A class of systems of differential equations and its treatment with matrix methods. III. Contiguous systems.

To the fiftieth birthday of Prof. Otto Varga.

By E. MAKAI (Budapest).

1.[1]) In Part I. of the present paper [1] I investigated the properties of the matrix differential equation

$$\mathbf{X}\mathbf{Y}' = \mathbf{A}\mathbf{Y} \tag{1.1}$$

($\mathbf{X}$ a diagonal matrix with the elements $x-a_1, \ldots, x-a_n$, $\mathbf{A}$ an n by n constant matrix) or what amounts to the same, those of the vector differential equation

$$\mathbf{X}\mathbf{y}' = \mathbf{A}\mathbf{y} \tag{1.2}$$

where $\mathbf{y}$ is a column of the matrix $\mathbf{Y}$.

It was found among others that several classical linear differential equations are equivalent to a system of type (1. 2). For instance the system

$$\begin{bmatrix} x-1 & 0 \\ 0 & x \end{bmatrix} \mathbf{y}' = \begin{bmatrix} \gamma-\alpha-\beta & \gamma-\alpha \\ \beta-\gamma & -\gamma \end{bmatrix} \mathbf{y} \tag{1.3}$$

has the solution

$$\mathbf{y}_{\alpha,\beta,\gamma} = \begin{bmatrix} \gamma F(\alpha, \beta, \gamma, x) \\ (\beta-\gamma) F(\alpha, \beta, \gamma+1, x) \end{bmatrix}$$

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[1]) Notations:

$\mathbf{a} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \{a_1, a_2, \ldots, a_n\}$, $\mathbf{b}, \ldots$ column vectors

$\mathbf{a}^* = [a_1, \ldots, a_n]$, $\mathbf{b}^*, \ldots$, row vectors

$\mathbf{A}, \mathbf{B}, \ldots$ matrices

$\langle a_1, a_2, \ldots, a_n \rangle$ diagonal matrix

$\mathbf{I} = \langle 1, 1, \ldots, 1 \rangle$ unit matrix where $F(a, b, c, x)$ is Gauss' hypergeometric function.[1a]) A simpler instance of an equation of type (1. 1) is the case when $n=1$:

$$(x-a_1)y' = a_{11}y. \tag{1.4}$$

Returning to the general case we shall term two matrix differential equations $\mathbf{XY}' = \mathbf{AY}$ and $\mathbf{XZ}' = \mathbf{BZ}$ ($\mathbf{A}$ and $\mathbf{B}$ constant n by n matrices) to be *contiguous* if there exists a matrix $\mathbf{M}(x)$ linear in x (i. e. its elements are linear functions of x) such that

$$\mathbf{Y} = \mathbf{M}(x)\mathbf{Z} \tag{1.5}$$

is a solution of $\mathbf{XY}' = \mathbf{AY}$. This implies that if $\mathbf{y}$ and $\mathbf{z}$ are the kth columns of $\mathbf{Y}$ and $\mathbf{Z}$ respectively, then

$$\mathbf{y} = \mathbf{M}(x)\mathbf{z}. \tag{1.6}$$

For instance, the differential equation $(x-a_1)z' = (a_{11}-1)z$ is contiguous in the sense defined above to (1. 4), for $(x-a_1)z$ is a solution of (1. 4).

Another example of a pair of contiguous differential equations is (1. 3) and the equation which one gets from (1. 3) by replacing in it α by $\alpha+1$, or rather the corresponding matrix differential equations of type (1. 1). Indeed one may readily verify using Gauss' formulae between contiguous hypergeometric functions that

$$\mathbf{y}_{\alpha,\beta,\gamma} = \begin{bmatrix} x-1 & x \\ \frac{\gamma-\beta}{\alpha-\gamma}(x-1) & \frac{\gamma-\beta}{\alpha-\gamma}x + \frac{\alpha}{\gamma-\alpha} \end{bmatrix} \mathbf{y}_{\alpha+1,\beta,\gamma}, \tag{1.7}$$

which remains true for any continuation of the vectors $\mathbf{y}_{\alpha,\beta,\gamma}$ and $\mathbf{y}_{\alpha+1,\beta,\gamma}$ on

