On nuclei of groupoids.

To Professor O. Varga on his 50th birthday.

By A. KERTÉSZ (Debrecen) and A. SADE (Marseille).

§ 1. Introduction.

Let $(G,\cdot)$ be a groupoid.[1]) The set A_λ of all x in G such that $(xa)b = x(ab)$ for all $a, b (\in G)$ is often called the *left nucleus* of $(G,\cdot)$ (see e. g. BRUCK [4]). The *middle nucleus* A_μ of $(G,\cdot)$ is similarly defined as the set of all $x(\in G)$ such that $(ax)b = a(xb)$, and the *right nucleus* A_ϱ in terms of $(ab)x = a(bx)$. The *nucleus* A of $(G,\cdot)$ is defined as $A = A_\lambda \cap A_\mu \cap A_\varrho$. These examples, as well as some others, e. g. the center and the Moufang nucleus (see BRUCK [3], p. 288) suggest the following more general definition of nuclei.

Let $\psi_1(x_1, \dots, x_k)$, $\psi_2(x_1, \dots, x_k)$ be two single valued functions in the variables $x_1, \dots, x_k$, defined on the groupoid $(G,\cdot)$ and taking values in G. The set X_i of all elements $x(\in G)$ such that

$$(\mathfrak{X}) \qquad \psi_1(a_1, \dots, a_{i-1}, x, a_{i+1}, \dots, a_k) = \psi_2(a_1, \dots, a_{i-1}, x, a_{i+1}, \dots, a_k)$$

for all $a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_k (\in G)$ will be called the $\mathfrak{X}_i$-*nucleus* $(i = 1, \dots, k)$ of $(G,\cdot)$. The $\mathfrak{X}$-*nucleus* X is defined as $X = \bigcap_{i=1}^{k} X_i$. Clearly with this definition the nuclei $A_\lambda, A_\mu, A_\varrho$ and A are exactly the $\mathfrak{A}$-nuclei of $(G,\cdot)$ corresponding to

$$(\mathfrak{A}) \qquad (x_1x_2)x_3 = x_1(x_2x_3)$$

and $\lambda = 1$, $\mu = 2$, $\varrho = 3$; moreover the center Z of $(G,\cdot)$ coincides with the $\mathfrak{Z}_1$-, $\mathfrak{Z}_2$- and $\mathfrak{Z}$-nuclei corresponding to

$$(\mathfrak{Z}) \qquad x_1x_2 = x_2x_1.$$

The purpose of the present paper is to define some special nuclei and to investigate the relation between them. In view of the great number of

---

[1]) For terminology and notations see § 2.

problems which can be raised, we found it impossible to aim at completeness even in studying these special cases, and had to content ourselves with a clarification of the most fundamental relations.

§ 2. Terminology and notation

In this section we shall summarize the terminology and notation used throughout the paper.

$a \Longrightarrow b$: a implies b;
$a \Longleftrightarrow b$: a implies b and b implies a;
$a \in G$: a is an element of the set G;
$H \subseteq G$: the set H is contained in the set G;
$H \subset G$: H is a proper subset of G;
$H_1 \cap H_2$: the set of all elements common to H_1 and H_2;
$\emptyset$: the empty set;
$\varphi : a \rightarrow b$: φ is the mapping under which b is the image of a;
ι : the identity mapping;
$\overset{\text{def}}{=}$: equality by definition.

(i) A *groupoid* $(G, \times)$ is a system consisting of the non-empty set G and the (single-valued) binary operation $a \times b$ defined for all $a, b (\in G)$. When the operation is denoted by a dot "$\cdot$", we shall often write for the sake of brevity ab instead of $a \cdot b$ and $ab \cdot c$ instead of $(a \cdot b) \cdot c$.

The *center* Z of $(G, \times)$ is the set of all elements $c (\in G)$ such that

$$c \times a = a \times c$$

for all $a (\in G)$. If $Z = G$, we call the groupoid $(G, \times)$ abelian.

(ii) A *left quasigroup* $(G, \times)$ is a groupoid such that for each ordered pair $a, b (\in G)$ there exists one and only one $x (\in G)$ such that $x \times a = b$. The *right quasigroup* $(G, \times)$ is similarly defined in terms of $a \times y = b$. A *quasigroup* $(G, \times)$ is a groupoid which is both a left and a right quasigroup.

(iii) A *loop* $(G, \times)$ is a quasigroup with a unit element, i. e. an element e such that $a \times e = e \times a = a$ holds for each $a (\in G)$.

(iv) A *semigroup* $(G, \times)$ is a groupoid such that the "associative law"

$$(a \times b) \times c = a \times (b \times c)$$

holds for all $a, b, c (\in G)$.

(v) A *homomorphism* φ of a groupoid $(G, \times)$ into (onto) a groupoid $(H, \circ)$ is a single-valued mapping of G into (onto) H, such that $(a\varphi) \circ (b\varphi) = (a \times b)\varphi$ for all $a, b (\in G)$. If φ is one-to-one onto H, then φ is called an *isomorphism* of $(G, \times)$ with $(H, \circ)$. An *endomorphism* of $(G, \times)$ is a homomorphism of $(G, \times)$ into itself, while an *automorphism* of $(G, \times)$ is an isomorphism of $(G, \times)$ with itself.

(vi) The groupoids $(G, \times)$ and $(H, \circ)$ are said to be *isotopic* if there exists an ordered triple $\{\xi, \eta, \zeta\}$ of one-to-one mappings ξ, η, ζ of G onto H such that

$$(a\xi)\circ(b\eta) = (a\times b)\zeta$$

for all $a, b (\in G)$. Clearly isomorphism is a special type of isotopism, namely that corresponding to the case $\xi = \eta = \zeta$. An isotopism $\{\xi, \eta, \iota\}$ of $(G, \times)$ onto $(G, *)$ (i. e. an isotopism the third component ζ of which is the identity mapping) is called a *principal isotopism*.

§ 3. Nuclei related to the associative and similar laws.

Let us consider all such "laws", which arise from the associative law by a permutation of the elements and by a (possible) rearrangement of brackets: [2])

($\mathfrak{A}$)	$x_1x_2\cdot x_3 = x_1\cdot x_2x_3$;	($\mathfrak{H}$)	$x_1\cdot x_2x_3 = x_2\cdot x_1x_3$;
($\mathfrak{B}$)	$x_1x_2\cdot x_3 = x_1\cdot x_3x_2$;	($\mathfrak{L}$)	$x_1\cdot x_2x_3 = x_2\cdot x_3x_1$;
($\mathfrak{C}$)	$x_1x_2\cdot x_3 = x_2\cdot x_1x_3$;	($\mathfrak{M}$)	$x_1\cdot x_2x_3 = x_3\cdot x_2x_1$;
($\mathfrak{D}$)	$x_1x_2\cdot x_3 = x_2\cdot x_3x_1$;	($\mathfrak{N}$)	$x_1x_2\cdot x_3 = x_1x_3\cdot x_2$;
($\mathfrak{F}$)	$x_1x_2\cdot x_3 = x_3\cdot x_1x_2$;	($\mathfrak{O}$)	$x_1x_2\cdot x_3 = x_2x_1\cdot x_3$;
($\mathfrak{I}$)	$x_1x_2\cdot x_3 = x_3\cdot x_2x_1$;	($\mathfrak{S}$)	$x_1x_2\cdot x_3 = x_2x_3\cdot x_1$;
($\mathfrak{J}$)	$x_1\cdot x_2x_3 = x_1\cdot x_3x_2$;	($\mathfrak{Q}$)	$x_1x_2\cdot x_3 = x_3x_2\cdot x_1$.

Let $(G, \cdot)$ be any groupoid and let $\mathfrak{Y}$ denote an arbitrary but fixed letter from the set $\mathfrak{A}$—$\mathfrak{Q}$ above.

