a, b^{-1}h\varphi b, c^{-1}\{a^{-1}\otimes b\}c\{h^{-1}\otimes b\}\{a\otimes b\}\{a\otimes h^{-1}\varphi\}) \end{aligned}$$

by Lemma 8. 3. Thus we arrive finally at the form

$$(a^{-1}h^{-1}a, b^{-1}h\varphi b, \{h^{-1}\otimes b\}\{a\otimes h^{-1}\varphi\})$$

for n, as required.

As a temporary abbreviation, we shall denote the element n of Lemma 8. 8 by $f(a, b, h)$.

Lemma 8. 9. *Every element of N has an expression of the form*

$$\prod_{i=1}^{r} f(a_{2i-1}, b_{2i-1}, h_{2i-1})\{f(a_{2i}, b_{2i}, h_{2i}^{-1})\}^{-1},$$

for suitable choices of the a_i, b_i, h_i, $(i=1, 2, \ldots, 2r)$.

PROOF. By Lemma 8. 8, if m is any element of N,

$$m = \prod_{i=1}^{s} f(a_i', b_i', h_i')^{\lambda_i},$$

for some a_i', b_i', h_i'. In this expression the λ_i are integers, which we may straight away assume to be ± 1. Since $f(1, 1, 1) = 1$, we can insert $f(1, 1, 1)$ or $\{f(1, 1, 1)\}^{-1}$ where necessary to bring this expression to the form

$$m = \prod_{i=1}^{r} f(a_{2i-1}, b_{2i-1}, h_{2i-1})\{f(a_{2i}, b_{2i}, h_{2i})\}^{-1}.$$

Further we can replace h_{2i} by h_{2i}^{-1} for each i, and arrive at an expression of the required sort for m.

Lemma 8. 10. *Every element of N has an expression of the form (a, b, c) where*

$$a = \prod_{i=1}^{2r} a_i^{-1}h_i^{-1}a_i, \quad b = \prod_{i=1}^{2r} b_i^{-1}h_i\varphi b_i,$$

$$c = \prod_{i=1}^{2r} h_i^{-1}\otimes b_i \prod_{i=1}^{2r} a_i\otimes h_i^{-1}\varphi \prod_{i=1}^{r} h_{2i}\otimes h_{2i}\varphi \prod_{1\le k<j\le 2r} h_j\otimes h_k\varphi,$$

for suitable choices of a_i, b_i, h_i, $(i=1, 2, \ldots, 2r)$.

Then using throughout the fact that $[X, Y]$ is central,

$$\begin{aligned}\prod_{i=1}^{s+1} x_i^{y_i} &= \prod_{i=s}^{1} x_i^{y_i}\, x_{s+1}^{y_{s+1}} \prod_{1\leqq p<q\leqq s} [x_p, x_q] = \\ &= \prod_{i=s+1}^{1} x_i^{y_i} \left[\prod_{i=s}^{1} x_i^{y_i}, x_{s+1}^{y_{s+1}}\right] \prod_{1\leqq p<q\leqq s} [x_p, x_q] = \\ &= \prod_{i=s+1}^{1} x_i^{y_i} \prod_{i=s}^{1} [x_i, x_{s+1}] \prod_{1\leqq p<q\leqq s} [x_p, x_q] = \\ &= \prod_{i=1}^{s+1} x_i^{y_i} \prod_{1\leqq p<q\leqq s+1} [x_p, x_q].\end{aligned}$$

This completes the induction, and proves the lemma.

Lemma 8. 12. *Let X be a subgroup of the group Y such that $[[X, Y], Y]=E$. Then if $s\geqq 2$ and $x_1, x_2, \ldots, x_s$ are arbitrary elements of X and $y_1, y_2, \ldots, y_s$ are arbitrary elements of Y,*

$$\prod_{i=1}^{s} [y_i, x_i] = \prod_{i=1}^{s} y_i^{-1} x_i^{-1} y_i \prod_{i=1}^{s} x_i \prod_{1\leqq p<q\leqq s} [x_p, x_q^{-1}].$$

The proof proceeds by an inductive argument very similar to that used in the proof of Lemma 8. 11.

We are now in a position to prove the following theorem, which is a proof of the sufficiency of the conditions so far derived as being necessary for the tidiness of N. The notation used in the theorem has been listed earlier in this section.

