On the definitions of direct product in universal algebra.

To Professor Ottó Varga on his 50th birthday.

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§ 1.

The most customary manner for introducing the direct product $\times A_\lambda$ of the abstract algebras A_λ is the following: we form the set-theoretical product of the sets A_λ and we define the operations component-wise. (The notion „direct product" will mean always the *unrestricted* or *complete* direct product.) If each algebra A_λ has a unit element e_λ, then there is also an inner characterization of this concept which proceeds by identifying the element $a \in A_\lambda$ with the element of $\times A_\lambda$ which has a as its λ-component and any other component of which is the respective unit element. Thus we got the algebra $\times A_\lambda$ as the direct product of certain of its *subalgebras.*

In more recent investigations there arose also other ideas for the inner characterization of this concept. The definition by a system of *endomorphisms* appeared to be reasonable for abelian groups in the papers [6] (p. 311) and [4] (p. 159) of T. SZELE, J. SZENDREI and A. KERTÉSZ. The two manners of introduction mentioned above are in the following connection: we consider in the direct product $\times A_\lambda$ the endomorphism ε_λ which assignes to each $a \in \times A_\lambda$ the λ-component of a (for any λ).

Another possibility is touched in G. BIRKHOFF's book „Lattice theory". He wants to give a connection between *congruence relations* and direct decompositions into a finite number of algebras in a theorem of his book ([2] p. 87, Theorem 4; cited in [5] p. 211 and [3] p. 87), however, this theorem is not correct. A similar (but true) result is due to J. HASHIMOTO ([3], p. 96) for an infinity of factors. The fundamental idea is to define the congruence relation ϱ_λ by the rule: $a \equiv b \pmod{\varrho_\lambda}$ holds if and only if a and b have the same λ-component.

In § 2 we define certain systems of congruence relations and certain systems of endomorphisms in an algebra A; these systems can serve for characterizing the direct decompositions of A. These systems will be the $(\mathfrak{S}\mathfrak{C})$-systems of congruence relations and the $(\mathfrak{I}\mathfrak{O}\mathfrak{S}\mathfrak{C})$-systems of endomorphisms. The § 3 is devoted to elucidate the connection among the three manners: in Theorem 1 we investigate the many-one correnspondence between the $(\mathfrak{S}\mathfrak{C})$-systems of congruence relations and the corresponding systems of subalgebras, the Theorem 2 states that there is a natural one-one mapping between the $(\mathfrak{I}\mathfrak{O}\mathfrak{S}\mathfrak{C})$-systems of endomorphisms and the $(\mathfrak{S}\mathfrak{C})$-systems of congruence relations.

It seems that the characterization by congruence relations has the greatest importance from universal-algebraic point of view: it can be extended for algebras without unit element, and it can be modified into inner characterization of subdirect products (see [3]). In § 4 we investigate two automorphism groups of an algebra A which leave fix (in different senses) an $(\mathfrak{S}\mathfrak{C})$-system of congruence relations of A. Theorem 3 contains a result about the embedding of the stricter group into the wider one, and determines the stricter automorphism group. This theorem in an analogon of the earlier results of [1], so we can omit the details of the proof in consequence the analogy. § 5 contains a counter-example (due to L. FUCHS and G. SZÁSZ) disproving the statement mentioned above of BIRKHOFF.

§ 2.

The notion of *algebra* is defined in [2] or [3]. The assumption that the operations are finitary is not necessary for us. The element e is called the *unit element* of the algebra A if it forms a subalgebra of A and any other subalgebra of A contains at least two elements. The existence of a unit element assures that we can speak about a *null endomorphism* (denoted by o) of A.

The bracket in an expression of the form $\{\varrho_\lambda\}$ denotes that the index λ runs through a set Λ (of arbitrary cardinality), this set of indices is supposed to be the same throughout the paper. We make no distinction between the systems of congruence relations $\{\varrho_\lambda\}$ and $\{\varrho'_\lambda\}$ of there is a permutation $\lambda \to \lambda'$ of Λ such that $\varrho_{\lambda'} = \varrho'_\lambda$ holds for any λ. If ϱ is a congruence relation and α is an automorphism of A, then $\varrho\alpha$ means the congruence relation defined by the rule: $a \equiv b \pmod{\varrho\alpha}$ if and only if $a\alpha^{-1} \equiv b\alpha^{-1} \pmod{\varrho}$.