Definitions. The set Y_λ of all elements $x(\in G)$ for which ($\mathfrak{Y}$) holds if we substitute $x_1 = x$, $x_2 = a$, $x_3 = b$ (a, b being arbitrary element of G), will be called the *left* $\mathfrak{Y}$*-nucleus* or simply the $\mathfrak{Y}_\lambda$*-nucleus* of the groupoid $(G, \cdot)$.

Similarly, we define the *middle* $\mathfrak{Y}$*-nucleus* ($\mathfrak{Y}_\mu$*-nucleus*) Y_μ in terms of $x_1 = a$, $x_2 = x$, $x_3 = b$, and the *right* $\mathfrak{Y}$-nucleus ($\mathfrak{Y}_\varrho$-nucleus) Y_ϱ in terms of $x_1 = a$, $x_2 = b$, $x_3 = x$.

$Y \overset{\text{def}}{=} Y_\lambda \cap Y_\mu \cap Y_\varrho$ is called the $\mathfrak{Y}$*-nucleus* of $(G, \cdot)$.

If Z denotes the center of $(G, \cdot)$, then $Z^* \overset{\text{def}}{=} Z \cap A$ is called the *Bruck center* of $(G, \cdot)$.

---

[2]) The relation of these conditions to group axioms was investigated by T. Faragó [5].

A subset H of $(G,\cdot)$ is said to be *characteristic* if for every $h(\in H)$ and for every automorphism φ of $(G,\cdot)$ the element $h\varphi$ is also in H.

Theorem 1 (BRUCK [4], pp. 250—255).[3]) *If $(G,\cdot)$ is any groupoid, then*

(i) *$(A_\lambda,\cdot)$, $(A_\mu,\cdot)$, $(A_\varrho,\cdot)$, $(A,\cdot)$ are semigroups and $(Z^*,\cdot)$ an abelian semigroup, provided they are non-empty;*

(ii) *these sets are characteristic in $(G,\cdot)$;*

(iii) *if further $(G,\cdot)$ is a loop, then $(A_\lambda,\cdot)$, $(A_\mu,\cdot)$, $(A_\varrho,\cdot)$, $(A,\cdot)$ are groups and $(Z^*,\cdot)$ an abelian group.*

If $(G,\cdot)$ and $(H,\circ)$ are isotopic groupoids with unit, then $(A_\lambda,\cdot)$, $(A_\mu,\cdot)$, $(A_\varrho,\cdot)$ and $(Z^*,\cdot)$ are respectively isomorphic to the corresponding entities defined for $(H,\circ)$.

Theorem 2. *If $(G,\cdot)$ is any groupoid with unit e, then*

(I) $Z\cap A_\mu = B_\mu = C_\lambda = D_\lambda$; *this set with the operation "$\cdot$" forms an abelian semigroup with unit (provided it is non-empty), and it is characteristic in $(G,\cdot)$; in particular if $(G,\cdot)$ is a loop, then $(Z\cap A_\mu,\cdot)$ is an abelian group;*

(II) $Z\cap A_\lambda = C_\mu = N_\mu = N_\varrho = P_\lambda$;

(III) $Z\cap A_\varrho = B_\varrho = K_\lambda = K_\mu = L_\lambda$;

(IV) $Z\cap B_\lambda = D_\mu = P_\varrho = Q_\lambda = Q_\varrho$;

(V) $Z\cap C_\varrho = D_\varrho = L_\mu = M_\lambda = M_\varrho$;

(VI) $Z\cap J_\lambda = Z\cap O_\varrho = I_\varrho$;

(VII) $Z\cap K_\varrho = L_\varrho = M_\mu$;

(VIII) $Z\cap N_\lambda = P_\mu = Q_\mu$;

(IX) $Z = F_\varrho = J_\mu = J_\varrho = O_\lambda = O_\mu$;

(X) $Z\supseteq F_\lambda = F_\mu = I_\lambda = I_\mu$.

Addendum 1. $Z^* = B_\mu\cap B_\varrho = C_\lambda\cap C_\mu$; $B\subseteq Z^*$; $C\subseteq Z^*$; *if they are non-empty, $(B,\cdot)$ and $(C,\cdot)$ are abelian semigroups and are characteristic in $(G,\cdot)$. In particular, if $(G,\cdot)$ is a loop, then $B=Z^*$ or $B=\emptyset$ and $C=Z^*$ or $C=\emptyset$.*

Addendum 2. *Let Y denote an arbitrarily chosen one of the sets $B, C, I, J, K, L, M, N, O, P, Q$. If Y has an element which can be always cancelled on one side, then $(G,\cdot)$ is abelian.*

---

[3]) For a proof of this theorem we refer to [3].

Addendum 3. *If* $(G, \cdot)$ *is abelian, then*

(α) $A = A_\lambda = A_\varrho = B_\lambda = B_\varrho = C_\mu = C_\varrho = D_\mu = D_\varrho = K_\lambda =$
$= K_\mu = L_\lambda = L_\mu = M_\lambda = M_\varrho = N_\mu = N_\varrho = P_\lambda = P_\varrho = Q_\lambda = Q_\varrho$;

(β) $A_\mu = B_\mu = C_\lambda = D_\lambda = K_\varrho = L_\varrho = M_\mu = N_\lambda = P_\mu = Q_\mu$;

(γ) $G = F_\lambda = F_\mu = F_\varrho = F = I_\lambda = I_\mu = I_\varrho = I = J_\lambda = J_\mu = J_\varrho =$
$= J = O_\lambda = O_\mu = O_\varrho = O$;

(δ) $A = B = C = D = K = L = M = N = P = Q$.

Addendum 4. *Let* W *denote any one of the letters* B, C, K, L, M, N, P, Q, *and* τ *any one of the letters* λ, μ, ϱ. *If* $W_\tau = G$, *then* $(G, \cdot)$ *is an abelian semigroup with unit.*

REMARK 1. The assertions of Theorem 2 show that

$$(*) \quad Z \supseteq B_\mu, B_\varrho, C_\lambda, C_\mu, D_\lambda, D_\mu, D_\varrho, F_\lambda, F_\mu, F_\varrho, I_\lambda, I_\mu, I_\varrho,$$
$$J_\mu, J_\varrho, K_\lambda, K_\mu, L_\lambda, L_\mu, L_\varrho, M_\lambda, M_\mu, M_\varrho, N_\mu, N_\varrho, O_\lambda,$$
$$O_\mu, P_\lambda, P_\mu, P_\varrho, Q_\lambda, Q_\mu, Q_\varrho.$$

We shall make use of this fact several times in the course of the proof and therefore we want to point out that it can easily be proved also directly (i. e. independently from the theorem). Let for instance $x \in B_\mu$ and let a be an arbitrary element of G. Then $ex \cdot a = e \cdot ax$, i. e. $xa = ax$. Thus we have in fact $B_\mu \subseteq Z$. The validity of the other assertions in (*) can be shown similarly.

REMARK 2. The equalities $J_\mu = J_\varrho$; $K_\lambda = K_\mu$; $M_\lambda = M_\varrho$; $N_\mu = N_\varrho$; $O_\lambda = O_\mu$; $Q_\lambda = Q_\varrho$ hold without the assumption that $(G, \cdot)$ has a unit. Similarly the inclusions $Z \subseteq F_\varrho, J_\mu, J_\varrho, O_\lambda, O_\mu$ are also valid in the general case.

REMARK 3. We have examples which show that the sets $B_\mu, C_\mu, B_\varrho, D_\mu, D_\varrho, I_\varrho, L_\varrho, P_\mu, F_\varrho, F_\lambda$ are in general pairwise different, among them only B_μ is closed under the operation; finally $Z \supset F_\lambda$ is possible. Thus Theorem 2 gives a complete clarification of the relations in which the $\mathfrak{Y}_\tau$-nuclei $(\tau = \lambda, \mu, \varrho)$ stand to each other and to the center; in the relations (I)—(X) each nucleus occurs exactly once.