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1a) We remark here as a supplement to 5, Part I of this paper, that an example of a vector differential equation of the type

$$(\mathbf{K}-\mathbf{L}x)\mathbf{z}' = \mathbf{M}\mathbf{z} \tag{3.1}$$

of Part I. ($\mathbf{K}, \mathbf{L}, \mathbf{M}$ constant square matrices) is

$$\begin{bmatrix} -x & 1 \\ 1 & -x \end{bmatrix} \mathbf{z}' = \begin{bmatrix} -\nu-1 & 0 \\ 0 & \lambda+\nu+\frac{1}{2} \end{bmatrix} \mathbf{z}.$$

One of its solutions is

$$\mathbf{z} = \begin{bmatrix} P_{\nu+1}^{(\lambda)}(x) \\ P_{\nu}^{(\lambda)}(x) \end{bmatrix},$$

the components of the solution vector being ultraspherical polynomials in Szegő's notation (Orthogonal Polynomials, New York, 1939). This can be verified by the use of formula (4. 7. 28) of Szegő's book. (Cf. problem 4783 by L. Carlitz in *Amer. Math. Monthly* **65** (1958), p. 288.)

their (common) Riemann surface securing thereby the contiguity of the two matrix differential equations.

The subsequent analysis will show that in the hypergeometric case there exist three other formulae of the type (1. 6) similar to (1. 7) and connecting $\mathbf{y}_{\alpha,\beta,\gamma}$ with $\mathbf{y}_{\alpha,\beta+1,\gamma}$, $\mathbf{y}_{\alpha+1,\beta,\gamma+1}$, $\mathbf{y}_{\alpha,\beta+1,\gamma+1}$ respectively.

Each of these formulae can be verified directly with the help of Gauss' 15 relations between contiguous hypergeometric functions, and conversely from these formulae the entire set of Gauss' relations may be derived.

As a generalization of the foregoing we will show the following

Theorem. *Given a matrix differential equation of the form* (1. 1), *if* (*a*) *there exists a characteristic value of* $\mathbf{A}$ *different from* 1 *and* 0 *and* (*b*) *among the quantities* $a_1, a_2, \ldots, a_n$ *at least two are differing, then there exists a matrix differential equation* $\mathbf{X}\mathbf{Z}' = \mathbf{B}\mathbf{Z}$ *such that between the solution matrices* $\mathbf{Y}$ *and* $\mathbf{Z}$ *there exists a relation of type* (1. 5).

In other words the solution of (1. 1) may be written as a product of two matrix factors so that the first factor is linear in x and the second is a matrix $\mathbf{Z}$ which satisfies a matrix differential equation similar to (1. 1).

The construction which follows shows that $\mathbf{B}$ is not uniquely determined; indeed if one supposes that $\mathbf{A}$ is diagonalizable and none of its characteristic values is equal to 0 or 1, further each coordinate of every characteristic vector of $\mathbf{A}$ is different from 0, then there are at least n^2 matrix differential equations, contiguous to $\mathbf{X}\mathbf{Y}' = \mathbf{A}\mathbf{Y}$. Each of these n^2 contiguous equations, say $\mathbf{X}\mathbf{Z}' = \mathbf{B}^{(p,q)}\mathbf{Z}$ where p and q are any natural numbers not greater than n, may be characterized as follows: if the set of the diagonal elements and of the characteristic values of $\mathbf{A}$ and $\mathbf{B}^{(p,q)}$ are $\{a_{ii}\}, \{\lambda_i\}$ and $\{b_{ii}^{(p,q)}\}, \{\mu_i^{(p,q)}\}$ respectively $(i = 1, 2, \ldots, n)$ then

$$b_{ii} = a_{ii} - \delta_{ip}\,; \; \mu_i^{(p,q)} = \lambda_i - \delta_{iq} \qquad (i = 1, 2, \ldots, n)$$

where δ_{ik} is the Kronecker delta.

