Some finite summation formulas of arithmetic character.

To Professor O. Varga on his 50th birthday.

By L. CARLITZ (Durham).

1. Put

$$\frac{te^{xt}}{e^t-1}=\sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!},\ B_n=B_n(0), \tag{1.1}$$

so that $B_n(x)$ is the Bernoulli polynomial of degree n in the notation of NÖRLUND [6]. Also let $\bar{B}_n(x)$ denote the corresponding Bernoulli function defined by

$$\bar{B}_n(x)=B_n(x)\ (0\leqq x<1),\quad \bar{B}_n(x+1)=\bar{B}_n(x).$$

MIKOLÁS [4] has proved the elegant formula

$$\int_0^1 \bar{B}_r(ax)\bar{B}_r(bx)dx=(-1)^{r-1}\frac{r!\,r!}{(2r)!}\left(\frac{(a,b)}{[a,b]}\right)^r B_{2r}, \tag{1.2}$$

where (a,b), $[a,b]$ denote, respectively, the greatest common divisor and the least common multiple of the integers a, b. More generally he has proved that the HURWITZ zeta-function defined for $R(s)>1$ by

$$\zeta(s,x)=\sum_{n=0}^{\infty}\frac{1}{(x+n)^s}$$

satisfies, for $R(s)>\frac{1}{2}$,

$$\int_0^1 \zeta(1-s,\{ax\})\zeta(1-s,\{bx\})\,dx=\frac{2\Gamma^2(s)\zeta(2s)}{(2\pi)^{2s}}\left(\frac{(a,b)}{[a,b]}\right)^s, \tag{1.3}$$

where $\{x\}=x-[x]$ denotes the fractional part of the real number x.

Quite recently MORDELL [5] has proved that if $f_1(x), f_2(x), \ldots, f_n(x)$ are functions of x such that for any positive integer k there exists the relation

$$\sum_{r=0}^{k-1} f_i\left(x+\frac{r}{k}\right) = C_i^{(k)} f_i(kx) \qquad (i=1,\ldots,n), \tag{1.4}$$

where $C_i^{(k)}$ is independent of x, and $a_1, \ldots, a_n$ are positive integers that are relatively prime in pairs, then if the integrals exist

$$\begin{aligned}\int_0^A f_1\left(\left\{\frac{x}{a_1}\right\}\right)\cdots f_n\left(\left\{\frac{x}{a_n}\right\}\right)dx &= A\int_0^1 f_1\left(\left\{\frac{Ax}{a_1}\right\}\right)\cdots f_n\left(\left\{\frac{Ax}{a_n}\right\}\right)dx = \\ &= C_1^{(a_1)}\cdots C_n^{(a_n)}\int_0^1 f_1(x)\cdots f_n(x)dx,\end{aligned} \tag{1.5}$$

where $A = a_1 a_2 \ldots a_n$.

It is noted that both $B_r(x)$ and $\zeta(s,x)$ sa isfy relations of the form (1.4). Specializing (1.5), MORDELL obtains

$$\int_0^1 \overline{B}_p(ax)\overline{B}_q(bx)dx = (-1)^{p-1}\frac{p!\,q!}{(p+q)!}\frac{(a,b)^{p+q}}{a^q b^p}B_{p+q} \qquad (p+q \geqq 2). \tag{1.6}$$

For $p=q=r$, (1.6) evidently reduces to (1.2).