PROOF. *To* (I). Let a and b be in the sequel arbitrary elements of G. If $x \in Z \cap A_\mu$, then $ax \cdot b = a \cdot xb = a \cdot bx$, and consequently $x \in B_\mu$. If $x \in B_\mu$, then using (*) we get $xa \cdot b = ax \cdot b = a \cdot bx = a \cdot xb$, and thus $x \in C_\lambda$. If $x \in C_\lambda$, then using (*) we get $xa \cdot b = a \cdot xb = a \cdot bx$, and thus $x \in D_\lambda$. Let now be

$x \in D_\lambda$. Then by $(*)$ $x \in Z$ and using this $ax \cdot b = xa \cdot b = a \cdot bx = a \cdot xb$ and so $x \in Z \cap A_\mu$.

Let $x, y \in B_\mu$. Then $(a \cdot xy)b = (ay \cdot x)b = ay \cdot bx = a(bx \cdot y) = a(b \cdot yx) = = a(b \cdot xy)$, accordingly B_μ is closed with respect to the operation; as $B_\mu \subseteq A_\mu$ it is a semigroup and clearly $e \in B_\mu$. Moreover, if φ is an automorphism of $(G, \cdot)$, then

$$[a(x\varphi)]b = \{[(a\varphi^{-1})x](b\varphi^{-1})\}\varphi = \{(a\varphi^{-1})[(b\varphi^{-1})x]\}\varphi = a[b(x\varphi)],$$

which gives that B_μ is characteristic in $(G, \cdot)$.

Let now $(G, \cdot)$ be a loop. In order to establish that B_μ is an abelian group, on account of the preceding paragraph, the equality $B_\mu = Z \cap A_\mu$ and Theorem 1 it suffices to show that if $x \in B_\mu$ and the element x^{-1} satisfies $xx^{-1} = e$, then $x^{-1} \in Z$. Let us multiply the equality $xa = ax$ by x^{-1} from the right. We get $xa \cdot x^{-1} = ax \cdot x^{-1} = a \cdot xx^{-1} = ae = a$, and so

$$a = xa \cdot x^{-1}. \tag{1}$$

Now we multiply (1) by x^{-1} from the left. Then

$$\begin{aligned} x^{-1}a = x^{-1}(xa \cdot x^{-1}) = x^{-1}(ax \cdot x^{-1}) = x^{-1}(a \cdot xx^{-1}) = \\ = x^{-1}(a \cdot x^{-1}x) = x^{-1}(ax^{-1} \cdot x) = x^{-1}(x \cdot ax^{-1}) = \\ = x^{-1}x \cdot ax^{-1} = xx^{-1} \cdot ax^{-1} = e \cdot ax^{-1} = ax^{-1}. \end{aligned}$$

So we have in fact $x^{-1} \in Z$.

To (II). If $x \in Z \cap A_\lambda$ then $ax \cdot b = xa \cdot b = x \cdot ab$, and so $x \in C_\mu$. If $x \in C_\mu$, then making use of $(*)$ we get $ax \cdot b = x \cdot ab = ab \cdot x$, hence $x \in N_\mu$. The equality $ax \cdot b = ab \cdot x$ shows that $N_\mu = N_\varrho$. If $x \in N_\varrho$, then using $(*)$ we get $xa \cdot b = ax \cdot b = ab \cdot x$ and so $x \in P_\lambda$. Let now $x \in P_\lambda$. Then by $(*)$ $x \in Z$, and using this we get $xa \cdot b = ab \cdot x = x \cdot ab$ and consequently $x \in Z \cap A_\lambda$.

Simple computations of a similar kind serve to establish (III), (IV), (V), (VI), (VII), (VIII) and (X). It is easy to see that all elements of the center are contained in all the sets $F_\varrho, J_\mu, J_\varrho, O_\lambda, O_\mu$. Thus, making use of $(*)$, we get also (IX).

To Addendum 1. We first show that

$$A_\lambda \cap A_\mu \cap Z = A_\lambda \cap A_\varrho \cap Z = A_\mu \cap A_\varrho \cap Z = Z^*. \tag{2}$$

Let $x \in A_\lambda \cap A_\mu \cap Z$; then for any elements $a, b (\in G)$

$$ab \cdot x = x \cdot ab = xa \cdot b = ax \cdot b = a \cdot xb = a \cdot bx$$

and consequently $x \in A_\varrho$. Similar considerations show that if $x \in A_\lambda \cap A_\varrho \cap Z$ then $x \in A_\mu$, and if $x \in A_\mu \cap A_\varrho \cap Z$ then $x \in A_\lambda$.

Now, making use of (2) and applying (I) and (III), (I) and (II) respectively we get

(3) $$B_\mu \cap B_\varrho = Z \cap A_\mu \cap A_\varrho = Z^*,$$

(4) $$C_\lambda \cap C_\mu = Z \cap A_\mu \cap A_\lambda = Z^*,$$

so

$$B \subseteq Z^* \quad \text{and} \quad C \subseteq Z^*.$$

Let $x, y \in B$. Then, by (3) and by Theorem 1 $xy \in B_\mu \cap B_\varrho$. Moreover, for all elements $a, b (\in G)$

$$(xy \cdot a)b = (x \cdot ya)b = x(b \cdot ya) = x(by \cdot a) = x(yb \cdot a) = x(y \cdot ba) = xy \cdot ba,$$

so that $xy \in B_\lambda$, and thus $xy \in B$. Similarly, if $x, y \in C$ then by (4) and by Theorem 1 $xy \in C_\lambda \cap C_\mu$ and for any $a, b (\in G)$ we have

$$ab \cdot xy = (ab \cdot x)y = (b \cdot ax)y = b(ax \cdot y) = b(a \cdot xy),$$

so $xy \in C_\varrho$, and consequently $xy \in C$.

Let φ be an automorphism of $(G, \cdot)$, and let x be any element of B. Then by (2) and by Theorem 1 $x\varphi \in B_\mu \cap B_\varrho$. For arbitrary elements $a, b (\in G)$

$$[(x\varphi)a]b = \{[x(a\varphi^{-1})](b\varphi^{-1})\}\varphi = \{x[(b\varphi^{-1})(a\varphi^{-1})]\}\varphi = (x\varphi)(b\,a).$$

Hence $x\varphi \in B_\lambda$, and so $x\varphi \in B$. Similar considerations serve to show that C is characteristic in $(G, \cdot)$.

Let $(G, \cdot)$ be a loop, and let $y_1 \in B, x \in Z^*$; $a, b \in G$ and z_1 be the element of G for which $y_1 z_1 = a$ holds. Then $xa \cdot b = (x \cdot y_1 z_1)b = x(y_1 z_1 \cdot b) = x(y_1 \cdot bz_1) = x(bz_1 \cdot y_1) = x(b \cdot z_1 y_1) = x \cdot ba$. Thus $x \in B_\lambda$, $Z^* \subseteq B_\lambda$. From this there follows the equality $B = B_\lambda \cap Z^* = Z^*$. — Similarly, if $y_2 \in C$, then there exists an element z_2 of G, for which $z_2 y_2 = b$. Then $ab \cdot x = (a \cdot z_2 y_2)x = (az_2 \cdot y_2)x = (z_2 \cdot ay_2)x = (z_2 \cdot y_2 a)x = (z_2 y_2 \cdot a)x = ba \cdot x = b \cdot ax$. Thus $x \in C_\varrho$, $Z^* \subseteq C_\varrho$. Hence $C = C_\varrho \cap Z^* = Z^*$ follows.