Theorem 8. 13. *Suppose that there exist homomorphisms δ, δ° of D, D° respectively onto $[A, H], [B, H\varphi]$ defined by*

$$(a\otimes h\varphi)\delta = [a, h^{-1}], \quad (h\otimes b)\delta^\circ = [b, h\varphi].$$

Suppose further that for any $d\in D\cap D^\circ$, $d\delta\in H, d\delta^\circ = d\delta\varphi$. Then N is tidy in G_0 with respect to A and B.

Proof. By Lemma 8. 10, every element of N has the form (a, b, c) where

$$a=\prod_{i=1}^{2r} a_i^{-1} h_i^{-1} a_i, \quad b=\prod_{i=1}^{2r} b_i^{-1} h_i\varphi b_i,$$

$$c=\prod_{i=1}^{2r} h_i^{-1}\otimes b_i \prod_{i=1}^{2r} a_i\otimes h_i^{-1}\varphi \prod_{i=1}^{r} h_{2i}\otimes h_{2i}\varphi \prod_{1\leqq k<j\leqq 2r} h_j\otimes h_k\varphi,$$

for suitable $a_i\in A$, $b_i\in B$, $h_i\in H$, $(i=1, 2, \ldots, 2r)$. Suppose now that this element has the form $(a, b, 1)$, that is, suppose that $c=1$; then we conclude that

$$d=\prod_{i=1}^{2r} a_i\otimes h_i^{-1}\varphi \prod_{i=1}^{r} h_{2i}\otimes h_{2i}\varphi = \prod_{i=1}^{r} h_i\otimes b_i \prod_{1\leqq k<j\leqq 2r} h_j^{-1}\otimes h_k\varphi.$$

It will now be seen that $d \in D \cap D^{\circ}$, so that by the data $d\vartheta \in H$ and $d\vartheta^{\circ} = d\vartheta\varphi$.

Next,

$$d\vartheta = \prod_{i=1}^{2r} [a_i, h_i] \prod_{i=1}^{r} [h_{2i}, h_{2i}^{-}] = \prod_{i=1}^{2r} [a_i, h_i].$$

Further, from Theorem 8 7 we see that $[[A, H], A] = E$, and therefore by Lemma 8. 12,

$$d\vartheta = \prod_{i=1}^{2r} [a_i, h_i] = \prod_{i=1}^{2r} a_i^{-1} h_i^{-1} a_i \prod_{i=1}^{2r} h_i \prod_{1 \leqq p < q \leqq s} [h_p, h_q^{-1}].$$

But as we just saw, $d\vartheta \in H$; this means that

$$a = \prod_{i=1}^{2r} a_i^{-1} h_i^{-1} a_i \in H.$$

Furthermore

$$d\vartheta^{\circ} = \prod_{i=1}^{2r} [b_i, h_i\varphi] \prod_{1 \leqq k < j \leqq 2r} [h_k\varphi, h_j^{-1}\varphi].$$

Thus from Theorem 8. 7 and Lemma 8. 12,

$$d\vartheta^{\circ} = \prod_{i=1}^{2r} b_i^{-1} h_i^{-1} \varphi b_i \prod_{i=1}^{2r} h_i\varphi \prod_{1 \leqq p < q \leqq 2r} [h_p\varphi, h_q^{-1}\varphi] \prod_{1 \leqq k < j \leqq 2r} [h_k\varphi, h_j^{-1}\varphi].$$

Moreover, $d\vartheta\varphi = d\vartheta^{\circ}$; so that since

$$d\vartheta\varphi = a\varphi \prod_{i=1}^{2r} h_i\varphi \prod_{i \leqq p < q \leqq 2r} [h_p\varphi, h_q^{-1}\varphi],$$

we deduce from the fact that $[B, H\varphi]$ is central that

$$a\varphi = \prod_{i=1}^{2r} b_i^{-1} h_i^{-1} \varphi b_i \prod_{1 \leqq k < j \leqq 2r} [h_k\varphi, h_j^{-1}\varphi] \tag{8. 14}$$

Now $b = \prod_{i=1}^{2r} b_i^{-1} h_i \varphi b_i$, so that by Lemma 8. 11,

$$b = \prod_{i=2r}^{1} b_i^{-1} h_i \varphi b_i \prod_{1 \leqq p < q \leqq 2r} [h_p\varphi, h_q\varphi].$$

Therefore since $[B, H\varphi]$ is central,

$$b^{-1} = \prod_{i=1}^{2r} b_i^{-1} h_i^{-1} \varphi b_i \prod_{1 \leqq p < q \leqq 2r} [h_p\varphi, h_q^{-1}\varphi].$$

Equation (8. 14) now gives us that $b^{-1} = a\varphi$. Thus we have shown that any element of N which is of the form $(a, b, 1)$ must be of the form $(h^{-1}, h\varphi, 1)$; therefore that N is tidy in G_0 with respect to A and B.

