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On a generalized Pexider equation connected with the iteration theory

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Abstract. Let X, Y, Z be arbitrary nonempty sets. We consider the following functional equation (iterative) of Pexider type $F_{st} = k_{st} \circ H_s \circ G_t$ for (s,t) belonging to the domain of a binary operation on a groupoid K, where $\{F_t\}_{t\in K} \subset Z^X$, $\{G_t\}_{t\in K} \subset Y^X$, $\{H_t\}_{t\in K} \subset Z^Y$ are unknown families of functions and k_{st} belongs to the group of all bijections of Z onto Z. It is shown that, in the case when there exists a unit element $e \in K$ and H_e is an injection, G_e is a surjection, the equation can be reduced to the Cauchy equation. Some conditions are established under which the Cauchy equation has continuous solutions. Finally, as an application, using some facts from the iteration theory, we give solutions of the Pexider equation in some special cases.

Notations and definitions

Let K be a nonempty set endowed with a binary operation (i.e. a mapping of a subset $D(K)$ of $K \times K$ into K). The set K with the binary operation is called a groupoid (cf. [5]). If $(s,t) \in D(K)$ then we say that the product st is defined. An element $e \in K$ will be called a unit if for every $t \in K$ the products te and et are defined and $te = et = t$.

By $\mathrm{In}(X,Y)$, $(\mathrm{Sur}(X,Y),\ \mathrm{Bij}(X,Y))$ we denote the set of all injections (surjections, bijections) of a set X into (onto) a set Y.

$\mathrm{Ran}\, f$ $(\mathrm{Dom}\, f)$ means the range (the domain) of the function f and id_X stands for the identity function on the set X.

Let X, Y be topological spaces. We say that $f : X \to Y$ is an open map if f maps open subsets of X onto open subsets of Y. The set of all homeomorphisms of X onto Y will be denoted by $\mathrm{Hom}(X,Y)$.

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To simplify the notations, we write $\operatorname{In} X$, $\operatorname{Sur} X$, $\operatorname{Bij} X$, $\operatorname{Hom} X$ in the case when $X = Y$.

Further, by $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ we will denote, as usual, the sets of rationals, reals and complex numbers, respectively.

Let X be a subset of $\mathbb{R}^n$. A family of continuous functions $\{T_t : X \to X,\ t \in \mathbb{R}\}$ ($\{T_t : X \to X,\ t \in \mathbb{Q}\}$) is said to be an iteration group (resp., a rational iteration group) if

$$T_{s+t} = T_s \circ T_t \qquad \text{for } s, t \in \mathbb{R} \quad (\text{resp., for } s, t \in \mathbb{Q}).$$

If for every $x \in X$ the mapping $t \mapsto T_t(x)$ is continuous then the iteration group is said to be continuous (cf. [8] or [11]). The set $\{T_t(x),\ t \in \mathbb{R}$ (or $t \in \mathbb{Q}$)$\}$ is called the orbit of x.

Let K be a groupoid and X, Y, Z be arbitrary nonempty sets. We shall consider the following functional equation of the iterative type (i.e. the equation in which compositions of unknown functions appear)

$$\text{(P)} \qquad F_{st} = k_{st} \circ H_s \circ G_t, \qquad (s,t) \in D(K),$$

where $\{F_t\}_{t\in K} \subset Z^X$, $\{G_t\}_{t\in K} \subset Y^X$, $\{H_t\}_{t\in K} \subset Z^Y$ are unknown families of functions and $k_{st} \in \operatorname{Bij} Z$ for $(s,t) \in D(K)$. Similar problems have been also studied in [2], [3], [7], [10], [12], [13], [14].

Main results

Theorem 1. *Let K be a groupoid such that there exists a unit element e in K. If $\{F_t\}_{t\in K} \subset Z^X$, $\{G_t\}_{t\in K} \subset Y^X$, $\{H_t\}_{t\in K} \subset Z^Y$ satisfy the equation* (P)*, where $k_{st} \in \operatorname{Bij} Z$ for every $(s,t) \in D(K)$ and $H_e \in \operatorname{In}(Y,Z)$, $G_e \in \operatorname{Sur}(X,Y)$, then there exist functions $a \in \operatorname{In}(Y,Z)$, $b \in \operatorname{Sur}(X,Y)$ and a family of functions $\{T_t\}_{t\in K} \subset Y^Y$ such that*

$$(1) \qquad T_{st} = T_s \circ T_t, \qquad (s,t) \in D(K)$$

and

$$(2) \qquad \begin{cases} F_t = k_t \circ a \circ T_t \circ b, \\ G_t = T_t \circ b, \\ H_t = a \circ T_t, \qquad t \in K. \end{cases}$$