To Addendum 2. Let y be an element of Y such that it can always be cancelled on one side. As by (*) $Y \subseteq Z$, this implies that y can always be cancelled on either side. Now if a, b are any two elements of G and $Y = B$, then $ab \cdot y = y \cdot ab = yb \cdot a = by \cdot a = b \cdot ay = b \cdot ya = ba \cdot y$ and so the cancellation of y yields $ab = ba$, proving that $(G, \cdot)$ is abelian. That $ab = ba$ and so that $(G, \cdot)$ is abelian can be similarly deduced from $ab \cdot y = b \cdot ay = b \cdot ya = yb \cdot a = by \cdot a = y \cdot ba = ba \cdot y$ if $Y = C$; from $ab \cdot y = y \cdot ba = ba \cdot y$ if $Y = I$; from $y \cdot ab = y \cdot ba$ if $Y = J$; from $y \cdot ab = a \cdot yb = a \cdot by = b \cdot ay = b \cdot ya = y \cdot ba$ if $Y = K$; from $y \cdot ab = a \cdot by = a \cdot yb = y \cdot ba$ if $Y = L$; from $y \cdot ab = b \cdot ay = b \cdot ya = a \cdot yb = a \cdot by = y \cdot ba$ if $Y = M$; from $ab \cdot y = ay \cdot b = ya \cdot b = yb \cdot a =$

$= by \cdot a = ba \cdot y$ if $Y = N$; from $ab \cdot y = ba \cdot y$ if $Y = O$; from $ab \cdot y = by \cdot a = = yb \cdot a = ba \cdot y$ if $Y = P$; and from $ab \cdot y = yb \cdot a = by \cdot a = ay \cdot b = ya \cdot b = = ba \cdot y$ if $Y = Q$.

To Addendum 3. Let $(G, \cdot)$ be abelian. and let a, b be arbitrary elements in G. Then, owing to (II), (III), (IV) and (V), in order to prove (α) it will be sufficient to show that

$$A = A_\lambda = A_\varrho = B_\lambda = C_\varrho .$$

If $x \in A$, then $x \in A_\lambda$. If $x \in A_\lambda$, then $ab \cdot x = x \cdot ab = x \cdot ba = xb \cdot a = = a \cdot xb = a \cdot bx$, hence $x \in A_\varrho$. If $x \in A_\varrho$, then $xa \cdot b = b \cdot xa = b \cdot ax = ba \cdot x = = x \cdot ba$ and consequently $x \in B_\lambda$. If $x \in B_\lambda$, then $ab \cdot x = x \cdot ab = x \cdot ba = xa \cdot b = = ax \cdot b = b \cdot ax$ and so $x \in C_\varrho$. Let finally $x \in C_\varrho$. Then $xa \cdot b = ax \cdot b = b \cdot ax = = ab \cdot x = x \cdot ab$, i. e. $x \in A_\lambda$; $ax \cdot b = b \cdot ax = ab \cdot x = ba \cdot x = a \cdot bx = a \cdot xb$, i. e. $x \in A_\mu$; $ab \cdot x = ba \cdot x = a \cdot bx$; i. e. $x \in A_\varrho$; consequently $x \in A$.

In order to prove (β), on account of (I), (VII) and (VIII), it will be sufficient to show that

$$A_\mu = K_\varrho = N_\lambda .$$

If $x \in A_\mu$, then $a \cdot bx = bx \cdot a = b \cdot xa = b \cdot ax$, hence $x \in K_\varrho$. If $x \in K_\varrho$, then $xa \cdot b = ax \cdot b = b \cdot ax = a \cdot bx = bx \cdot a = xb \cdot a$, hence $x \in N_\lambda$. If $x \in N_\lambda$, then $ax \cdot b = xa \cdot b = xb \cdot a = a \cdot xb$, i. e. $x \in A_\mu$.

Since ($\mathfrak{F}$), ($\mathfrak{I}$), ($\mathfrak{J}$) and ($\mathcal{O}$) are consequences of the commutativity, (γ) is clear

(δ) is an immediate consequence of (α) and (β).

To Addendum 4. Let $W_\tau = G$. Then $W = G$, hence $e \in W$. Since e can always be cancelled, by Addendum 2 $(G, \cdot)$ is abelian and, by (δ) of Addendum 3, it is a semigroup.

This completes the proof of Theorem 2 and its addenda.

§ 4. Endomorphizer nuclei.

1. Let $(G, \cdot)$ be a groupoid, let $\varphi : x \to x'$ $(x, x' \in G)$ be a single-valued mapping of G into itself, and consider the equation

$$(\mathcal{E}) \qquad x_1' x_2' = (x_1 x_2)' .$$

The set of all elements $x (\in G)$ such that

$$x' a' = (xa)'$$

holds for all $a (\in G)$, will be called the *left endomorphizer nucleus* (for φ) of $(G, \cdot)$ (for short the *$\mathcal{E}_\lambda$-nucleus*) and it will be denoted by $\mathcal{E}_\lambda[G, \varphi]$. Similarly, the *right endomorphizer nucleus* (for φ) of $(G,\cdot)$ (for short the *$\mathscr{E}_\varrho$-nucleus*) is the set of all elements $x(\in G)$ such that

$$a'x'=(ax)'$$

holds for all $a(\in G)$. We denote this set by $\mathscr{E}_\varrho[G,\varphi]$. Finally

$$\mathscr{E}[G,\varphi]\overset{\text{def}}{=}\mathscr{E}_\lambda[G,\varphi]\cap\mathscr{E}_\varrho[G,\varphi]$$

is called the *endomorphizer nucleus* (for φ) of G (for short *$\mathscr{E}$-nucleus*).

Let m be an arbitrary fixed element of G and suppose that for any $x(\in G)$ there exists exactly one $x'(\in G)$ such that $xx'=m$. The mapping $\Phi_m: x\to x'$ $(x\in G)$ is a single-valued mapping of G into itself; we call it the *quotient mapping (for m)*. If, in particular, $\mathscr{E}_\lambda[G,\Phi_m]=G$, then Φ_m is an endomorphism of $(G,\cdot)$, which we call the *quotient endomorphism (for m)*. In this case we can also say that $(G,\cdot)$ admits a quotient endomorphism for m. Moreover, if the mapping Φ_m is one-to-one and $\mathscr{E}_\lambda[G,\Phi_m]=G$, then Φ_m is an automorphism of $(G,\cdot)$, and we say that $(G,\cdot)$ admits a quotient automorphism for m.

Examples. (i) Let $(G,\times)$ be the set of vectors in 3-dimensional euclidean space with vector multiplication as operation, and let φ be the mapping which makes correspond to the elements of G their projections onto a fixed 1-dimensional subspace. It is easy to see that in this case $\mathscr{E}_\lambda[G,\varphi]$ is the set of all vectors of the 1-dimensional subspace considered.

(ii) For a group $(G,\cdot)$ $\mathscr{E}_\lambda[G,\Phi_m]$ coincides with the center Z of $(G,\cdot)$, or else it is empty, according as to whether $m=1$ or not.

Proof. Let $xx'=m$ $(x\in G)$; then $x'=x^{-1}m$, $x'y'=x^{-1}my^{-1}m$, $(xy)'=$ $=(xy)^{-1}m=y^{-1}x^{-1}m$. Thus, in order that $x\in\mathscr{E}_\lambda[G,\Phi_m]$ be fulfilled, it is necessary and sufficient that for any $y(\in G)$

$$y^{-1}x^{-1}m=x^{-1}my^{-1}m$$

or, what is the same

$$y^{-1}x^{-1}=x^{-1}my^{-1}$$

be valid. In the special case when $y=1$, the latter equality yields $x^{-1}=x^{-1}m$ and this implies $m=1$. So there remains the condition $y^{-1}x^{-1}=x^{-1}y^{-1}$, which shows that one needs in fact $x\in Z$. — On the other hand, if $m=1$ and $x\in Z$ then clearly $x\in\mathscr{E}_\lambda[G,\Phi_m]$.

(iii) For an arbitrary field $(K,+,\cdot)$ the quasigroup $(K,\circ)$ defined by

$$x\circ y=a(x-m)+b(y-m)+m \qquad (x,y,a,b,m\in K;\ a,b\neq 0)$$

admits a quotient automorphism for m, as can be shown by a simple computation.