As an application we shall deduce the (known) linear relations between a class of contiguous generalized hypergeometric functions. It will be seen that all but one of these relations are consequences of one and the same matrix formula (6. 4).[2])

2. We shall try to find necessary conditions relating to the form of $\mathbf{M} = \mathbf{M}(x)$. Suppose that there exists a relation $\mathbf{Y} = \mathbf{M}(x)\mathbf{Z}$ between the

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[2]) The simpler question whether one can find to a given $\mathbf{A}$ a *constant* matrix $\mathbf{M}$ and $\mathbf{B}$ having the same properties as above may be answered immediately. Take $\mathbf{M}$ a regular constant diagonal matrix. Then from $\mathbf{X}\mathbf{Y}' = \mathbf{A}\mathbf{Y}$ it follows $\mathbf{X}\mathbf{M}\mathbf{Y}' = \mathbf{M}\mathbf{A}\mathbf{M}^{-1}\mathbf{M}\mathbf{Y}$ and $\mathbf{B} = \mathbf{M}\mathbf{A}\mathbf{M}^{-1}$, $\mathbf{Z} = \mathbf{M}\mathbf{Y}$.

solutions $\mathbf{Y}$ and $\mathbf{Z}$ of the systems $\mathbf{XY}'=\mathbf{AY}$ resp. $\mathbf{XZ}'=\mathbf{BZ}$ (det $\mathbf{Y}\neq 0$, det $\mathbf{Z}\neq 0$ if $x\neq a_i$, $i=1,2,\dots,n$) where each element of $\mathbf{M}$ is a linear function of x. Let x be different from each of the quantities a_i. Then det $\mathbf{M}=$ det $\mathbf{Y}$/det $\mathbf{Z}\neq 0$, further the matrices $\mathbf{K}=\mathbf{X}^{-1}\mathbf{A}$ and $\mathbf{L}=\mathbf{X}^{-1}\mathbf{B}$ exist and from the equations $\mathbf{Y}'=\mathbf{KY}, \mathbf{Z}'=\mathbf{LZ}$ and $\mathbf{Y}=\mathbf{MZ}$ we infer that

$$\mathbf{M}'\mathbf{Z}+\mathbf{MZ}'=\mathbf{KMZ},$$

$$\mathbf{M}'+\mathbf{ML}=\mathbf{KM}$$

and

(2. 1) $$\mathbf{M}'+\mathbf{MX}^{-1}\mathbf{B}=\mathbf{X}^{-1}\mathbf{AM}.$$

Denoting the general term of $\mathbf{A}, \mathbf{B}, \mathbf{M}, \mathbf{M}'$ by a_{ik}, b_{ik}, $m_{ik}(x)$, m'_{ik}, respectively, the last relation is equivalent to

(2. 1') $$\sum_l \frac{m_{il}(x)\, b_{lk}}{x-a_l}=\sum_l \frac{a_{il}\, m_{lk}(x)}{x-a_i}-m'_{ik}.$$

If one supposes that $a_k\neq a_i$ if $i\neq k$, this relation holds only if $x-a_l \mid m_{il}(x)$ $(l\neq i)$[3]) so that $\mathbf{M}(x)$ may be written in the from

$$\mathbf{M}=\begin{bmatrix} r_{11}\cdot(x-a_1) & r_{12}\cdot(x-a_2)\dots r_{1n}\cdot(x-a_n) \\ \vdots & \vdots \qquad\qquad \vdots \\ r_{n1}\cdot(x-a_1) & r_{n2}\cdot(x-a_2)\dots r_{2n}\cdot(x-a_n)\end{bmatrix}+\begin{bmatrix} s_1 & & 0 \\ & \ddots & \\ 0 & & s_n\end{bmatrix}$$

where the r_{ik}'s and s_i's are constants. If we introduce the matrix $\mathbf{R}$ with elements r_{ik} and the matrix $\mathbf{S}$ with the elements $s_i\delta_{ik}$ (δ_{ik} is the Kronecker delta) the last relations may be written in the form

(2. 2) $$\mathbf{M}=\mathbf{RX}+\mathbf{S} \qquad (\mathbf{R} \text{ and } \mathbf{S} \text{ const. matrices, } \mathbf{S} \text{ diag.}).$$

Substituting this into (2. 1) we have after suitable transformations

(2. 3) $$\mathbf{XR}+\mathbf{XRB}+\mathbf{SB}=\mathbf{ARX}+\mathbf{AS}$$

as $\mathbf{M}'=\mathbf{R}$ *and* $\mathbf{XS}=\mathbf{SX}$.