Conversely, if $a \in Z^Y$, $b \in Y^X$, $k_{st} \in Z^Z$ *for* $(s,t) \in D(K)$ *and* $\{T_t\}_{t\in K} \subset Y^Y$ *satisfies* (1) *then the functions* F_t, G_t, H_t *given by* (2) *satisfy equation* (P).

Proof. Put $F_t =: F(t)$, $G_t =: G(t)$, $H_t =: H(t)$, $k_t := k(t)$. Setting $t = e$ in (P) and then $s = e$ we get

$$F(s) = k(s) \circ H(s) \circ G(e), \qquad s \in K, \tag{3}$$

$$F(t) = k(t) \circ H(e) \circ G(t), \qquad t \in K. \tag{4}$$

Hence $\mathrm{Ran}(k(t)^{-1} \circ F(t)) \subset \mathrm{Dom}(H(e)^{-1})$. Thus we have

$$(k(t) \circ H(e))^{-1} \circ F(t) = H(e)^{-1} \circ k(t)^{-1} \circ F(t), \qquad t \in K. \tag{5}$$

Comparing the right hand sides of (3) and (4) for $s = t$, we obtain

$$H(t) \circ G(e) = H(e) \circ G(t), \qquad t \in K. \tag{6}$$

Hence, by the relation $G(e) \in \mathrm{Sur}(X, Y)$, we infer that $\mathrm{Ran}\, H(t) \subset \mathrm{Dom}(H(e)^{-1})$ for $t \in K$.

Now, introduce on X an equivalence relation $\mathcal{R}$:

$$x\mathcal{R}y \quad \text{iff} \quad G(e)(x) = G(e)(y).$$

Let $\widetilde{X} = X/\mathcal{R}$ and let g be an invertible mapping such that $g([x]) \in [x]$, where $[x]$ stands for an equivalence class containing x. Thus the function $G(e) \circ g : \widetilde{X} \to Y$ is a bijection. From (3) we obtain

$$F(t) \circ g = k(t) \circ H(t) \circ G(e) \circ g, \qquad t \in K,$$

whence

$$H(t) = k(t)^{-1} \circ F(t) \circ g \circ (G(e) \circ g)^{-1}, \qquad t \in K. \tag{7}$$

Hence (P) may be written as follows:

$$F(st) = k(st) \circ k(s)^{-1} \circ F(s) \circ g \circ (G(e) \circ g)^{-1} \circ G(t) \quad \text{for } (s,t) \in D(K). \tag{8}$$

By (4) and (5) we have

$$G(t) = H(e)^{-1} \circ k(t)^{-1} \circ F(t), \qquad t \in K. \tag{9}$$

Next (6) implies

$$G(t) = H(e)^{-1} \circ H(t) \circ G(e), \qquad t \in K. \tag{10}$$

Define

$$T(t) := H(e)^{-1} \circ k(t)^{-1} \circ F(t) \circ g \circ (G(e) \circ g)^{-1}, \quad t \in K. \tag{11}$$

Hence, calculating $k(st)^{-1} \circ F(st)$ from (8) and by (9) we can write

$$\begin{aligned} T(st) &= H(e)^{-1} \circ k(st)^{-1} \circ F(st) \circ g \circ (G(e) \circ g)^{-1} \\ &= H(e)^{-1} \circ k(s)^{-1} \circ F(s) \circ g \circ (G(e) \circ g)^{-1} \circ G(t) \circ g \circ (G(e) \circ g)^{-1} \\ &= H(e)^{-1} \circ k(s)^{-1} \circ F(s) \circ g \circ (G(e) \circ g)^{-1} \circ H(e)^{-1} \circ k(t)^{-1} \\ &\quad \circ F(t) \circ g \circ (G(e) \circ g)^{-1} \\ &= T(s) \circ T(t). \end{aligned}$$

Then (1) holds, where $T_t := T(t)$, $t \in K$. By (7) and (11) we have

$$H(t) = H(e) \circ T(t), \quad t \in K \tag{12}$$

and from, (12) and (10),

$$G(t) = T(t) \circ G(e), \quad t \in K. \tag{13}$$

Substituting (13) into (4) we obtain

$$F(t) = k(t) \circ H(e) \circ T(t) \circ G(e), \quad t \in K. \tag{14}$$

Putting $a := H(e)$, $b := G(e)$ we get from (14), (13) and (12) the formula (2).