(iv) Consider the groupoid $(\mathfrak{J}, -)$ of the ordinary rational integers under substraction, and let φ be the mapping of $(\mathfrak{J}, -)$ into itself defined as follows:

$$\varphi : x \to x'; \; x' = \begin{Bmatrix} 0 \text{ for } x \text{ even} \\ 1 \text{ for } x \text{ odd} \end{Bmatrix} \qquad (x \in \mathfrak{J}).$$

It is easy to see that $\mathfrak{E}_\lambda[(\mathfrak{J}, -), \varphi)]$ coincides with the set of all odd integers, $\mathfrak{E}_\varrho[(\mathfrak{J}, -), \varphi]$ with the set of all even integers, and so $\mathfrak{E}[(\mathfrak{J}, -), \varphi]$ is empty.

2. Example (iv) shows that $\mathfrak{E}_\lambda[G, \varphi]$ is not necessarily closed with respect to the operation considered. We have however the following theorem:

Theorem 3. *If $(G, \cdot)$ is a semigroup then for any single-valued mapping $\varphi : x \to x'$ $(x, x' \in G)$ each of the sets $\mathfrak{E}_\lambda[G, \varphi]$, $\mathfrak{E}_\varrho[G, \varphi]$ and $\mathfrak{E}[G, \varphi]$, which is non-empty, is a subsemigroup of $(G, \cdot)$. If $(G, \cdot)$ is a group then each of the sets $\mathfrak{E}_\lambda[G, \varphi], \mathfrak{E}_\varrho[G, \varphi], \mathfrak{E}[G, \varphi]$ which is non-empty is a subgroup.*

PROOF. Let $x, y \in \mathfrak{E}_\lambda[G, \varphi]$ and let c be an arbitrary element of G. Then

$$(xy \cdot c)' = (x \cdot yc)' = x'(yc)' = x' \cdot y'c' = x'y' \cdot c' = (xy)'c'$$

shows that $xy \in \mathfrak{E}_\lambda(G, \varphi]$. Similarly, for $x, y \in \mathfrak{E}_\varrho[G, \varphi]$, $c \in G$ we have $(c \cdot xy)' = = c'(xy)'$ and so $xy \in \mathfrak{E}_\varrho[G, \varphi]$.

Let now $(G, \cdot)$ be a group, its unit e, and $x \in \mathfrak{E}_\lambda[G, \varphi]$. Then by the first part of the theorem $x^2 \in \mathfrak{E}_\lambda[G, \varphi]$; further, $x' = (xe)' = x'e'$ implies $e' = e$. If g is any element of G then

$$x'g' = (xg)' = (x^2 \cdot x^{-1}g)' = (x^2)'(x^{-1}g)' = (x')^2(x^{-1}g)'$$

whence $(x')^{-1}g' = (x^{-1}g)'$. The special case $g = e$ yields $(x')^{-1} = (x^{-1})'$, and thus we may write $(x^{-1})'g' = (x^{-1}g)'$ proving that $x^{-1} \in \mathfrak{E}_\lambda[G, \varphi]$. This, together with the first part of the theorem, shows that $\mathfrak{E}_\lambda[G, \varphi]$ is a subgroup; so are, for similar reasons, $\mathfrak{E}_\varrho[G, \varphi]$ and $\mathfrak{E}[G, \varphi]$.

REMARK. Since in the first half of our proof we used only the relations $\mathfrak{E}_\lambda[G, \varphi[\subseteq \mathfrak{A}_\mu(G)$ and $\mathfrak{E}_\varrho[G, \varphi] \subseteq \mathfrak{A}_\mu(G)$ respectively, the following more general statement is also valid:[4])

If for the groupoid $(G, \cdot)$ $\mathfrak{E}_\lambda[G, \varphi] \subseteq \mathfrak{A}_\mu(G)$ $(\mathfrak{E}_\varrho[G, \varphi] \subseteq \mathfrak{A}_\mu(G))$ then $\mathfrak{E}_\lambda[G, \varphi]$ $(\mathfrak{E}_\varrho[G, \varphi])$ is closed under the operation considered. If both of these inclusions hold then $\mathfrak{E}[G, \varphi]$ is also closed.

Theorem 4. *Let $(Q, \cdot)$ be a quasigroup. The intersection $N = A \cap E$ of the $\mathfrak{A}$-nucleus A and the $\mathfrak{E}$-nucleus E (for Φ_m, where $m \in A$) of $(Q, \cdot)$ is a*

---

[4]) By $\mathfrak{A}_\mu(G)$ we denote here the $\mathfrak{A}_\mu$-nucleus of the groupoid $(G, \cdot)$.

group $(N,\cdot)$ *or else it is empty. The intersection* $N^* = E \cap Z^*$ *of* E *with the Bruck center* Z^* *of* $(Q,\cdot)$ *is an abelian group* $(N^*,\cdot)$ *or else it is empty.*

PROOF.[5]) (I) If $x, y \in A$, then for arbitrary elements $a, b (\in Q)$ the following relations hold:

$$\alpha)\quad (xy\cdot a)b = (x\cdot ya)b = x(ya\cdot b) = x(y\cdot ab) = xy\cdot ab\,;$$

$$\beta)\quad (a\cdot xy)b = (ax\cdot y)b = ax\cdot yb = a(x\cdot yb) = a(xy\cdot b)\,;$$

$$\gamma)\quad ab\cdot xy = (ab\cdot x)y = (a\cdot bx)y = a(bx\cdot y) = a(b\cdot xy),$$

and consequently $x, y \in A \Longrightarrow xy \in A$.

(II_1) Let $x, xy \in A$ and $a, b \in Q$. Then

$$\alpha)\quad x(ya\cdot b) = (x\cdot ya)b = (xy\cdot a)b = xy\cdot ab = x(y\cdot ab),$$

and cancellation of x on the left yields

$$ya\cdot b = y\cdot ab\,;$$

β) since $(Q,\cdot)$ is a left quasigroup, there exists a $v(\in Q)$ such that $vx = a$, and thus

$$ay\cdot b = (vx\cdot y)b = (v\cdot xy)b = v(xy\cdot b) = v(x\cdot yb) = vx\cdot yb = a\cdot yb\,;$$

γ) since $(Q,\cdot)$ is a left quasigroup, there exists a $w(\in Q)$ such that $wx = b$, and thus

$$ab\cdot y = (a\cdot wx)y = (aw\cdot x)y = aw\cdot xy = a(w\cdot xy) = a(wx\cdot y) = a\cdot by,$$

hence $x, xy \in A \Longrightarrow y \in A$.

(II_2) $y, xy \in A \Longrightarrow x \in A$. The proof of this is similar to that of (II_1).

(III) Let $x, y \in N$ and let $\Phi_m : q \to q'$ $(q, q' \in Q;\ m \in A)$. First of all we remark that in view of $qq' = m \in A$ and of (II_1) $q' \in A$ holds for $q \in A$. Thus in particular $x', y' \in A$. Now, if a is an arbitrary element of Q, then

$$\alpha)\quad (xy)'a' = x'y'\cdot a' = x'\cdot y'a' = x'(ya)' = (x\cdot ya)' = (xy\cdot a)'\,;$$

$$\beta)\quad a'(xy)' = a'\cdot x'y' = a'x'\cdot y' = (ax)'y' = (ax\cdot y)' = (a\cdot xy)'\,;$$

and hence $x, y \in N \Longrightarrow xy \in E$.

(IV_1) Let $x, xy \in N$ and $a \in Q$. Then by (II_1) one has $y \in A$. We have still to show that $y \in E$.

---

5) A reference to Theorem 1 would slightly shorten the proof, but for the sake of completeness we do not make use of this.

α) By the conditions imposed on x and on xy, and in view of the remark made in (III) concerning the element q, we get

$$x'\cdot y'a' = x'y'\cdot a' = (xy)'a' = (xy\cdot a)' = (x\cdot ya)' = x'(ya)',$$

and hence, by cancellation of x',

$$(ya)' = y'a'.$$

β) Let $t(\in Q)$ be an element, such that $tx = a$. Then $t'x' = a'$. Taking this into account, and making use again of the remark made in (III), we get

$$a'y' = t'x'\cdot y' = t'\cdot x'y' = t'(xy)' = (t\cdot xy)' = (tx\cdot y)' = (ay)'.$$

Hence $x, xy \in N \Longrightarrow y \in N$.