2. In the present note we consider in place of the integrals occurring in (1.2), (1.3), (1.5), (1.6) certain finite sums. We first prove the following

Theorem 1. *Let $n \geqq 1$ and let $a_1, \ldots, a_n$ be positive integers that are relatively prime in pairs and put $A = a_1 a_2 \ldots a_n$. Let $f_1(x), \ldots, f_n(x)$ be functions of x of period 1 that satisfy (1.4). Then if k is an arbitrary positive integer, we have*

$$\begin{aligned}&\sum_{r=0}^{kA-1} f_1\left(\frac{r}{a_1 k}\right) f_2\left(\frac{r}{a_2 k}\right)\cdots f_n\left(\frac{r}{a_n k}\right) = \\ &= C_1^{(a_1)} C_2^{(a_2)}\cdots C_n^{(a_n)} \sum_{r=0}^{k-1} f_1\left(\frac{r}{k}\right) f_2\left(\frac{r}{k}\right)\cdots f_n\left(\frac{r}{k}\right).\end{aligned} \tag{2.1}$$

The proof is very similar to the proof of MORDELL's theorem. Put

$$A_s = a_1 a_2 \ldots a_s \quad (1 \leqq s \leqq n), \quad A = A_n. \tag{2.2}$$

Then if S denotes the left member of (2.1) we get, on replacing r by $mkA_{n-1}+r$ and noting that $(A_{n-1}, a_n) = 1$, so that mA_{n-1} runs through a complete residue system (mod a_n),

$$S=\sum_{r=0}^{kA_{n-1}-1} f_1\left(\frac{r}{a_1k}\right)\cdots f_{n-1}\left(\frac{r}{a_{n-1}k}\right)\sum_{m=0}^{a_n-1} f_n\left(\frac{mA_{n-1}}{a_n}+\frac{r}{a_nk}\right)=$$

$$=\sum_{r=0}^{kA_{n-1}-1} f_1\left(\frac{r}{a_1k}\right)\cdots f_{n-1}\left(\frac{r}{a_{n-1}k}\right)\sum_{m=0}^{a_n-1} f_n\left(\frac{m}{a_n}+\frac{r}{a_nk}\right)=$$

$$=C_n^{(a_n)}\sum_{r=0}^{kA_{n-1}-1} f_1\left(\frac{r}{a_1k}\right)\cdots f_{n-1}\left(\frac{r}{a_{n-1}k}\right)f_n\left(\frac{r}{k}\right)= \qquad \text{(by (1.4))}$$

$$=C_{n-1}^{(a_{n-1})}C_n^{(a_n)}\sum_{r=0}^{kA_{n-2}-1} f_1\left(\frac{r}{a_1k}\right)\cdots f_{n-2}\left(\frac{r}{a_{n-2}k}\right)f_{n-1}\left(\frac{r}{k}\right)f_n\left(\frac{r}{k}\right)=$$

$$=C_2^{(a_2)}\cdots C_n^{(a_n)}\sum_{r=0}^{kA_1-1} f_1\left(\frac{r}{a_1k}\right)f_2\left(\frac{r}{k}\right)\cdots f_n\left(\frac{r}{k}\right).$$

For the final step replace r by $mk+r$, where $0\leqq m<a_1$, $0\leqq r<k$. Then

$$S=C_2^{(a_2)}\cdots C_n^{(a_n)}\sum_{r=0}^{k-1}\sum_{m=0}^{a_1-1} f_1\left(\frac{m}{a_1}+\frac{r}{a_1k}\right)f_2\left(\frac{r}{k}\right)\cdots f_n\left(\frac{r}{k}\right)=$$

$$=C_1^{(a_1)}C_2^{(a_2)}\cdots C_n^{(a_n)}\sum_{r=0}^{k-1} f_1\left(\frac{r}{k}\right)f_2\left(\frac{r}{k}\right)\cdots f_n\left(\frac{r}{k}\right),$$

where we have again used (1.4) in the inner summation. This proves the theorem.

We remark that for $n=1$, a_1 is an arbitrary positive integer.

If we divide both sides of (2.1) by k and let $k\to\infty$, then provided the limits exist it is evident that we get (1.5). We may think of (2.1) as a finite analog of (1.5), analogous to GOOD's formula [3]

$$P_n(x)=\frac{1}{k}\sum_{r=0}^{k-1}\left(x+\sqrt{x^2-1}\cos\frac{2\pi r}{k}\right)^n \qquad (k>n)$$

for the Legendre polynomial. For other formulas of this kind see [1], [2].