The converse statement is easy to check.

Remark 1. As an immediate consequence of Theorem 1 we have Theorem 1 from [3] concerning the solutions of the equation

$$F_{st} = H_s \circ G_t, \quad (s,t) \in D(K).$$

To see this it is enough to take $k_s = \mathrm{id}_Z$ for $s \in K$ in formulas (2).

Let X, Y be topological spaces. Let $f \in \mathrm{Sur}(X, Y)$ and $\mathcal{R}$ be an equivalence relation on the space X defined by

$$x \mathcal{R} y \quad \text{iff} \quad f(x) = f(y).$$

Denote by $X/\mathcal{R}$ the quotient space $X/\mathcal{R}$ endowed with the usual quotient topology i.e. the final topology determined by the canonical surjection $k : X \ni x \mapsto [x] \in X/\mathcal{R}$ (cf. e.g. [6]). Let g be an invertible map such that $g([x]) \in [x]$. Define a map $\tilde{f} : X/\mathcal{R} \to Y$ putting $\tilde{f} = f \circ g$. It is easy to see that $\tilde{f}$ is a bijective map.

Using the obvious fact that k is a continuous map, one can easily show the following Lemma (see e.g. [6], Chap. 7).

Lemma. *If the map f is continuous (resp., open) then the map $\tilde{f}$ is continuous (resp., open).*

Let K be a group with a unit element e and X, Y, Z be arbitrary topological spaces and let $\{F_t\}_{t\in K} \subset Z^X$, $\{G_t\}_{t\in K} \subset Y^X$, $\{H_t\}_{t\in K} \subset Z^Y$. Assume the following hypotheses:

(a) H_t is a continuous map for $t \in K$ and H_e is an open map;
(b) G_t is a continuous map for $t \in K$ and G_e is an open map;
(c) F_t is a continuous map for $t \in K$, F_e is an open map and H_e is a continuous open map;
(d) G_t is a continuous map for $t \in K$ and G_e, H_e are open maps.

Proposition. *Let $\{F_t\}_{t\in K} \subset Z^X$, $\{G_t\}_{t\in K} \subset Y^X$, $\{H_t\}_{t\in K} \subset Z^Y$ be families of functions satisfying the equation* (P) *for $s, t \in K$, where $k_t \in \operatorname{Hom} Z$ for $t \in K$. Suppose that $H_e \in \operatorname{In}(Y, Z)$, $G_e \in \operatorname{Sur}(X, Y)$. If one of the hypotheses* (a), (b) *or* (c) *holds then there exist $a \in \operatorname{In}(Y, Z)$, $b \in \operatorname{Sur}(X, Y)$ and a family of mappings $\{T_t\}_{t\in K} \subset \operatorname{Hom} Y$ satisfying equation* (1) *for $s, t \in K$ such that* (2) *holds.*

Moreover, if the hypothesis (c) *holds then the families $\{F_t\}_{t\in K}$, $\{G_t\}_{t\in K}$, $\{H_t\}_{t\in K}$ are families of continuous mappings.*

Proof. Note that by the proof of Theorem 1, we obtain (cf. (14))

$$(15)\qquad \begin{cases} F_t = k_t \circ H_e \circ T_t \circ G_e, \\ G_t = T_t \circ G_e, \qquad \text{for} \quad t \in K, \\ H_t = H_e \circ T_t, \end{cases}$$

where $\{T_t\}_{t\in K} \subset Y^Y$ satisfies (1) for $s, t \in K$.

Thus it suffices to prove that T_t is a homeomorphism for every $t \in K$ and $\{F_t\}_{t\in K}$, $\{G_t\}_{t\in K}$, $\{H_t\}_{t\in K}$ are families of continuous mappings in the case (c).

