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On the bounded and totally bounded sets

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Abstract. It is well known that finite unions of bounded (totally bounded) sets are bounded (totally bounded). In this paper we will generalize thesee elementary facts to the bounded (totally bounded) families in the Pompeiu-Hausdorff metric (uniform) spaces of subsets. We also give the characterization of boundness (totally boundness) in the spaces of subsets, and we will apply these results to the theory of multifunctions.

1. Notations and premilinares

Let $(X,\mathcal{U})$ be a uniform space. Denote by $\mathcal{P}_0(X)$ (resp. $2^X, \mathcal{T}B(X)$) the family of all non void (resp. closed, totally bounded) subsets of X. Define the uniformity $\mathcal{U}^\uparrow$ on $\mathcal{P}_0(X)$ by the basis $\{U^\uparrow \mid U \in \mathcal{U}\}$, where

$$U^\uparrow = \{(A,B) \in \mathcal{P}_0(X) \times \mathcal{P}_0(X) \mid A \subset U(B), B \subset U(A)\}.$$

If the uniformity $\mathcal{U}$ is given by metric d, then Pompeiu-Hausdorff metric on 2^X is defined as

$$\delta(A,B) = \inf\{\varepsilon > 0,\ A \subset \bigcup_{x\in B} S_d(x,\varepsilon),\ B \subset \bigcup_{y\in A} S_d(y,\varepsilon)\}$$

where $S_d(z,\varepsilon) = \{u \in X \mid d(u,z) < \varepsilon\}$.

We will denote the family of all non-void bounded subsets of metric space (X,d) by $\mathcal{B}(X)$.

2. Boundness and totally boundness of a union of sets

Theorem 1. *Let (X,d) be a metric space and let $\mathcal{A}$ be a family of closed subsets of X, bounded in the metric space $(2^X,\delta)$. If there exists an $A_0 \in \mathcal{A}$, bounded in (X,d), then the union $\bigcup_{A\in\mathcal{A}} A$ is bounded.*

Proof. Let $x_0 \in A_0$ be fixed. Since A_0 is bounded there exists $r > 0$ such that $A_0 \subset S_d(x_0, r)$. Because $\mathcal{A}$ is bounded in the space $(2^X, \delta)$, there exists $r_1 > 0$ such that $\mathcal{A} \subset S_\delta(A_0, r_1)$. Then for every $A \in \mathcal{A}$ relation $A \subset \bigcup\limits_{A \in \mathcal{A}_0} S_d(x, r_1)$ holds good. We will show that $B = \bigcup\limits_{A \in \mathcal{A}} A \subset S_d(x_0, r + r_1)$.

Let $x \in B$. Then there exists $A \in \mathcal{A}$ such that $x \in A$, that is there is an element y of A_0 with property $x \in S_d(y, r_1)$. Therefore

$$d(x, x_0) \leq d(x, y) + d(y, x_0) < r_1 + r,$$

thus $x \in S_d(x_0, r + r_1)$. It results $B \subset S_d(x_0, r + r_1)$, which is what we set out to prove.

Remark. The condition that the bounded family $\mathcal{A}$ from the space (S^X, δ) to containe lest one bounded element is essentially for boundness of the union $\bigcup\limits_{A \in \mathcal{A}} A$. We illustrate this fact by the following example:

Let $X = R$ and $d(x, y) = |x - y|$, $x, y \in R$. Taking $\mathcal{A} = \big\{[a, +\infty), a \in [0, 1]\big\}$ it is easy to see that this set is bounded in the space $(2^R, \delta)$, but the set $\bigcup\limits_{A \in \mathcal{A}} A = [0, \infty)$ is unbounded in the space (R, d).

Theorem 2. *Let $(X,\mathcal{U})$ be an uniform space, and let $\mathcal{A}$ be a family of totally bounded subsets of X. If $\mathcal{A}$ is totally bounded in the space $(\mathcal{P}_0(X), \mathcal{U}^\uparrow)$, then the union $\bigcup\limits_{A \in \mathcal{A}} A$ is totally bounded set in the space $(X,\mathcal{U})$.*

Proof. Let $B = \bigcup\limits_{A \in \mathcal{A}} A$ and let U be a member of uniformity $\mathcal{U}$. Then there exists a symmetric member V of $\mathcal{U}$ such that $V \circ V \subset U$. Since $\mathcal{A}$ is totally bounded in $(\mathcal{P}_0(X), \mathcal{U}^\uparrow)$ it results that there exists a finite number of elements $A_1, A_2, \ldots, A_n$ of $\mathcal{A}$ such that $\mathcal{A} \subset \bigcup\limits_{i=1}^{n} V^\uparrow(A_i)$. Since the sets $A_1, A_2, \ldots, A_n$ are totally bounded in the space $(X,\mathcal{U})$, it follows that there exist $Y_i = \{x_i^1, x_i^2, \ldots, x_i^{m(i)}\} \subset A_i$, $i \in \overline{1, n}$ with property $A_i \in \bigcup\limits_{j=1}^{m(i)} V(x_i^j)$, $i \in \overline{1, n}$.

Let $Y = \bigcup\limits_{i=1}^{n} Y_i$. It is evident that Y is a finite subset of X. We will show that $B \subset U(Y)$.