Let us now introduce the diagonal matrix $\mathbf{C}=\langle a_1,\dots,a_n\rangle$. Then

(2. 4) $$\mathbf{X}=\mathbf{I}x-\mathbf{C}.$$

[3]) The possibility that $x-a_l \nmid m_{il}(x)$ and hence $b_{lk}=0$ $(l\neq i, k=1,2,\dots,n)$ may be discarded, for in this case the lth equation of the system $\mathbf{XZ}'=\mathbf{BZ}$ is simply $(x-a_l)z'_l=0$. This shows that the place $x=a_l$ is no singular place of the solution of the system $\mathbf{XZ}'=\mathbf{BZ}$.

Substituting this in (2.3) and using the principle of equal coefficients we have

$$\mathbf{R}+\mathbf{RB}=\mathbf{AR} \tag{2.5}$$

and

$$-\mathbf{CR}-\mathbf{CRB}+\mathbf{SB}=-\mathbf{ARC}+\mathbf{AS}. \tag{2.6}$$

The last equation may be substituted obviously by

$$\mathbf{SB}-\mathbf{AS}=\mathbf{CAR}-\mathbf{ARC}. \tag{2.7}$$

This enables us to enounce the following. If $a_i \neq a_k$ $(i \neq k)$ and if the relations $\mathbf{XY}'=\mathbf{AY}, \mathbf{XZ}'=\mathbf{BZ}, \mathbf{Y}=\mathbf{MZ}$ ($\mathbf{M}$ linear in x) hold, then (2. 2), (2. 3), (2. 5) and (2. 7) are necessarily true. To a given $\mathbf{A}$ and a given $\mathbf{C}$ one may find therefore a $\mathbf{B}$ and an $\mathbf{M}$ (or $\mathbf{R}$ and $\mathbf{S}$) if one solves equations (2. 5) and (2. 7).

If any two a_i's coincide the conditions given above are not necessary, as from (2. 1') the formula (2. 2) does not follow. Yet if in this case too one may find to a given $\mathbf{A}$ matrices $\mathbf{R}, \mathbf{S}$ and $\mathbf{B}$ satisfying (2. 2), (2. 3), (2. 6) and (2. 7), then our problem is again solved.

3. The next step is to find a solution of the system of the two matrix equations (2. 5) and (2. 7) with given $\mathbf{C}$ and $\mathbf{A}$. Unknowns are the three matrices $\mathbf{R}$, $\mathbf{B}$ and the diagonal matrix $\mathbf{S}$. I didn't succeed in finding the general solution of this system.

However, if $\lambda \neq 1, 0$ is a characteristic value of $\mathbf{A}$, one gets a particular solution in the following way. Let $\mathbf{u}$ be a characteristic column vector of $\mathbf{A}$ corresponding to λ or in the case of a multiple root one of these vectors. We may suppose without restriction of the generality that $u_1 \neq 0$ and particularly that $u_1 = 1$.

Suppose now (a) that $\mathbf{v}^* = [v_1, \ldots, v_n] = [1, a_{12}, \ldots, a_{1n}]$ is a characteristic row vector of the unknown matrix $\mathbf{B}$ corresponding to the characteristic number $\lambda - 1$; (b) $\mathbf{R}$ is a one-dyad matrix and $r_{ik} = u_i v_k$; (c) $s_1 = 0$; (d) none of the quantities $a_2, \ldots, a_n$ is equal to a_1. We are going to show that it is possible to find a solution of the system (2. 5), (2. 7) satisfying these assumptions.