Let ϱ be an equivalence relation on the topological space X defined by:

$$x \varrho y \quad \text{iff} \quad G_e(x) = G_e(y).$$

Denote by $\widetilde{X}$ the quotient space X/ϱ endowed with the quotient topology. Let g be an invertible mapping such that $g([x]) \in [x]$. Define the maps $\widetilde{G}_t : \widetilde{X} \to Y$ and $\widetilde{F}_t : \widetilde{X} \to \operatorname{Ran} F_t$ setting

$$\widetilde{G}_t = G_t \circ g, \quad \widetilde{F}_t = F_t \circ g$$

for $t \in K$. Substituting in (P) $s = e$ and then $t = e$ we get respectively:

$$F_t = k_t \circ H_e \circ G_t, \qquad t \in K, \tag{16}$$

$$F_s = k_s \circ H_s \circ G_e, \qquad s \in K. \tag{17}$$

Comparing the right hand sides of (16) and (17) for $s = t$ we have

$$H_e \circ G_t = H_t \circ G_e, \qquad t \in K. \tag{18}$$

Note that by (15) we get

$$T_t = H_e^{-1} \circ H_t, \qquad t \in K. \tag{19}$$

Setting $t = e$ in (19) we obtain

$$T_e = \mathrm{id}_Y . \tag{20}$$

Using (20) and the fact that $\{T_t\}_{t\in K}$ satisfies (1) for $s, t \in K$ one can easily check (cf. e.g. [15], Remark 1, p. 218) that T_t is a bijection for $t \in K$. By virtue of (18) and (19) we have

$$T_t = \widetilde{G}_t \circ (\widetilde{G}_e)^{-1}, \qquad t \in K. \tag{21}$$

Having disposed of the preliminary steps, we proceed to investigate the three cases separately.

First assume that hypothesis (a) holds. Then the map $H_e : Y \to \operatorname{Ran} H_e$ is a homeomorphism. Thus, by (19), we get the continuity of the map T_t for every $t \in K$.

Suppose that hypothesis (b) holds. By the Lemma, the map $\widetilde{G}_e$ is a homeomorphism. Fix $t \in K$, $t \neq e$. On account of (18) and (19) we have $G_t = T_t \circ G_e$. Whence G_t is a surjection. From (19) we deduce that the function H_t is an injective map. Hence, by (18), we obtain

$$G_t(x) = G_t(y) \quad \text{iff} \quad G_e(x) = G_e(y), \qquad x, y \in X.$$

Consequently, by the Lemma, $\widetilde{G}_t$ is a continuous map. Now (21) implies the continuity of T_t.

Finally suppose that hypothesis (c) holds. On account of (16) we get

$$G_t = H_e^{-1} \circ k_t^{-1} \circ F_t, \qquad t \in K.$$

Hence G_t is a continuous map for $t \in K$. Moreover G_e is an open map since F_e is an open map and $H_e \colon Y \to \operatorname{Ran} H_e$ is a homeomorphism.

By the Lemma, the bijective map $\widetilde{G}_e$ is a homeomorphism. From (17) and the injectivity of H_t we infer that $\widetilde{F}_t : \widetilde{X} \to \operatorname{Ran} F_t$ is a bijective map for $t \in K$ and in view of the Lemma, $\widetilde{F}_t$ is continuous. Now note that (17) yields

$$H_t = k_t^{-1} \circ \widetilde{F}_t \circ (\widetilde{G}_e)^{-1}, \qquad t \in K.$$

Consequently H_t is a continuous map for $t \in K$. This implies, by (19), the continuity of T_t for $t \in K$.

To finish the proof, observe that T_t is a homeomorphism for $t \in K$, since $(T_t)^{-1} = T_{t^{-1}}$ and each T_t is a continuous function.

Remark 2. From the above proof, one can see that when the hypothesis (a) holds, it suffices to assume that K is a groupoid with a unit element $e \in K$, to obtain the continuity of T_t for $t \in K$.

Applications

Using the above results and some facts from the iteration theory we will solve the Pexider functional equation

$$F_{s+t} = k_{s+t} \circ H_s \circ G_t, \qquad s, t \in \mathbb{R}, \tag{22}$$

(or $s, t \in \mathbb{Q}$), where $\{F_t\}$, $\{G_t\}$, $\{H_t\}$ are unknown families of functions which map a real interval (or the unit circle, or a subset of $\mathbb{R}^n$ space) into itself and k_t is a homeomorphism for $t \in K$. More precisely, we shall show under some additional assumptions, that the functions F_t, G_t, H_t satisfying the above equation, are conjugate (in some sense) with some special families of functions.