We have therefore, by Lemma 8. 2, found necessary and sufficient conditions that the generalised free GN_2 product of A and B amalgamating H and $H\varphi$ exist. Let us examine a few examples to see how the conditions work in practice.

Example 8. 15. In this example both A and B are dihedral groups of order 8;

$$A = \mathrm{Gp}(a, b;\ a^4 = b^2 = (ab)^2 = 1),$$
$$B = \mathrm{Gp}(c, d;\ c^4 = d^2 = (cd)^2 = 1).$$

We take H to be the subgroup $\mathrm{Gp}(b)$ and $H\varphi$ to be $\mathrm{Gp}(d)$.

It is not difficult to show that $A \otimes B$ is elementary abelian of order 16 and is generated by the elements

$$a \otimes c,\ a \otimes d,\ b \otimes c,\ b \otimes d.$$

Next, $D = \mathrm{Gp}(a \otimes d, b \otimes d)$, $D^\circ = \mathrm{Gp}(b \otimes c, b \otimes d)$ and also $D \cap D^\circ = \mathrm{Gp}(b \otimes d)$. Lastly

$$[A, H] = \mathrm{Gp}([a, b]) = \mathrm{Gp}(a^2),$$
$$[B, H] = \mathrm{Gp}([c, d]) = \mathrm{Gp}(c^2).$$

In fact therefore there do exist homomorphisms δ, δ° of D, D° onto $[A, H]$, $[B, H\varphi]$ respectively defined by

$$(a \otimes d)\delta = [a, b^{-1}],\quad (b \otimes d)\delta = 1,$$
$$(b \otimes c)\delta^\circ = [c, d],\quad (b \otimes d)\delta^\circ = 1,$$

which are consonant on $D \cap D^\circ$: $d^\circ \delta = d^\circ \delta^\circ = 1$ for any $d^\circ \in D \cap D^\circ$. Thus we can apply the theorem to form the generalised free GN_2 product of A and B amalgamating H and $H\varphi$. It is

$$\mathrm{Gp}(a, c, b;\ a^4 = c^4 = b^2 = (ab)^2 = (cb)^2 = [[x, y], z] = 1),$$

where each of x, y, z takes the values a, c, b. In fact it is easy to see that this group if of order 64 and nilpotent of class 2.

In general it is not easy to apply the conditions here derived, in fact it is usually difficult; but occasionally it is very easy to see that they are not satisfied, as in the following example (already used once). Incidentally, it is moderatelly easy to apply conditions (7. 5); and any example worth considering will satisfy them.

Example 8. 16. We take our two groups as follows:

$$A = \mathrm{Gp}(a, b;\ a^4 = b^2 = (ab)^2 = 1),$$
$$B = \mathrm{Bp}(c;\ c^8 = 1),$$

and we let $H = \mathrm{Gp}(a)$, $H\varphi = \mathrm{Gp}(c^2)$, φ being the isomorphism defined by

$a\varphi = c^2$. In this case, $D = \mathrm{Gp}(a \otimes c^2, b \otimes c^2)$. But

$$a \otimes c^2 = a^2 \otimes c = [a, b] \otimes c = 1,$$
$$b \otimes c^2 = b^2 \otimes c = 1 \otimes c = 1.$$

In other words, $D = E$, so that it cannot be mapped homomorphically onto $[A, H]$, which is the non-trivial group $\mathrm{Gp}(a^2)$.

§ 9. Normal forms and necessary and sufficient MN_2 conditions.

In any discussion of generalised products of groups with amalgamations, an investigation as to the existence of normal forms for the elements of these products naturally plays an important role, for it is by use of the normal forms that we hope to construct the product from the groups which generate it. This has been done for the generalised free and the generalised direct products of groups (see [9]); we shall give here some normal forms for the elements of the generalised free GN_2 product of two groups· with amalgamation. These normal forms can be used to construct the product, but the construction is so complicated that it is almost worthless, and is omitted.

The considerations of this section depend on the following theorem.