It is easy to show that each matrix $\mathbf{B}$ and $\mathbf{R}$ satisfying the above assumptions fulfils equation (2. 5) or

$$r_{ik} + \sum_l r_{il} b_{lk} - \sum_l a_{il} r_{lk} = 0, \qquad (i, k = 1, 2, \ldots, n),$$

for

$$r_{ik}+\sum_l r_{il}b_{lk}-\sum_l a_{il}r_{lk}=u_i v_k+u_i\sum_l v_l b_{lk}-\sum_l a_{il}u_l v_k=$$
$$=u_i v_k+u_i(\lambda-1)v_k-\lambda u_i v_k=0.$$

Further equation (2. 7) is equivalent to

$$s_i b_{ik}-s_k a_{ik}=(a_i-a_k)\sum_l a_{il}r_{lk} \qquad (i,k=1,2,\ldots,n)$$

and by virtue of (a) and (b)

$$s_i b_{ik}-s_k a_{ik}=(a_i-a_k)\sum_l a_{il}u_l v_k=(a_i-a_k)\lambda u_i v_k$$

or using (c) and (a)

$$(2.\,8_1) \qquad s_i b_{i1}=(a_i-a_1)\lambda u_i$$

$$(2.\,8_k) \qquad s_i b_{ik}-s_k a_{ik}=(a_i-a_k)\lambda u_i a_{1k} \qquad (k=2,3,\ldots,n).$$

Now let be $i=1$ in $(2.\,8_k)$. Then $[s_k+(a_1-a_k)\lambda]a_{1k}=0$ and this equation is always satisfied by

$$(2.\,9) \qquad s_k=\lambda(a_k-a_1) \qquad (k=2,3,\ldots,n).$$

Moreover by virtue of our assumption (c) (2. 9) remains true if $k=1$. Substituting the value of s_i and s_k from (2. 9) into (2. 8) one sees that these equations are always satisfied if

$$(2.\,10) \qquad b_{ik}=a_{ik}\frac{a_k-a_1}{a_i-a_1}+a_{1k}u_i\frac{a_i-a_k}{a_i-a_1} \qquad (i,k=2,3,\ldots,n)$$

and

$$(2.\,11) \qquad b_{i1}=u_i \qquad (i=2,3,\ldots,n).$$

The remaining first row of the matrix **B** may be calculated from assumption (a):

$$\sum_i v_i b_{ik}=(\lambda-1)v_k.$$

It is

$$b_{1k}=(\lambda-1)v_k-\sum_{i=2}^{n} v_i b_{ik}$$

and especially

$$b_{11}=a_{11}-1.$$

Thus we found a set of matrices **R**, **S**, **B** which satisfy the assumptions (a) — (d) and form a solution of the system (2. 5), (2. 7). We have yet to show that the matrix $\mathbf{M}=\mathbf{RX}+\mathbf{S}$ thus constructed is regular at each regular place of the system. Indeed if $x \neq a_j$, then a simple calculation shows that

$$\det \mathbf{M} = \lambda^{n-1}(x-a_1)(a_2-a_1)\cdots(a_n-a_1) \neq 0.$$

It is here that we make use of the restriction that $\lambda \neq 0$.

It is to be mentioned that in the case $\det \mathbf{M} \neq 0$ the relation (1.6) remains true but it may become meaningless if $\mathbf{M}(x)\mathbf{z}$ happens to be identically 0.

4. As a corollary one may show that the spectra *of the matrices* $\mathbf{A}$ *and* $\mathbf{B}$ *connected by the relation of the preceding chapter differ only in one characteristic number.*

Let indeed be

$$\mathbf{U} = \begin{bmatrix} 1 & 0 \cdots 0 \\ u_2 & 1 \quad 0 \\ \vdots & \ddots \quad \vdots \\ u_n & 0 \cdots 1 \end{bmatrix}, \quad \text{then } \mathbf{U}\mathbf{A}\mathbf{U}^{-1} = \begin{bmatrix} \lambda & a_{12} & \cdots a_{1n} \\ 0 & a_{22}-a_{12}u_2 & a_{2n}-a_{1n}u_2 \\ \vdots & & \vdots \\ 0 & a_{n2}-a_{12}u_n \cdots & a_{nn}-a_{1n}u_n \end{bmatrix}.$$