Theorem 2. *Let Δ be either a real open interval or the unit circle, $\{F_t\}_{t\in\mathbb{R}}$, $\{G_t\}_{t\in\mathbb{R}}$, $\{H_t\}_{t\in\mathbb{R}}$ be families of functions mapping Δ into Δ and satisfying the functional equation* (22)*,where $k_t \in \operatorname{Hom}\Delta$ for $t \in \mathbb{R}$. Suppose that $H_0 \in \operatorname{In}\Delta$, $G_0 \in \operatorname{Sur}\Delta$, the function $t \mapsto H_t(x)$ is continuous* *for every* $x \in \Delta$ *and* $H_0(x) \neq H_1(x)$ *for* $x \in \Delta$. *If one of the hypotheses* (a), (c) *or* (d) *holds then there exist* $a \in \operatorname{In}\Delta$, $b \in \operatorname{Sur}\Delta$ *such that*

$$\begin{cases} F_t = k_t \circ a \circ \varphi^{-1} \circ A_t \circ \varphi \circ b, \\ G_t = \varphi^{-1} \circ A_t \circ \varphi \circ b, \\ H_t = a \circ \varphi^{-1} \circ A_t \circ \varphi, \qquad t \in \mathbb{R}, \end{cases} \tag{23}$$

where, in the case where Δ *is a real open interval,* φ *is a homeomorphism of* Δ *onto* $\mathbb{R}$ *and* $A_t(x) = x + t$ *for* $x, t \in \mathbb{R}$ *and, in the case where* Δ *is the unit circle,* φ *is a homeomorphism of* Δ *onto* Δ *and* $A_t(x) = e^{2\pi i \alpha t}x$ *for* $x \in \Delta$, $t \in \mathbb{R}$ *and some* $\alpha \in \mathbb{R}$.

Conversely, if $a \in \Delta^\Delta$, $b \in \Delta^\Delta$, $k_t \in \Delta^\Delta$, $t \in \mathbb{R}$ *are arbitrary functions,* φ *is a bijection of* Δ *onto* $\mathbb{R}$ *(or* Δ *onto* Δ*),* $\{A_t\}_{t\in\mathbb{R}}$ *is a family of functions mapping* $\operatorname{Ran}\varphi$ *into* $\operatorname{Ran}\varphi$ *and satisfying the Cauchy equation* (1) *for* $s, t \in \mathbb{R}$, *then the functions* F_t, G_t, H_t *given by* (23) *satisfy the Pexider equation* (22).

Proof. Note that on account of the Proposition we have the representation

$$\begin{cases} F_t = k_t \circ a \circ T_t \circ b, \\ G_t = T_t \circ b, \\ H_t = a \circ T_t, \qquad t \in \mathbb{R}, \end{cases} \tag{24}$$

where $\{T_t\}_{t\in\mathbb{R}}$ is an iteration group on Δ such that $T_t \in \operatorname{Hom}\Delta$ for every $t \in \mathbb{R}$. Moreover, by (19), the mapping $t \mapsto T_t(x)$ is continuous for every $x \in \Delta$ and T_1 has no fixed points.

Suppose first that Δ is a real open interval. It is well known (see [9], Th. 6 or [11], p. 99) that such a continuous iteration group $\{T_t\}_{t\in\mathbb{R}}$ is conjugate with the group of translations $P_t(x) = x + t$; that is there exists a homeomorphism $\eta : \Delta \to \mathbb{R}$ such that

$$T_t = \eta^{-1} \circ P_t \circ \eta, \quad t \in \mathbb{R}. \tag{25}$$

Now, assume that Δ is the unit circle i.e. $\Delta = \{z \in \mathbb{C} : |z| = 1\}$. Then by Theorem 2 in [15], which states that every continuous iteration group of the homeomorphism T_1, such that T_1 has no fixed points or $T_1 = \mathrm{id}_\Delta$, is conjugate with a group of rotations of the circle $Q_t(x) = e^{2\pi i \alpha t}x$ for some $\alpha \in \mathbb{R}$, we infer that there exists a homeomorphism $\psi : \Delta \to \Delta$ such that

$$T_t = \psi^{-1} \circ Q_t \circ \psi, \qquad t \in \mathbb{R}. \tag{26}$$

Finally, substituting (25) and (26) into (24) respectively, we obtain the required formulae (23).

The converse statement is easy to check.