(IV_2) $y, xy \in N \Longrightarrow x \in N$. The proof is similar to that of (IV_1).

On the basis of (I) and of (III) N is closed under the operation considered; by (II_1), (II_2), (IV_1) and (IV_2) $(N,\cdot)$ has a bilateral division; as $N \subseteq A$ the associative law holds in $(N,\cdot)$; so $(N,\cdot)$ is a group.

(V) Let us denote by Z the center of $(Q,\cdot)$. Then by definition we have $Z^* = Z \cap A$. Let $x, y \in Z^*$ and $a \in Q$. Then $xy\cdot a = x\cdot ya = x\cdot ay = xa\cdot y = ax\cdot y = a\cdot xy$, i. e. $xy \in Z$; since at the same time we have by (I) $xy \in A$, the relation $xy \in Z^*$ follows.

Let now $x, xy \in Z^*$ and $a \in Q$. Then $x\cdot ay = xa\cdot y = ax\cdot y = a\cdot xy = xy\cdot a = x\cdot ya$ and cancelling x on the left we get $ay = ya$ i. e. $y \in Z$. On the other hand by (II_1) we get $y \in A$ and so $y \in Z^*$. A similar calculation can serves to establish $y, xy \in Z^* \Longrightarrow x \in Z^*$.

We have shown that $(Z^*,\cdot)$ is an abelian group. Since by what has previously been said $(A \cap E,\cdot)$ is a group and $N^* = Z^* \cap E = Z^* \cap (A \cap E)$, $(N^*,\cdot)$ is also an abelian group. — This completes the proof of Theorem 4.

From the proof of this theorem we can draw several consequences, e. g. the following:

The $\mathfrak{A}_\lambda$-nucleus of a right quasigroup is an associative right quasigroup, or else it is empty. (By (I α) and ($\mathrm{II}_1\,\alpha$).)

The $\mathfrak{A}$-nucleus of a quasigroup is a group or else it is empty. (By (I), (II_1) and (II_2).)

The intersection of the $\mathfrak{A}_\lambda$-nucleus A_λ with the $\mathfrak{E}_\lambda$-nucleus (for Φ_m, where $m \in A_\lambda$) of a right quasigroup is an associative right quasigroup, or else it is empty. (By (I α), ($\mathrm{II}_1\,\alpha$), (III α) and ($\mathrm{IV}_1\,\alpha$).)

The intersection of the $\mathfrak{A}$-nucleus A with the $\mathfrak{E}_\lambda$-nucleus (for Φ_m, where $m \in A$) of a right-quasigroup is an associative right quasigroup or else it is empty. (By (I), (II_1), (III α) and ($\mathrm{IV}_1\,\alpha$).)

Theorem 5. *If $(Q, \cdot)$ is a right quasigroup and ζ is a permutation of Q which is permutable with the mapping $\Phi_m : x \to x'$ (where $m(\in Q)$ is a fixed element) then the isotope of Q by the isotopism $\{\xi = \eta = \iota, \zeta\}$ is a right quasigroup $(Q, \circ)$, and $\mathfrak{E}_\lambda[(Q, \cdot), \Phi_m] = \mathfrak{E}_\lambda[(Q, \circ), \Phi_{m\zeta}]$.*

PROOF. Since

$$ax = b \Longleftrightarrow a \circ x = b\zeta \qquad (a, x, b \in Q) \tag{5}$$

and ζ is a permutation of Q, $(Q, \circ)$ is clearly a right quasigroup. Moreover from (5) it follows in particular that

$$xx' = m \Longleftrightarrow x \circ x' = m\zeta \qquad (x, x', m \in Q),$$

and this means that Φ_m of $(Q, \cdot)$ coincides with $\Phi_{m\zeta}$ of $(Q, \circ)$.

Suppose now that $x \in \mathfrak{E}_\lambda[(Q, \cdot), \Phi_m]$. Then for any $a(\in Q)$ we have

$$x' \circ a' = (x'a')\zeta = (xa)'\zeta = [(xa)\zeta]' = (x \circ a)',$$

so that $x \in \mathfrak{E}_\lambda[(Q, \circ), \Phi_{m\zeta}]$. Conversely, let $x \in \mathfrak{E}_\lambda[(Q, \circ), \Phi_{m\zeta}]$. Then for any element $a(\in Q)$ we have

$$(x'a')\zeta = x' \circ a' = (x \circ a)' = [(xa)\zeta]' = (xa)'\zeta,$$

and since ζ is a permutation of the set Q, $x'a' = (xa)'$. Thus $x \in \mathfrak{E}_\lambda[(Q, \cdot), \Phi_m]$

If $(G, \cdot)$ is a group with the unit element e, then any permutation ζ of G, which is permutable with Φ_e, yields a quasigroup $(G, \circ)$ and $\mathfrak{E}_\lambda[(G, \circ), \Phi_{e\zeta}] = Z$, where Z is the center of the group $(G, \cdot)$. Since

$$xa = ax = b \Longleftrightarrow x \circ a = a \circ x = b\zeta \qquad (x, a, b \in G)$$

the center of $(G, \circ)$ is also Z. — $(G, \circ)$ is in general not a group, not even a loop. For example, let G be a cyclic group of order 4, generated by a and define ζ as $e\zeta = a^2$, $a\zeta = a$, $a^2\zeta = e$, $a^3\zeta = a^3$. This ζ commutes with Φ_e but as the following multiplication-table shows, $(G, \circ)$ has no unit element.

$\circ$	e	a	a^2	a^3
e	a^2	a	e	a^3
a	a	e	a^3	a^2
a^2	e	a^3	a^2	a
a^3	a^3	a^2	a	e

If $(Q, \cdot)$ is a right quasigroup, then Φ_m exists for any $m(\in Q)$. It is not hard to give a necessary and sufficient condition for $(Q, \cdot)$ to admit a quotient endomorphism for any $m(\in Q)$:

A right quasigroup $(Q, \cdot)$ *admits a quotient endomorphism for any constant* $m(\in Q)$ *if and only if*

$$ab = ac \cdot bd \qquad (a, b, c, d \in Q) \tag{6}$$

holds whenever $ab = cd$.[6])

Let us first suppose that for any $m(\in Q)$ $\Phi_m : x \to x'$ is an endomorphism, and let $ab = cd = m$. Then with $b = a'$, $d = c'$ we have $ab = m = ac(ac)' = ac \cdot a'c' = ac \cdot bd$. On the other hand, if (6) is valid whenever $ab = cd$, then for any mapping $\Phi_m : x \to x'$ the relation

$$m = xy(xy)' = xx' = xy \cdot x'y'$$

holds, and by the left cancellation law

$$(xy)' = x'y'.$$

3. Now we are going to investigate the behaviour of $\mathfrak{E}$-nuclei in the case of certain groupoid constructions.

Let $P = (H, \cdot)$, $Q = (H, \circ)$ and $R = (H, \times)$ be three groupoids defined on the same set H. Similarly as in [6] (p. 232, N° 4), we define on the set H a groupoid $S = (H, *)$ with the operation

$$x * y = (x \cdot y) \circ (x \times y) \qquad (x, y \in H).$$

We say that S is *the product with respect to* Q of the groupoids P and R, and we write $S = P \circ R$.