We remark that the same argument leads to the following generalization of (2.1).

$$\text{(2.3)}\qquad \sum_{r=0}^{kA-1} f_1\left(x_1+\frac{r}{a_1k}\right)f_2\left(x_2+\frac{r}{a_2k}\right)\cdots f_n\left(x_n+\frac{r}{a_nk}\right)=$$

$$=C_1^{(a_1)}C_2^{(a_2)}\cdots C_n^{(a_n)}\sum_{r=0}^{k-1} f_1\left(a_1x_1+\frac{r}{k}\right)f_2\left(a_2x_2+\frac{r}{k}\right)\cdots f_n\left(a_nx_n+\frac{r}{k}\right),$$

where the x_i are arbitrary variables. For $k=1$, (2. 3) reduces to

$$\sum_{r=0}^{A-1} f_1\left(x_1+\frac{r}{a_1}\right) f_2\left(x_2+\frac{r}{a_2}\right)\cdots f_n\left(x_n+\frac{r}{a_n}\right)= \tag{2.4}$$
$$= C_1^{(a_1)} C_2^{(a_2)} \cdots C_n^{(a_n)} f_1(a_1x_1) f_2(a_2x_2)\cdots f_n(a_nx_n),$$

which may be compared with (1. 4).

3. We now suppose $n=2$ and consider first the case $f(x)=\bar{B}_m(x)$. Changing the notation slightly, (2. 1) becomes for $(a, b)=1$

$$\sum_{r=1}^{kab-1} \bar{B}_m\left(\frac{r}{ak}\right)\bar{B}_n\left(\frac{r}{bk}\right)=a^{1-m}b^{1-n}\sum_{r=0}^{k-1} B_m\left(\frac{r}{k}\right)B_n\left(\frac{r}{k}\right). \tag{3.1}$$

It is convenient to consider the slightly more general sum

$$S_{m,n}=S_{m,n}(x)=\sum_{r=0}^{k-1} B_m\left(x+\frac{r}{k}\right)B_n\left(x+\frac{r}{k}\right). \tag{3.2}$$

Making use of (1. 1) we get (as in MORDELL's proof of (1. 6))

$$\sum_{m,n=0}^{\infty} S_{m,n}\frac{u^m v^n}{m!\,n!}=\frac{uve^{x(u+v)}}{(e^u-1)(e^v-1)}\sum_{r=0}^{k-1} e^{r(u+v)/k}=\frac{uve^{x(u+v)}}{(e^u-1)(e^v-1)}\,\frac{e^{u+v}-1}{e^{(u+v)/k}-1}$$

$$(u+v)\sum_{m,n=0}^{\infty} S_{m,n}\frac{u^m v^n}{m!\,n!}=\frac{uv(u+v)e^{x(u+v)}}{e^{(u+v)/k}-1}\left(1+\frac{1}{e^u-1}+\frac{1}{e^v-1}\right)=$$
$$=\frac{(u+v)e^{x(u+v)}}{e^{(u+v)/k}-1}\left\{uv+\sum_{r=0}^{\infty}\frac{B_r}{r!}(u^r v+uv^r)\right\}=$$
$$=k\sum_{m,n=0}^{\infty}\frac{B_{m+n}(kx)}{k^{m+n}}\,\frac{u^m v^n}{m!\,n!}\left\{u+v+\sum_{r=2}^{\infty}\frac{B_r}{r!}(u^r v+uv^r)\right\}.$$

Comparison of coefficients yields

$$\frac{1}{k}\left(mS_{m-1,n}(x)+nS_{m,n-1}(x)\right)= \tag{3.3}$$
$$=(m+n)\frac{B_{m+n-1}(kx)}{k^{m+n-1}}+\sum_{r=2}^{\min(m,n)}\left\{\binom{m}{r}n+\binom{n}{r}m\right\}B_r\,\frac{B_{m+n-r-1}(kx)}{k^{m+n-r-1}}.$$