Theorem 9.1. *Let G be the generalised free GN_2 product of groups A and B with the subgroup H amalgamated. Then if a, b are elements of A, B respectively such that the product ab lies in the cartesian $[A, B]$ then that product must lie in the subgroup $[A, H][B, H]$.*

Proof. Let G_0 be the free regular second nilpotent product of groups $\hat{A}, \hat{B}$ which are isomorphic to A, B respectively under the respective isomorphisms θ_1 and θ_2. Then if we denote the subgroup $H\theta_1^{-1}$ of A by $\hat{H}$, it is clear that $\hat{H}\theta_1\theta_2^{-1}$ is a subgroup of B which is isomorphic with $\hat{H}$. Thus we have the familiar situation of the last section; we shall denote $\theta_1\theta_2^{-1}$ restricted to $\hat{H}$ by φ. Furthermore, θ_1 and θ_2 can be simultaneously extended to a homomorphism θ of G_0 onto G, whose kernel is the normal closure of all elements $\hat{h}^{-1}\hat{h}\varphi$, where $\hat{h} \in \hat{H}$.

Now by Lemma 8.10, if n is an arbitrary element of N, $n = \alpha\beta\gamma$ where $\alpha \in \hat{A}$, $\beta \in \hat{B}$, $\gamma \in [\hat{A}, \hat{H}\varphi]^{G_0}[\hat{B}, \hat{H}]^{G_0}$. Suppose now that a, b are elements of A, B respectively such that $ab = u \in [A, B]^G$. Then there exist $\hat{a}$, $\hat{b}, \hat{u}$ in $\hat{A}, \hat{B}, [\hat{A}, \hat{B}]^{G_0}$ respectively such that

$$\hat{a}\theta = a, \hat{b}\theta = b, \hat{u}\theta = u.$$

Thus $(\hat{a}\,\hat{b})\theta = \hat{u}\theta$ so that $\hat{a}\,\hat{b} = \hat{u}n$, where $n \in N$. Now $n = \alpha\beta\gamma$ as above, so that $\hat{a}\,\hat{b} = \alpha\beta\hat{u}\gamma$, since the cartesian is central. But owing to the uniqueness of such representations in G_0, we conclude that

$$\hat{a} = \alpha,\ \hat{b} = \beta,\ \hat{u}\gamma = 1.$$

Hence $\hat{u}=\gamma^{-1}\in[\hat{A}, \hat{H}\varphi]^{G_0}[\hat{B}, \hat{H}]^{G_0}$. But this means that $ab=(\hat{a}\,\hat{b})\theta=\hat{u}\theta$ lies in $([\hat{A}, \hat{H}\varphi]^{G_0}[\hat{B}, \hat{H}]^{G_0})$. But this subgroup is

$$[\hat{A}\theta, \hat{H}\varphi\theta]^{G_0\theta}[\hat{B}\theta, \hat{H}\theta]^{G_0\theta}=[A, H]^G[B, H]^G.$$

This completes the proof of the theorem.

It is easy to give examples to show that the conclusion of Theorem 9.1 is no longer true if the word "free" is removed.

An easy corollary of the theorem is the following result, which shows exactly how the constituents of a generalised free GN_2 product intersect the cartesian.

Lemma 9.2. *If G is the generalised free GN_2 product of A and B amalgamating the subgroup H, then in G,*

(i) $[A, B]\cap A=[A, H]([B, H]\cap H)$,

(ii) $[A, B]\cap B=[B, H]([A, H]\cap H)$.

(iii) $[A, B]\cap H=([A, H]\cap H)([B, H]\cap H)$.

PROOF. We prove the first of these equalites to exemplify the method of proof. Note first the obvious fact that

$$[A, B]\cap A\supseteqq[A, H]([B, H]\cap H).$$

Next let $a\in A\cap[A, B]$. Then by Theorem 9.1, $a=a\cdot 1=a'b'$, for some $a'\in[A, H]$, $b'\in[B, H]$. We then get that $b'=a'^{-1}a\in A\cap B=H$, so that $b'\in[B, H]\cap H$. Thus

$$[A, B]\cap B\subseteqq[A, H]([B, H]\cap H),$$

which together with the above reverse inclusion gives the answer.

We can now exhibit our normal forms.

Theorem 9.3. *Let G be the generalised free GN_2 product of the groups A and B amalgamating the subgroup H. Let further T_1 be a right transversal of B modulo H, and T_2 a right transversal of $[A, B]$ modulo $[A, H][B, H]$. Then every element of G has a unique expression*

$$g=atu,$$

where $a\in A$, $t\in T_1$, $u\in T_2$.