Under a stronger regularity assumption i.e. assuming the continuity of the function $(x,t) \mapsto H_t(x)$, it is possible to obtain similar results in the multi-dimensional real case, namely:

Theorem 3. *Let $D \subset \mathbb{R}^n$ be a nonempty set and $\{F_t\}_{t\in\mathbb{R}}$, $\{G_t\}_{t\in\mathbb{R}}$, $\{H_t\}_{t\in\mathbb{R}}$ be families of functions mapping D into D, satisfying the Pexider equation* (22), *where $k_t \in \operatorname{Hom} D$ for $t \in \mathbb{R}$, and such that $H_0 \in \operatorname{In} D$, $G_0 \in \operatorname{Sur} D$. Suppose that for every $x \in D$, $t \in \mathbb{R}$ the mapping $(x,t) \mapsto H_t(x)$ is continuous, $H_t(x) \neq H_0(x)$ for $x \in D$, $t \neq 0$ and there exists a hypersurface $\Gamma \subset D$ homeomorphic to $\mathbb{R}^{n-1}$ such that $H_0(\Gamma)$ has exactly one common point with every orbit of $\{H_t,\, t \in \mathbb{R}\}$. If one of the hypotheses* (a), (c) *or* (d) *holds, then there exist $a \in \operatorname{In} D$, $b \in \operatorname{Sur} D$ and a homeomorphism $\varphi : \mathbb{R}^n \to D$ such that* (23) *holds, where $\{A_t,\, t \in \mathbb{R}\}$ is the one-parameter group of translations i.e. $A_t(x) = x + t\alpha$ for $t \in \mathbb{R}$, $x \in D$ and an $\alpha \in \mathbb{R}^n$, $\alpha \neq 0$.*

Conversely, if $a \in D^D$, $b \in D^D$, $k_t \in D^D$ for $t \in \mathbb{R}$ are arbitrary functions, $\{A_t\}_{t\in\mathbb{R}}$ is a family of functions satisfying the equation (1) *for $s,t \in \mathbb{R}$, φ is a bijection from $\mathbb{R}^n$ onto D then the functions F_t, G_t, H_t given by* (23) *satisfy the equation* (22).

Proof. We shall use a result of M. C. Zdun ([17], Th. 1) which states that the iteration group $\{f^t\}_{t\in\mathbb{R}}$ defined on $D \subset \mathbb{R}^n$, such that the mapping $(x,t) \mapsto f^t(x)$ is continuous, is given by the formula

$$f^t(x) = \varphi^{-1}(\varphi(x) + \alpha t), \quad t \in \mathbb{R}, \quad x \in D,$$

where $\varphi : \mathbb{R}^n \to D$ is a homeomorphism, iff $f^0 = \mathrm{id}$, $f^t(x) \neq x$ for $x \in D$, $t \neq 0$ and there exists a hypersurface $\Gamma \subset D$ homeomorphic to $\mathbb{R}^{n-1}$ which has exactly one common point with every orbit of $\{f^t, t \in \mathbb{R}\}$.

By the Proposition we have the representation (24), where $\{T_t\}_{t\in\mathbb{R}}$ is an iteration group. Moreover $T_0 = \mathrm{id}_Y$, $T_t(x) \neq x$ for $x \in D$, $t \neq 0$ and, by (19), the mapping $(x,t) \mapsto T_t(x)$ is continuous since the mappings $(x,t) \mapsto H_t(x)$ and H_0^{-1} are continuous. Now, observe that by the injectivity of H_0 and (19) we can infer that the intersection $\Gamma \cap \{T_t,\, t \in \mathbb{R}\}$ is a single element set for every orbit of $\{T_t,\, t \in \mathbb{R}\}$. Thus, by the mentioned theorem of M. C. Zdun, we obtain

$$T_t = \varphi^{-1} \circ A_t \circ \varphi, \qquad t \in \mathbb{R}, \tag{27}$$

where $\varphi\colon \mathbb{R}^n \to D$ is a homeomorphism and $A_t(x) = x + \alpha t$ for $t \in \mathbb{R}$, $x \in D$ and some $\alpha \in \mathbb{R}^n$, $\alpha \neq 0$. Substituting formula (27) into (24) we get (23), as claimed.

Checking the converse statement is a mere calculation.