Now it is clear form (3. 2) that

$$\frac{d}{dx}S_{m,n}(x)=mS_{m-1,n}+nS_{m,n-1};$$

thus integration leads to

$$(3.4)\qquad \begin{aligned}\frac{1}{k} S_{m,n}(x) &= \frac{B_{m+n}(kx)}{k^{m+n}} + \\ &+ \sum_{r=2}^{\min(m,n)} \left\{ \binom{m}{r} n + \binom{n}{r} m \right\} B_r \frac{B_{m+n-r}(kx)}{(m+n-r)k^{m+n-r}} + C_{m,n}\end{aligned}$$

where $C_{m,n}$ is independent of x. But if we put $x=0$ and then let $k \to \infty$, (3.4) becomes

$$\int_0^1 B_m(y) B_n(y)\, dy = C_{m,n} \qquad (m+n \geqq 2),$$

so that [6, p. 31] or [5]

$$(3.5)\qquad C_{m,n} = (-1)^{n-1} \frac{m!\, n!}{(m+n)!} B_{m+n} \qquad (m+n \geqq 2).$$

In particular for $x=0$, (3.4) and (3.5) yield

$$(3.6)\qquad \begin{aligned}\frac{1}{k} \sum_{r=0}^{k-1} B_m\left(\frac{r}{k}\right) B_n\left(\frac{r}{k}\right) &= \frac{B_{m+n}}{k^{m+n}} + \\ + \sum_{r=2}^{\min(m,n)} \left\{ \binom{m}{r} n + \binom{n}{r} m \right\} &\frac{B_r B_{m+n-r}}{(n+n-r)k^{m+n-r}} + (-1)^{n-1} \frac{m!\, n!}{(m+n)!} B_{m+n} \\ &\qquad (m+n \geqq 2).\end{aligned}$$

Returning to (3.1), since

$$\sum_{r=0}^{dkab-1} \bar{B}_m\left(\frac{r}{ak}\right) \bar{B}_n\left(\frac{r}{bk}\right) = d \sum_{r=0}^{kab-1} \bar{B}_m\left(\frac{r}{ak}\right) \bar{B}_n\left(\frac{r}{bk}\right),$$

we readily obtain from (3.6) the following

Theorem 2. *For a, b arbitrary positive integers, $k \geqq 1, m+n \geqq 2$, $K = k\,[a,b]$, we have*

$$(3.7)\qquad \begin{aligned}\frac{1}{K} \sum_{r=0}^{K-1} \bar{B}_m\left(\frac{ar}{K}\right) \bar{B}_n\left(\frac{br}{K}\right) &= \\ = \frac{(a,b)^{m+n}}{a^n b^m} \left[\frac{B_{m+n}}{k^{m+n}} + \sum_{r=2}^{\min(m,n)} \left\{ \binom{m}{r} n + \binom{n}{r} m \right\} \right. &\frac{B_r B_{m+n-r}}{(m+n-r)k^{m+n-r}} + \\ &\left. + (-1)^{n-1} \frac{m!\, n!}{(m+n)!} B_{m+n} \right].\end{aligned}$$

For $k = \infty$, it is clear that (3.7) reduces to (1.6).

4. For the HURWITZ zeta-function, the case $n=2$ of (2. 1) implies

$$(4.1)\quad \sum_{r=0}^{kab-1}\zeta\left(1-s,\frac{r}{ak}\right)\zeta\left(1-t,\frac{t}{bk}\right)=a^{1-s}b^{1-t}\sum_{r=0}^{k-1}\zeta\left(1-s,\frac{r}{k}\right)\zeta\left(1-t,\frac{r}{k}\right).$$