Theorem 6. *If* $P = (H, \cdot)$, $R = (H, \times)$ *are two arbitrary groupoids on the same set* H, $Q = (H, \circ)$ *a groupoid on* H *which satisfies both cancellation laws and* $S \overset{\text{def}}{=} P \circ R \overset{\text{def}}{=} (H, *)$, *then for any endomorphism* $\varphi : x \to x'$ *of* Q *the equality* $M = N$ *holds, where* $M \overset{\text{def}}{=} (\mathfrak{E}_\lambda[P, \varphi] \cup \mathfrak{E}_\lambda[R, \varphi]) \cap \mathfrak{E}_\lambda[S, \varphi]$ *and* $N \overset{\text{def}}{=} \mathfrak{E}_\lambda[P, \varphi] \cap \mathfrak{E}_\lambda[R, \varphi]$. — *If, in particular, the mapping* φ *is such that there exist elements* $m, n(\in H)$ *for which the relations* $a \cdot a' = m$, $a \times a' = n$ *are satisfied with any* $a(\in H)$, *then*

$$(\mathfrak{E}_\lambda[P, \Phi_m] \cup \mathfrak{E}_\lambda[R, \Phi_n]) \cap \mathfrak{E}_\lambda[S, \Phi_{m \circ n}] = \mathfrak{E}_\lambda[P, \Phi_m] \cap \mathfrak{E}_\lambda[R, \Phi_n].$$

PROOF. Let us first suppose that $x \in M$. Then the element x belongs to at least one of the sets $\mathfrak{E}_\lambda[P, \varphi]$, $\mathfrak{E}_\lambda[R, \varphi]$. In view of the prevailing symmetry we may suppose that $x \in \mathfrak{E}_\lambda[P, \varphi]$. Then for any $a(\in H)$ we hawe

$$x' * a' = (x' \cdot a') \circ (x' \times a') = (x \cdot a)' \circ (x' \times a'),$$

---

[6]) If all elements x of $(Q, \cdot)$ are idempotent, i. e. if for any x the equality $x^2 = x$ holds, then in case $ab = cd$ (6) is exactly equivalent to the *equation of bisymmetry*

$$ab \cdot cd = ac \cdot bd$$

(sec e. g. [1] p. 180).

and on the other hand

$$(x*a)' = ((x\cdot a)\circ(x\times a))' = (x\cdot a)'\circ(x\times a)'.$$

Since by hypothesis $x'*a' = (x*a)'$, the relation

$$(x\cdot a)'\circ(x'\times a') = (x\cdot a)'\circ(x\times a)'$$

follows, from which we obtain by left cancellation

$$x'\times a' = (x\times a)'.$$

Thus $x\in\mathscr{E}_\lambda[R,\varphi]$, and so $M\subseteq N$.

Conversely, let us now suppose that $x\in N$. Then for any $a(\in H)$ we have

$$x'*a' = (x'\cdot a')\circ(x'\times a') = (x\cdot a)'\circ(x\times a)' = ((x\cdot a)\circ(x\times a))' = (x*a)',$$

and consequently $x\in\mathscr{E}_\lambda[S,\varphi]$. On the other hand $x\in\mathscr{E}_\lambda[P,\varphi]\cup\mathscr{E}_\lambda[R,\varphi]$, and thus $N\subseteq M$.

In order to establish the second assertion of the theorem, we suppose that for any $a(\in H)$ the equalities $aa' = m$, $a\times a' = n$ hold. Then

$$a*a' = (a\cdot a')\circ(a\times a') = m\circ n,$$

so φ coincides with Φ_m on P, with Φ_n on R, and with $\Phi_{m\circ n}$ on S. From this our assertion follows by the equality $M = N$.

Let $Q = (H,\circ)$ and $R = (H,\times)$ be two groupoids on the same set H. We define on the set H a new groupoid $S = (H,*)$ with the operation

$$x*y = x\circ(x\times y) \qquad (x,y\in H),$$

and we denote it by $S = H\circ R$. The groupoid $R\circ H$ is defined in an analogous way. The product $H\circ R$ is a special case of $P\circ R$. Indeed, let P be the groupoid $(H,\cdot)$ in which $x\cdot y = x$ for any $x,y(\in H)$.

We remark that *if Q and R are two right quasigroups, then $S = H\circ R$ too is a right quasigroup.* Indeed, when

$$a*x = a\circ(a\times x) = b \qquad (a,b\in H), \tag{7}$$

there exists exactly one y for which $a\circ y = b$ and exactly one x for which $a\times x = y$. This x clearly satisfies (7), and is the only element having this property.

Theorem 7. *If $Q = (H,\circ)$ is a groupoid which satisfies the left cancellation law, $R = (H,\times)$ is an arbitrary groupoid on the same set and $S \overset{\text{def}}{=} H\circ R \overset{\text{def}}{=} (H,*)$, then for any endomorphism $\varphi: x\to x'$ of Q the relations*

$$\mathscr{E}_\lambda[S,\varphi] = \mathscr{E}_\lambda[R,\varphi]$$

and

$$\mathfrak{E}_\varrho[S, \varphi] = \mathfrak{E}_\varrho[R, \varphi]$$

hold.

PROOF. Suppose first $x \in \mathfrak{E}_\lambda[S, \varphi]$ and let a be an arbitrary element of R. Then

$$x' \circ (x \times a)' = (x \circ (x \times a))' = (x * a)' = x' * a' = x' \circ (x' \times a').$$

By the left cancellation law

$$(x \times a)' = x' \times a'$$

and consequently $x \in \mathfrak{E}_\lambda[R, \varphi]$. — Conversely, if $x \in \mathfrak{E}_\lambda[R, \varphi]$ and a is an arbitrary element, then

$$(x * a)' = (x \circ (x \times a))' = x' \circ (x \times a)' = x' \circ (x' \times a') = x' * a',$$

and thus $x \in \mathfrak{E}_\lambda[S, \varphi]$.

Similar considerations serve to establish the validity of the relation

$$\mathfrak{E}_\varrho[S, \varphi] = \mathfrak{E}_\varrho[R, \varphi],$$

and so we see that the relation

$$\mathfrak{E}[S, \varphi] = \mathfrak{E}[R, \varphi]$$

also holds.

By the direct product of the groupoids $P = (H, \cdot)$ and $R = (K, \times)$ we mean the groupoid $S = (D, *)$ defined on the cartesian product D of the sets H and K by the operation

$$(a, b) * (c, d) = (a \cdot c, b \times d) \qquad (a, c \in H;\ b, d \in K).$$

Let φ, ψ be single-valued mappings of H, K respectively into themselves. Then $\vartheta : (x, y) \rightarrow (x\varphi, y\psi)$ is a single-valued mapping of D into itself. This mapping will be called the *direct product of the mappings φ and ψ.*

Theorem 8. *Let $S = (D, *)$ be the direct product of the groupoids $P = (H, \cdot)$ and $R = (K, \times)$, let φ, ψ be single-valued mappings of H, K respectively into themselves, and let ϑ be the direct product of the mappings φ and ψ. Then $\mathfrak{E}_\lambda[S, \vartheta]$ coincides with the cartesian product of the sets $\mathfrak{E}_\lambda[P, \varphi]$ and $\mathfrak{E}_\lambda[R, \psi]$. Similar statements are valid for the $\mathfrak{E}_\varrho$- and $\mathfrak{E}$-nuclei.*

Corollary 1. *ϑ is an endomorphism of S if and only if φ is an endomorphism of P and ψ an endomorphism of R.*

Corollary 2. *If $\varphi = \Phi_m$ and $\psi = \Phi_n$ then $\vartheta = \Phi_{(m, n)}$. — S admits a quotient endomorphism for (m, n) if and only if P admits one for m and R one for n.*

The proof clearly follows from the equalities

$$[(x, y)*(a, b)]\vartheta = ((x \cdot a)\varphi, (y \times b)\psi);$$
$$(x, y)\vartheta * (a, b)\vartheta = (x\varphi \cdot a\varphi, y\psi \times b\psi),$$

where the left hand sides are equal if and only if the right hand sides are equal.

§ 5. Semi-symmetrizer nuclei.

In [7] (see N° 18. 7) a semi-symmetric groupoid is defined as a groupoid $(G, \cdot)$, which satisfies

$$xy \cdot x = y$$

for any elements $x, y (\in G)$. Following this definition, we shall call a mapping $x \to \bar{x}$ of a groupoid $(G, \cdot)$ into itself *semi-symmetric*, if

$$xy \cdot \bar{x} = y$$

for any $x, y \in G$. Clearly, the groupoid is semi-symmetric if and only if the identity mapping on it is semi-symmetric.