Let $z \in B$. Then there exists $A \in \mathcal{A}$ such that $z \in A \in \mathcal{A} \subset \bigcup\limits_{A \in \mathcal{A}} V^{\uparrow}(A_i)$. Hence there exists $i_A \in \overline{1,n}$ with the property $A \in V^{\uparrow}(A_{i_A})$. Therefore

$$A \subset V(A_{i_A}) \subset \bigcup_{j=1}^{m(i_A)} (V \circ V)(x_{i_A}^j) \subset \bigcup_{j=1}^{m(i_A)} U(x_{i_A}^j) \subset U(Y_{i_A}) \subset U(Y)$$

Whence $z \in U(Y)$, which implies $B \subset U(Y)$. Hence B is totally bounded in the space $(X, \mathcal{U})$.

3. Properties of boundness and totally boundness on spaces of subsets

Theorem 3. *A metric space (X, d) is bounded if and only if so $(2^X, \delta)$ does.*

Proof. Suppose (X, d) is bounded, then there exist $x_0 \in X$ and $r > 0$ such that $X \subset S_d(x_0, r)$. We will show that

$$2^X \subset S_\delta(\{x_0\}, r).$$

Let $A \in 2^X$ and let us evaluate the distance $\delta(\{x_0\}, A)$. We have

$$\begin{aligned} \delta(\{x_0\}, A) &= \inf\{\varepsilon > 0 \mid \{x_0\} \subset \bigcup_{x \in A} S_d(x, \varepsilon), A \subset S_d(x_0, \varepsilon)\} = \\ &= \inf\{\varepsilon > 0 \mid A \subset S_d(x_0, \varepsilon)\} \leq r \end{aligned}$$

Therefore $(2^X, \delta)$ is bounded.

Converse. *Suppose $(2^X, \delta)$ is bounded. Then the family $\mathcal{A} = \{\{x\} \mid x \in X\}$ is bounded in $(2^X, \delta)$. Since $\{x\}$ is bounded in $(X, \mathcal{U})$ by Theorem 1, it results that $(X, \mathcal{U})$ is bounded.*

Theorem 4. *A uniform space $(X, \mathcal{U})$ is totally bounded if and only if so $(\mathcal{P}_0(X), \mathcal{U}^{\uparrow})$ does.*

Proof. Suppose $(X, \mathcal{U})$ is a totally bounded space. Let U be an arbitrary member of uniformity $\mathcal{U}$, and let $U^{\uparrow}$ be the corresponding member of the basis of the uniformity $\mathcal{U}^{\uparrow}$. Since the space $(X, \mathcal{U})$ is totally bounded, it results that there exists a finite set $\{x_i \mid i \in \overline{1,n}\} \subset X$ such that $X = \bigcup_{i=1}^{n} U(x_i)$. Denoting with F_j, $j \in \overline{1, 2^n - 1}$ the set of all non-empty subsets of $\{x_i \mid i \in \overline{1,n}\}$, we get $\mathcal{P}_0(X) = \bigcup_{j=1}^{2^n-1} U^{\uparrow}(F_j)$, that is $(\mathcal{P}_0(X), \mathcal{U}^{\uparrow})$ totally bounded.

Application. In the sequel we will show that the totally bounded property is preserved by an uniformly continuous multifunction, under certain conditions.

Let $(X,\mathcal{U}_1)$ and $(Y,\mathcal{U}_2)$ be uniform spaces. A multifunction on X into Y we will denote by $F\colon X \multimap Y$, and its corresponding function defined on X into $\mathcal{P}_0(Y)$ we wil denote by $\mathbf{F}$, $\mathbf{F}(x) = F(x)$. The image of $A \subset X$ by F is the subset $F(A) \subset Y$, defined as $F(A) = \bigcup_{x \in A} F(x)$.

Definition. The multifunction $F\colon (X,\mathcal{U}_1) \multimap (Y,\mathcal{U}_2)$ is said to be pointwise totally bounded iff the subsets $F(x)$ of Y are totally bounded for each $x \in X$

It is easy to show that a multifunction $F\colon (X,\mathcal{U}_1) \multimap (Y,\mathcal{U}_2)$ is uniformly continuous iff the corresponding function $\mathbf{F}\colon (X,\mathcal{U}_1) \to (\mathcal{P}_0(Y),\mathcal{U}_2)$ is uniformly continuous. Then we have:

Proposition. *Let $F\colon (X,\mathcal{U}_1) \multimap (Y,\mathcal{U}_2)$ be a pointwise totally bounded and uniformly continuous multifunction. If A is a totally bounded subset of X, then so $F(A)$ in $(Y,\mathcal{U}_2)$ does.*

Proof. Since F is an uniformly continuous multifunction it results that $\mathbf{F}$ is an uniformly continuous function.
Then $\mathbf{F}(A) = \{F(x) \mid x \in A\}$ is totally boundeed subset of $\mathcal{P}_0(Y)$ because A is supposed totally bounded. Taking into account that $F(A) = \bigcup_{x \in A} F(x)$ by Theorem 2, it results $F(A)$ is totally bounded subset of Y.

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