PROOF. (i) Firstly we show that every element has an expression of the required form. If g is an arbitrary element of G, it is easy to see (cf. for instance [2]) that $g=a_1b_1u_1$ for some $a_1\in A$, $b_1\in B$, $u_1\in[A, B]$. Express u_1 in the form

$$u_1=a'b'u,$$

where $a'\in[A, H]$, $b'\in[B, H]$, $u\in T_2$. Then since a' and b' are central in G, $g=a_1a'b_1b'u$. Next

$$b_1b'=ht$$

for some $h \in H$, $t \in T_1$; so that $g = a_1 a' h t u$. This is of the required form with $a = a_1 a' h \in A$.

(ii) Next suppose that g has two such expressions, say

$$g = a_1 t_1 u_1 = a_2 t_2 u_2.$$

Then

$$a_2^{-1} a_1 t_1 t_2^{-1} = u_2 u_1^{-1} \in [A, B],$$

since u_1 and u_2 are central. Therefore, by Theorem 9. 1,

$$a_2^{-1} a_1 t_1 t_2^{-1} \in [A, H]\,[B, H].$$

But then $u_2 u_1^{-1}$ lies in $[A, H]\,[B, H]$, which means that $u_1 = u_2$. We are now left with the equation $a_1 t_1 = a_2 t_2$, which gives

$$a_2^{-1} a_1 = t_2 t_1^{-1} \in A \cap B = H,$$

so that $t_1 = t_2$, $a_1 = a_2$. This completes the proof of the theorem.

The following theorems can be shown in a manner similar to that used in Theorem 9. 3.

Theorem 9. 4. *Let G be the generalised free GN_2 product of groups A and B amalgamating the subgroup H. Let further T_1 be a left transversal of A modulo H, and let T_2 be a right transversal of $[A, B]$ modulo $[A, H]\,[B, H]$. Then every element of G has a unique expression of the form*

$$g = tbu.$$

where $t \in T_1$, $b \in B$, $u \in T_2$.

Theorem 9. 5. *Let G be the generalised free GN_2 product of groups A and B amalgamating the normal subgroup H. Let further T_1 be a left transversal of A modulo H, and T_2 a left transversal of B modulo $[A, H]\,[B, H]$. Then every element of G has a unique expression of the form*

$$g = stu,$$

where $s \in T_1$, $t \in T_2$, $u \in [A, B]$.

These normal forms are not, as has been pointed out already, of any great use as tools for studying the generalised free GN_2 products. The normal form of the product of elements in normal form is obtained only after a fairly complicated set of equations, which makes manipulation very tiresome. However, it looks likely that there are no simpler normal forms.

Let us return now to a consideration of the generalised MN_2 products. It will certainly be asked where the theory of the last section breaks down for the MN_2 case; for if the generalised free MN_2 product of two groups A and B amalgamating the isomorphic subgroups H and $H\varphi$ exists, it is G_0/N, where these symbols have the same meaning as in § 8. However, even if N be tidy in G_0, we cannot state the generalised free MN_2 product is G_0/N — and indeed this product need not exist, as Example 6. 1 will be seen to demonstrate.

To find necessary and sufficient MN_2 conditions, we use the fact that the MN_2 product is simultaneously a GN_2 product, and that if they both exist, they coincide. Precisely, we prove the following theorem.

Theorem 9.6. *Let G be the generalised free GN_2 product of groups A and B amalgamating the subgroup H. Then G is the generalised free MN_2 product if and only if the following condition holds. Suppose that a'', b'', a', b', h are elements of $[A, A']$, $[B, B']$, $[A, H]$, $[B, H]$, H respectively, connected by the equations*

$$a'' = a'h, \ b'' = h^{-1}b'.$$

Then it must follow that $a'' \in H$, $b'' \in H$ and $a''b'' = 1$.

PROOF. The second member of the lower central series of G is, by Lemma 6.3,

$$[G, G'] = [A, A'][B, B'][G, [A, B]] = [A, A'][B, B'],$$

since the cartesian is central in G.

(i) Suppose that G is the generalised free MN_2 product of A and B, so that $[A, A'][B, B'] \cap [A, B] = E$. Then if a'', b'', a', b', h are as in the data, multiplying we get $a''b'' = a'b'$. But this means that

$$a''b'' \in [A, H][B, H] \cap [A, B] = E.$$

Then $a''b'' = 1$, and of course $a'' \in H$, $b'' \in H$.