Let $\varphi: x \to x'$ be a single-valued mapping of the groupoid $(G, \cdot)$ into itself. The set of all elements $x (\in G)$ for which

$$xa \cdot x' = a$$

holds with any a in G, will be called the semi-symmetrizer nucleus (for φ) of $(G, \cdot)$, for short the $\mathbb{S}$-nucleus, and it will be denoted by $\mathbb{S}[G, \varphi]$.

EXAMPLES. (i) The mapping $\varphi: x \to x^{-1}$ of the abelian group $(G, \cdot)$ is evidently semi-symmetric and thus $\mathbb{S}[G, \varphi] = G$.

(ii) Let us consider the quasigroup $(K, \circ)$, over an arbitrary field $(K, +, \cdot)$, defined by

$$x \circ y = ax + by + c \qquad (a, b, c \in K;\ a \neq 0 \neq b),$$

and let $\Phi_m: x \to x'$ $(m \in K)$ be a quotient mapping of $(K, \circ)$. Then by virtue of $x' = \dfrac{m - ax - c}{b}$ we have

$$(x \circ y) \circ x' = a(a-1)x + aby + m + ac,$$

which shows that $\mathbb{S}[(K, \circ), \Phi_m]$ is non-empty if and only if either $a = b = 1$, $c = -m$ or $ab = 1$, $a \neq 1$. In the first case the $\mathbb{S}$-nucleus of $(K, \circ)$ coincides with K and in the second case it consists of the single element $x = \dfrac{m + ac}{a(1-a)}$.

(iii) Let $(\mathfrak{J}, \cdot)$ be the multiplicative semigroup of the rational integers. One easily sees that in this case $\mathfrak{S}[(\mathfrak{J}, \cdot), \iota]$ coincides with the multiplicative group of the numbers $+1$ and -1.

Theorem 9. *Let $(G, \cdot)$ be an arbitrary groupoid. Then*

(α) *if for a mapping $\varphi: x \to x'$ of $(G, \cdot)$ into itself the inclusion $\mathfrak{A}_\mu(G) \subseteq \mathfrak{S}[G, \varphi]$ holds, then $\mathfrak{A}_\mu(G)$ is an associative right quasigroup, or else it is empty;*[4])

(β) *if $(G, \cdot)$ has a semi-symmetric permutation $\varphi: x \to x'$ then $(G, \cdot)$ is a quasigroup;*

(γ) *if Φ_m is a semi-symmetric permutation of $(G, \cdot)$ for some $m (\in G)$ then $(G, \cdot)$ is a loop;*[7])

(δ) *if $(G, \cdot)$ is semi-symmetric, then $(G, \cdot)$ is a quasigroup;*

(ε) *if $D \overset{\text{def}}{=} \mathfrak{A}_\mu(G) \cap \mathfrak{S}[G, \iota]$ is non-empty, then D is an (abelian) group, which is a direct product of groups of order* 2.[8])

Proof. (α) By (I, β) in the proof of Theorem 4 $\mathfrak{A}_\mu(G)$ is closed with respect to the operation in G. If there exists an element $x(\in \mathfrak{A}_\mu(G))$ such that

$$ax = b, \tag{8}$$

(a, b arbitrary elements in $\mathfrak{A}_\mu(G)$) then multiplying this equation on the right by a' we get $ax \cdot a' = ba'$, and since a is an element of the $\mathfrak{S}$-nucleus, we must have $x = ba'$. Conversely, $x = ba'$ is a solution of the equation (8), for in view of the fact that b belongs to the $\mathfrak{A}_\mu$-nucleus, one obtains $a \cdot ba' = ab \cdot a' = b$.

(β) First of all we show that the mapping φ is an endomorphism. Indeed, for arbitrary elements $x, y (\in G)$ one has

$$(xy \cdot x')(xy)' = x',$$

and so

$$y(xy)' = x'.$$

Multiplying this equation by y' on the right, we get

$$[y(xy)']y' = x'y',$$

and so

$$(xy)' = x'y'.$$

In view of

$$x \cdot a'x' = (ax \cdot a')a'x' = (ax \cdot a')(ax)' = a' = xa' \cdot x',$$

---

[7]) This is a loop with the special property of Artzy [2].

[8]) We include the case of a group having only one element.

we have

(9) $$x\cdot a'x' = xa'\cdot x'$$

for any $x, a \in G$.

Consider the equation

(10) $$ax = b \qquad (a, b \in G).$$

If x is a solution of this equation then necessarily $x = ba'$ holds. On the other hand, since φ is a permutation, there exists an element $b^*(\in G)$ such that $(b^*)' = b$. Thus by the substitution $x = ba'$ and using (9) we get

$$ax = a\cdot ba' = a[(b^*)'a'] = [a(b^*)']a' = ab\cdot a' = b,$$

and this shows that $x = ba'$ is a solution of the equation (10).

Consider now the equation

(11) $$ya = b \qquad (a, b \in G),$$

and let a^*, y^* be elements of G, such that $(a^*)' = a$, $(y^*)' = y$. Then, by (9) equation (11) yields

$$y = (y^*)' = [a^*(y^*)'](a^*)' = a^*[(y^*)'(a^*)'] = a^*\cdot ya = a^*b.$$

On the other hand, let $y = a^*b$. Then

$$ya = a^*b\cdot a = (a^*b)(a^*)' = b,$$

and so we have proved solvability and the uniqueness of the solution also for equation (11).

(γ) We show that m is the unit element of $(G, \cdot)$. Denote by x^* the element for which $(x^*)' = x$. Since for any $x(\in G)$ the relation $xx' = m$ holds, using (9) we get

$$xm = x\cdot xx' = x\cdot(x^*)'x' = x(x^*)'\cdot x' = (x^*)' = x$$

and

$$mx = x^*x\cdot x = x.$$

(δ) This is the special case of (β) in which φ is the identity mapping.

(ε) First we show that D is closed with respect to the operation. Let $x, y \in D$. Then for any $a \in G$

$$(xy\cdot a)(xy) = (x\cdot ya)(xy) = [(x\cdot ya)x]y = ya\cdot y = a$$

holds and consequently $xy \in \mathfrak{S}[G, \iota]$. On the other hand, by (I. β) in the proof of Theorem 4, $xy \in \mathfrak{A}_\mu(G)$ and so $xy \in D$. — Since the operation is associative in D and $(D, \cdot)$ is a semi-symmetrical groupoid, by (δ) $(D, \cdot)$ is a group.

Finally for any elements $x(\in D)$ and $a(\in G)$ one has $xa \cdot x = a$, and thus in particular for $a = 1$ ($=$ the unit element of $(D, \cdot)$) $x^2 = 1$. This shows that the order of any element of D different from 1 is equal to 2, so $(D, \cdot)$ is abelian and a direct product of groups of order 2.

As an immediate corollary to (ε), we get the following:

A semi-symmetric semigroup is a group, which is a direct product of groups of order 2.[8])

Bibliography.

[1] J. Aczél, Einige aus Funktionalgleichungen zweier veränderlichen ableitbare Differentialgleichungen, *Acta Sci. Math. Szeged* **13** (1950), 179—189.

[2] R. Artzy, On loops with a special property, *Proc. Amer. Math. Soc.* **6** (1955), 448—453.

[3] R. H. Bruck, A survey of binary systems, *Berlin—Göttingen—Heidelberg*, 1958.

[4] R. H. Bruck, Contributions to the theory of loops, *Trans. Amer. Math. Soc.* **60** (1946), 245—354.

[5] T. Faragó, Contribution to the definition of group, *Publ. Math. Debrecen* **3** (1953), 133—137.

[6] A. Sade, Groupoides orthogonaux, *Publ. Math. Debrecen* **5** (1958), 229—240.

[7] A. Sade, Quasigroupes parastrophiques. Expressions et identités, *Math. Nachr.* **20** (1959), 73—106.

(Received July 17, 1958; in revised form March 18, 1959.)