Finally, we shall show that in the case where the families $\{F_t\}$, $\{G_t\}$, $\{H_t\}$ mapping a real open interval into itself and satisfying the Pexider equation are indexed by rationals, we do not need any regularity assumptions to obtain representations like (23). More precisely, the following result holds:

Theorem 4. *Let I be a real open interval and $\{F_t\}_{t\in\mathbb{Q}}$, $\{G_t\}_{t\in\mathbb{Q}}$, $\{H_t\}_{t\in\mathbb{Q}}$ be families of functions mapping I into I and satisfying the equation*

$$F_{s+t} = k_{s+t} \circ H_s \circ G_t \qquad \text{for } s,t \in \mathbb{Q}, \tag{28}$$

where $k_t \in \operatorname{Hom} I$ for $t \in \mathbb{Q}$. Suppose that $H_0 \in \operatorname{In} I$, $G_0 \in \operatorname{Sur} I$ and $H_0(x) \neq H_1(x)$ for $x \in I$. If one of the hypotheses (a), (b) *or* (c) *holds then there exist $a \in \operatorname{In} I$, $b \in \operatorname{Sur} I$ such that*

$$\begin{cases} F_t = k_t \circ a \circ \gamma^{-1} \circ B_t \circ \gamma \circ b, \\ G_t = \gamma^{-1} \circ B_t \circ \gamma \circ b, \\ H_t = a \circ \gamma^{-1} \circ B_t \circ \gamma, \qquad t \in \mathbb{Q}, \end{cases} \tag{29}$$

where either $\{B_t\}_{t\in\mathbb{Q}}$ is the group of rational translations and γ is a homeomorphism mapping I onto $\mathbb{R}$, or $\{B_t\}_{t\in\mathbb{Q}}$ is a special rational iteration group of piecewise linear homeomorphisms mapping I onto I and γ is a homeomorphism of I onto I.

Conversely, if $a \in I^I$, $b \in I^I$, $k_t \in I^I$ for $t \in \mathbb{Q}$ are arbitrary functions, γ is a bijection mapping I onto I (or I onto $\mathbb{R}$) and $\{B_t\}_{t\in\mathbb{Q}}$ is a family of functions mapping $\operatorname{Ran}\gamma$ into $\operatorname{Ran}\gamma$ satisfying the Cauchy equation (1) *for $s,t \in \mathbb{Q}$, then the functions F_t, G_t, H_t given by* (29) *satisfy the Pexider equation* (28).

Proof. By the Proposition we get the representation (24), where $\{T_t\}_{t\in\mathbb{Q}}$ is a rational iteration group. Observe that by (19), T_1 has no fixed points since $H_0(x) \neq H_1(x)$ for $x \in I$. Consequently for every $t \in \mathbb{Q}$, T_t has no fixed points (see e.g. [1]). Now consider the set $L(x)$ of the limit points of the orbit $\{T_t,\, t \in \mathbb{Q}\}$. In paper [16](Th. 1) (see also [18], Prop. 1) it has been proved that the set $L(x)$ does not depend on x. Denote this set by L. Further, either

1) $L = \mathrm{cl}\, I$ or

2) *L is perfect and nowhere dense in I.*

In case 1) the rational iteration group $\{T_t\}_{t\in\mathbb{Q}}$ can be embedded in a real iteration group (see [16], Th. 4) which is conjugate with the group of translations. Thus, there exists a homeomorphism $\varphi : I \to \mathbb{R}$ such that

$$T_t = \varphi^{-1} \circ A_t \circ \varphi, \qquad t \in \mathbb{Q}, \tag{30}$$

where $A_t(x) = x + t, \quad x \in I, \ t \in \mathbb{Q}$.

In case 2), by Theorem 2 in [4], the rational iteration group $\{T_t\}_{t\in\mathbb{Q}}$ is conjugate with a special rational iteration group $\{p_t\}_{t\in\mathbb{Q}}$ of piecewise linear, fixed-point-free homeomorphisms. More precisely, there exists a homeomorphism $\psi : I \to I$ such that $\psi(x) = x$ for $x \in L$ and

$$T_t = \psi^{-1} \circ p_t \circ \psi, \qquad t \in \mathbb{Q}. \tag{31}$$

Substituting (31) and (30) into (24) we obtain the desired formulae.

The converse statement is easy to check.

Remark 3. The iteration group $\{p_t, t \in \mathbb{R} \quad (\text{or} \ \ t \in \mathbb{Q})\}$ has been constructed and examined by M. C. Zdun in [18] (see also [4]).

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