If we recall the formula [7, p. 269]

$$\zeta(1-s.a)=\frac{2\Gamma(s)}{(2\pi)^s}\left\{\cos\frac{s\pi}{2}\sum_{1}^{\infty}\frac{\cos 2\pi an}{n^s}+\sin\frac{s\pi}{2}\sum_{1}^{\infty}\frac{\sin 2\pi an}{n^s}\right\}=$$
$$=\frac{e^{\pi is/2}\Gamma(s)}{(2\pi)^s}\sum_{\infty}^{\infty}{}'\frac{e^{-2\pi ani}}{n^s},$$

where the prime indicates that $n\neq 0$, then since

$$\sum_{r=0}^{k-1}e^{2\pi rmi/k}=\begin{cases}k & (k\mid m)\\ 0 & (k\nmid m),\end{cases}$$

we get

$$\frac{1}{k}\sum_{r=0}^{k-1}\zeta\left(1-s,\frac{r}{k}\right)\zeta\left(1-t,\frac{r}{k}\right)=e^{\pi(s+t)i/2}\frac{\Gamma(s)\Gamma(t)}{(2\pi)^{s+t}}\sum_{k\mid m+n}{}'\frac{1}{m^sn^t},$$

the summation extending over all $m,n\neq 0$ such that $k\mid m+n$. Then exactly as in the proof of Theorem 2, we obtain the following

Theorem 3. *For a,b arbitrary positive integers, $k\geqq 1$, $K=k\,[a,b]$, s,t complex, $R(s)>1, R(t)>1$, we have*

$$(4.2)\quad \frac{1}{K}\sum_{r=0}^{K}\zeta\left(1-s,\left\{\frac{ar}{K}\right\}\right)\zeta\left(1-t,\left\{\frac{br}{K}\right\}\right)=$$
$$=\frac{(a,b)^{s+t}}{a^tb^s}e^{\pi(s+t)i/2}\frac{\Gamma(s)\Gamma(t)}{(2\pi)^{s+t}}\sum_{k\mid m+n}{}'\frac{1}{m^sn^t}.$$

In particular for $k=\infty$, (4. 2) reduces to

$$(4.3)\quad \int_0^1\zeta(1-s,\{ax\})\,\zeta(1-t,\{bx\})=$$
$$=2\frac{(a,b)^{s+t}}{a^ta^s}\cos\frac{(s-t)\pi}{2}\frac{\Gamma(s)\Gamma(t)}{(2\pi)^{s+t}}\zeta(s+t)\qquad (R(s)>1, R(t)>1).$$

For $s=t$, the right member of (4. 3) reduces to

$$2\frac{(a,b)^s}{[a,b]^s}\frac{\Gamma^2(s)}{(2\pi)^{2s}}\zeta(2s),$$

which is in agreement with (1. 3).

5. We remark that

$$\zeta(1-p,x)=-\frac{1}{p}B_p(x) \qquad (p=1,2,3,\ldots),$$

so that (1. 6) is contained in (4. 3) and similarly Theorem 2 is contained in Theorem 3. As pointed out by a refere, the sums

$$\sum_{r=0}^{k-1} B_m\left(\frac{r}{k}\right)B_n\left(\frac{r}{k}\right) \quad \text{and} \quad \sum_{r=0}^{k-1}\zeta\left(1-s,\frac{r}{k}\right)\left(1-t,\frac{r}{k}\right)$$

are generalized Dedekind sums and are closely related to the results in a recent paper by M. MIKOLÁS, On certain sums generating the Dedekind sums and their reciprocity theorems, *Pacific J. Math.* **7** (1957), 1167—1178.

The referee has also called the writer's attention to another paper by MIKOLÁS, Mellinsche Transformation und Orthogonalität bei $\zeta(s,u)$; Verallgemeinerung der Riemannschen Funktionalgleichung von $\zeta(s)$, *Acta Sci. Math. Szeged* **17** (1956), 143—164 in which (pp. 158—159) certain formulas involving $\zeta(s,u)$ are proved that imply both MORDELL's formula (1. 6) as well as (4. 3) under the restriction

$$\min(R(s),1)+\min(R(t),1)>0.$$

Thus (1. 6) and (4. 3) cannot be regarded as new.

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