Further let be

$$\mathbf{V} = \begin{bmatrix} 1 & v_2 \cdots v_n \\ 0 & 1 \quad 0 \\ \vdots & \ddots \\ 0 & 0 \cdots 1 \end{bmatrix}, \quad \text{then } \mathbf{V}\mathbf{B}\mathbf{V}^{-1} = \begin{bmatrix} \lambda-1 & 0 & \cdots a \\ b_{21} & b_{22}-b_{21}v_2 & b_{2n}-b_{21}v_n \\ \vdots & & \vdots \\ b_{n1} & b_{n2}-b_{n1}v_2 \cdots & b_{nn}-b_{n1}v_n \end{bmatrix}.$$

As by (2. 10), (2. 11) and assumption (a)

$$b_{ik}-b_{i1}v_k = (a_i-a_1)^{-1}(a_{ik}-a_{1k}u_i)(a_k-a_1) \qquad (i, k > 1)$$

the submatrices of $\mathbf{U}\mathbf{A}\mathbf{U}^{-1}$ and $\mathbf{V}\mathbf{B}\mathbf{V}^{-1}$ obtained by omitting the first rows and columns are similar and have therefore the same characteristic numbers. As the spectra of $\mathbf{A}$ and $\mathbf{B}$ consist of the spectra of these submatrices and of the number λ resp. $\lambda-1$, our assertion is proved.

5. Applying the preceding considerations to the system (1. 3), i. e. to the system which corresponds to the hypergeometric equation we see that the characteristic numbers of $\mathbf{A}$ are $-\alpha$ and $-\beta$. Here we suppose that neither α nor β are equal to γ. If we choose $\lambda = -\alpha$ and $s_1 = 0$ and perform the calculations of **3** we arrive at (1. 7) while the assumptions $\lambda = -\beta$, $s_1 = 0$; $\lambda = -\alpha$, $s_2 = 0$; $\lambda = -\beta$, $s_2 = 0$, respectively, lead to other three similar relations. These four relations differ only in form from Gauss' relations between contiguous hypergeometric functions and all of the latter relations may be obtained from them. We do not write these relations in full length, for in **6**

we shall establish them in a more general form in connection with the generalized hypergeometric functions, restricting ourselves to the case $s_1=0$.

6. The generalized hypergeometric function

$$(6.1)\quad {}_pF_{n-1}={}_pF_{n-1}(\alpha_1,\alpha_2,\dots,\alpha_p;\ \beta_2,\beta_3,\dots,\beta_n,x)=\sum_{\nu=0}^{\infty}\frac{(\alpha_1)_\nu(\alpha_2)_\nu\cdots(\alpha_p)_\nu}{(\beta_2)_\nu(\beta_3)_\nu\cdots(\beta_n)_\nu}\frac{x^\nu}{\nu!}$$

$[(c)_\nu=c(c+1)\cdots(c+\nu-1)$, $\alpha_j\neq 0$, β_j is neither 0, nor a negative integer and finally no β is equal to any $\alpha]$ and its contiguous[4]) functions satisfy a set of linear relations.

From this set one may choose a subset of $2p+n-1$ linearly independent relations with the aid of which one can construct each linear relation between generalized hypergeometric functions. The complete list of these relations was given by E. D. RAINVILLE [2]. Here we will give another deduction of these relations restricting ourselves to the case $p=n$.

Let the function (6. 1) be denoted by y_1^0 and let be $y_i^0={}_nF_{n-1}(\beta_i+)$. It was shown in Part I. of this paper (p. 21) that

$$(6.2)\quad \begin{aligned} (x-1)y_1^{0\prime}&=a y_1^0-U_2y_2^0-\cdots-U_ny_n^0\\ x y_2^{0\prime}&=\beta_2y_1^0-\beta_2y_2^0\\ &\dots\\ x y_n^{0\prime}&=\beta_ny_1^0\qquad\qquad\qquad -\beta_ny_n^0 \end{aligned}$$

where

$$(6.3)\quad a=\sum_{i=2}^{n}\beta_i-\sum_{i=1}^{n}\alpha_i,\ U_j=\frac{(\alpha_1-\beta_j)(\alpha_2-\beta_j)\cdots(\alpha_n-\beta_j)}{\beta_j(\beta_2-\beta_j)\cdots(\beta_{j-1}-\beta_j)(\beta_{j+1}-\beta_j)\cdots(\beta_n-\beta_j)}.$$