(ii) On the other hand, let g be any element of $[A, A'][B, B'] \cap [A, B]$. Then $g = a''b''$ for some $a'' \in [A, A']$, $b'' \in [B, B']$, and by Theorem 9.1,

$$a''b'' = a'b'$$

for some $a' \in [A, H]$, $b' \in [B, H]$. This gives

$$(a')^{-1}a'' = b'(b'')^{-1} = h \in H,$$

and $a'' = a'h$, $b'' = h^{-1}b'$. But from the data we now conclude that $a''b'' = 1$, and therefore that $[G, G'] \cap [A, B] = E$. This finishes the proof of the theorem.

We now state the following set of necessary and sufficient MN_2 conditions for the set-up as in § 8.

1. There exist homomorphisms δ, δ° of D, D° onto $[A, H]$, $[B, H\varphi]$ respectively defined by

$$(a \otimes h\varphi)\delta = [a, h^{-1}], \quad (b \otimes h)\delta^\circ = [h\varphi, b].$$

2. If d is any element of $D \cap D^\circ$, $d\delta \in H$ and $d\delta\varphi = d\delta^\circ$.

3. If, in the amalgam of A and B amalgamating H and $H\varphi$ according to φ, we have elements a'', b'', a', b', h of $[A, A']$, $[B, B']$, $[A, H]$, $[B, H]$, H respectively, connected by the equations

$$a'' = a'h, \ b'' = h^{-1}b',$$

then $a'' \in H$, $b'' \in H$ and $a''b'' = 1$.

For 1. and 2. give the existence of the generalised free GN_2 product, and by the theorem just proved, 3. makes it the MN_2 product.

Appendix. The GN_3 products.

The problem of finding necessary and sufficient GN_3 conditions is beset by the difficulty engendered by the fact that the cartesian is no longer central. I have been able to work out necessary conditions similar to those of Lemmas 8. 4, 8. 5, 8. 6, though any proof that these are also sufficient would at best be enormously complicated. We shall content ourselves with merely stating our necessary conditions, as the derivation is simple if rather tedious.

Let A and B be groups, H and $H\varphi$ subgroups of A, B respectively, isomorphic under the mapping φ. Let further G_0 be the free regular third nilpotent product of A and B, where for clarity we denote the commutator of an element a of A with an element b of B, in G_0, by $|a, b|$, and so on. We define D to be the subgroup of $|A, B|$ generated by all commutators $|a, h\varphi|$, and D° that generated by all $|h, b|$. Then the following are necessary for the existence of the generalised free GN_3 product of A and B amalgamating H and $H\varphi$.

1. There exist homomorphisms δ, δ° of D, D° onto $[A, H]$, $[B, H\varphi]$ respectively defined by

$$|a, h\varphi|\delta = [h, a], \quad |h, b|\delta^\circ = [b, h\varphi].$$

2. If d is any element of $D \cap D^\circ$, $d\delta \in H$, $d\delta\varphi = d\delta^\circ$.

To give even these necessary conditions the same sort of weight as that possessed by the necessary GN_2 conditions, we should have to give an abstract representation of the third nilpotent cartesian $|A, B|$. B. H. NEUMANN has conjectured that the following group, which he naturally calls the "elevensor product" and denotes by $A\,(\mathrm{xi})\,B$, will provide this abstract representation.

$A\,(\mathrm{xi})\,B$ is generated by "formal commutators" $|a, b|$, $|a, b, c|$, (where a ranges over A, b over B, c over their direct product $A \times B$) subject to the relations:

$$\begin{aligned}
|a_1 a_2, b| &= |a_1, b|\,|a_2, b|\,|a_1, b, a_2|, \\
|a, b_1 b_2| &= |a, b_1|\,|a, b_2|\,|a, b_1, b_2|, \\
|a_1 a_2, b, c| &= |a_1, b, c|\,|a_2, b, c|, \\
|a, b_1 b_2, c| &= |a, b_1, c|\,|a, b_2, c|, \\
|a, b, c_1 c_2| &= |a, b, c_1|\,|a, b, c_2|, \\
|a_1, b_1|\,|a_2, b_2| &= |a_2, b_2|\,|a_1, b_1|.
\end{aligned}$$

for all $a, a_1, a_2 \in A$, $b, b_1, b_2 \in B$, $c, c_1, c_2 \in A \times B$.

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