If the matrix of the right hand side of the system (6. 2) is denoted by $\mathbf{A}$ and $\mathbf{X}=\langle x-1,x,\dots,x\rangle$ then the solution of the system $\mathbf{X}\mathbf{y}'=\mathbf{A}\mathbf{y}$ characterised by the initial condition $\mathbf{y}(0)=\{1,1,\dots,1\}$ is the vector $\mathbf{y}^0=\{y_1^0,y_2^0,\dots,y_n^0\}$.[5])

Now we shall try to find a system $\mathbf{X}\mathbf{z}'=\mathbf{B}\mathbf{z}$ contiguous to the above system in the sense of **1.** The characteristic values of $\mathbf{A}$ are now $-\alpha_1,-\alpha_2,\dots,-\alpha_n$. Let the λ of **3** be $-\alpha_l$. (We suppose of course that

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[4]) A function contiguous to (6. 1) differs from it only by changing one and only one of its parameters α_i or β_j into α_i+1, α_i-1 resp. β_j+1, or β_j-1. As customary, these functions will be denoted by $F(\alpha_i+),\dots,F(\beta_j-)$. Further, the notation $F(\alpha_l+,\beta_j+)$ will be used for the function which one gets from (6. 1) by replacing in it α_l and β_j by α_l+1 and β_j+1.

[5]) If $p<n$ then the vector $\mathbf{y}^0$ satisfies a differential equation which does not belong to the type (1. 2) and therefore the reasoning of **3** is to be modified.

$-\alpha_l \neq 0, 1$). Then performing the calculations indicated above we have

$$\mathbf{u}=\left\{1, \frac{\beta_2}{\beta_2-\alpha_l}, \ldots, \frac{\beta_n}{\beta_n-\alpha_l}\right\}, \ \mathbf{v}^*=[1, -U_2, \ldots, -U_n]; \ \mathbf{S}=\langle 0, \alpha_l, \ldots, \alpha_l\rangle;$$

$$\mathbf{B}=\begin{bmatrix} a-1 & (\alpha_l-\beta_2+1)v_2 & (\alpha_l-\beta_3+1)v_3 & \cdots & (\alpha_l-\beta_n+1)v_n \\ \frac{\beta_2}{\beta_2-\alpha_l} & -\beta_2 & 0 & & 0 \\ \frac{\beta_3}{\beta_3-\alpha_l} & 0 & -\beta_3 & & 0 \\ \vdots & & & & \vdots \\ \frac{\beta_n}{\beta_n-\alpha_l} & 0 & 0 & \cdots & -\beta_n \end{bmatrix}.$$

Let now be $\mathbf{N}=\langle 1, \beta_2-\alpha_l, \ldots, \beta_n-\alpha_l\rangle$. Then $\mathbf{B}=\mathbf{N}^{-1}\overline{\mathbf{B}}\mathbf{N}$ where

$$\overline{\mathbf{B}}=\begin{bmatrix} \bar{a} & -\overline{U}_2 & \cdots & -\overline{U}_n \\ \beta_2 & -\beta_2 & & 0 \\ \vdots & & & \vdots \\ \beta_n & 0 & \cdots & -\beta_n \end{bmatrix}$$

and $\bar{a}, \overline{U}_i$ differ from the corresponding quantities a, U_i in (6.3) only by interchanging α_l by α_l+1. Consequently a solution of $\mathbf{X}\overline{\mathbf{y}}'=\overline{\mathbf{B}}\overline{\mathbf{y}}$ is $\overline{\mathbf{y}}^0=\{\bar{y}_1^0, \ldots, \bar{y}_n^0\}$ where $\bar{y}_i^0$ differs from y_i^0 only by a change of α_l into α_l+1. This solution corresponds to the initial condition $\bar{y}(0)=\{1, 1, \ldots, 1\}$.

Now (cfr. footnote 2) a solution of $\mathbf{X}\mathbf{z}'=\mathbf{B}\mathbf{z}$ is $\mathbf{z}^0=\mathbf{N}^{-1}\overline{\mathbf{y}}^0$ and this corresponds to the initial conditions

$$\mathbf{z}(0)=\{1, (\beta_2-\alpha_l)^{-1}, \ldots, (\beta_n-\alpha_l)^{-1}\}.$$

We state that $-\mathbf{z}^0$ is just that solution of $\mathbf{X}\mathbf{z}'=\mathbf{B}\mathbf{z}$ which is equal to $\mathbf{M}(x)\mathbf{y}^0=(\mathbf{u}\mathbf{v}^*\mathbf{X}+\mathbf{S})\mathbf{y}^0$. It suffices to verify the relation $\mathbf{y}^0(0)=-\mathbf{M}(0)\mathbf{z}^0(0)$ or

$$\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}=-\begin{bmatrix} -1 & 0 & \cdots & 0 \\ -\frac{\beta_2}{\beta_2-\alpha_l} & \alpha_l & & 0 \\ \vdots & & \ddots & \\ -\frac{\beta_n}{\beta_n-\alpha_l} & 0 & & \alpha_l \end{bmatrix}\begin{bmatrix} 1 \\ \frac{1}{\beta_2-\alpha_l} \\ \vdots \\ \frac{\beta_n}{\beta_n-\alpha_l} \end{bmatrix}$$

which is clearly true. The relation

$$\mathbf{y}^0=-\mathbf{M}(x)\mathbf{z}^0 \tag{6.4}$$

when written in full length is equivalent to the following n^2 relations:

$$(6.4_l) \qquad F=(1-x)F(\alpha_l+)+x\sum_{j=2}^{n}\frac{U_j}{\beta_j-\alpha_l}F(\alpha_l+,\beta_j+) \qquad (l=1,2,\ldots,n);$$

$$(6.4_{li}) \qquad \begin{aligned} F(\beta_i+)=(1-x)\frac{\beta_i}{\beta_i-\alpha_l}F(\alpha_l+)-\frac{\alpha_l}{\beta_i-\alpha_l}F(\alpha_l+,\beta_i+)+ \\ +\frac{\beta_i}{\beta_i-\alpha_l}x\sum_{j=2}^{n}\frac{U_j}{\beta_j-\alpha_l}F(\alpha_l+,\beta_j+) \qquad \begin{pmatrix} l=1,2,\ldots,n \\ i=2,3,\ldots,n \end{pmatrix}. \end{aligned}$$

One can show that these formulae contain all but one of Rainville's relations quoted above. For the n relations (6.4_l) are equivalent to Rainville's formula (21), while eliminating the term with Σ from (6.4_{li}) and (6.4_l) we get

$$(6.5) \qquad (\beta_i-\alpha_l)F(\beta_i+)-\beta_iF+\alpha_lF(\alpha_l+,\beta_i+)=0 \qquad (i\neq 1).$$

If $l=1$ we have

$$(6.6) \qquad (\beta_i-\alpha_1)F(\beta_i+)-\beta_iF+\alpha_1F(\alpha_1+,\beta_i+)=0 \qquad (i\neq 1).$$

This is equivalent to relation (15) of Rainville while subtracting the expression (6.6) from (6.5) we get

$$(\alpha_1-\alpha_l)F(\beta_i+)+\alpha_lF(\alpha_l+,\beta_i+)-\alpha_1F(\alpha_1+,\beta_i+)=0 \qquad (l=2,\ldots,n).$$

These $n-1$ relations are the equivalents of Rainville's formula (14).

There remains formula (19) of Rainville which is in our notation

$$(1-x)\alpha_1[F-F(\alpha_1+)]=x[\alpha F-\sum_{j=2}^{n}U_jF(\beta_j+)].$$

It is not contained in the matrix relation $\mathbf{y}^0=-\mathbf{M}(x)\mathbf{z}^0$ but one deduces it at once from the easily verifiable differential-difference equation

$$\alpha_1[F-F(\alpha_1+)]+xF'=0$$

and from the first equation of the system (6.2).

Bibliography.

[1] E. Makai, A class of systems of differential equations and its treatment with matrix methods, *Publ. Math. Debrecen* 5 (1957), 5—37.

[2] E. D. Rainville, The continuous function relations for ${}_pF_q$ etc., *Bull. Amer. Math. Soc.* **51** (1945), 714–723.

(Received July 5, 1958.)