We define the following two properties for systems of congruence relations:

$(\mathfrak{S})$ (Property of separability.) If $a \equiv b \pmod{\varrho_\lambda}$ holds for any $\lambda \in \Lambda$ $(a, b \in A)$, then $a = b$. [1])

[1]) This property occurs frequently in the lattice-theoretical form $\bigcap_{\lambda \in \Lambda} \varrho_\lambda = 0$. (Cf. [2], [3], [5].)

($\mathfrak{C}$) (Property of completeness.) Any system of congruences of the form $x \equiv a_\lambda \pmod{\varrho_\lambda}$ has a solution in A (λ runs through Λ, the a_λ's are arbitrary elements of A).

A system of congruence relations having the properties ($\mathfrak{S}$) and ($\mathfrak{C}$) is said to be an ($\mathfrak{S}\mathfrak{C}$)-system. We shall use similar notations also for other collections of properties.

For any endomorphism ε of A the congruence relation $\varrho[\varepsilon]$ defined by $a \equiv b \pmod{\varrho[\varepsilon]}$ if and only if $a\varepsilon = b\varepsilon$ ($a, b \in A$) is called the *congruence relation induced by* ε. We define the equality between systems of endomorphisms similarly to the equality of systems of congruence relations. The system of endomorphisms $\{\varepsilon_\lambda\}$ is said to have the property ($\mathfrak{S}$) if the congruence relations induced by the ε_λ's form an ($\mathfrak{S}$)-system. Similarly, we define the completeness of a system of endomorphisms by the completeness of the congruence relations induced by them. (One can get easily a direct definition.)

We define two further properties for systems of endomorphisms; the property ($\mathcal{O}$) is defined only in algebras containing a unit element.

($\mathfrak{J}$) (Property of idempotency.) $\varepsilon_\lambda^2 = \varepsilon_\lambda$ holds for any endomorphism of the system.

($\mathcal{O}$) (Property of orthogonality.) If ε_λ and ε_μ are distinct endomorphisms of the system, then $\varepsilon_\lambda \varepsilon_\mu = o$.

If some congruence relations form an ($\mathfrak{S}\mathfrak{C}$)-system $\{\varrho_\lambda\}$ of A, the A is said to be the *direct product* of the factor algebras A/ϱ_λ. If A has a unit element e, then let us define the subalgebras A_λ of A by what follows: $a \in A_\lambda$ if and only if $\mu \neq \lambda$ implies $a \equiv e \pmod{\varrho_\mu}$ ($\mu \in \Lambda$). These A_λ's are called the *subalgebras corresponding to the* ϱ_λ's.

§ 3.

Throughout this § A denotes an algebra containing a unit element e, and any factor algebra A/ϱ_λ must contain at least two elements.

Theorem 1. *Let $\{\varrho_\lambda\}$ and $\{\varrho'_\lambda\}$ be two ($\mathfrak{S}\mathfrak{C}$)-systems of congruence relations of A, let $\{A_\lambda\}$ and $\{A'_\lambda\}$ be the systems of corresponding subalgebras, respectively. If $\{A_\lambda\}$ and $\{A'_\lambda\}$ coincide, then there exists exactly one automorphism α of A leaving fixed any element of each A_λ for which*

$$\{\varrho_\lambda \alpha\} = \{\varrho'_\lambda\} \tag{1}$$

holds. Conversely, if α satisfies (1) *and leaves fixed all elements of the A_λ's, then the systems $\{A_\lambda\}$ and $\{A'_\lambda\}$ are the same.*

The proof of the theorem will consist of four parts A), B), C) and D). In parts A), B), C) the first statement of the theorem will be verified, and we shall prove in D) the converse statement. In part A) the automorphism α will be constructed, in B) we shall show that α has the properties exposed in the theorem, and in C) the unicity of α will be verified.

A) Let $\{A_\lambda\}=\{A'_\lambda\}$ be true for the $(\mathfrak{S}\mathfrak{C})$-systems $\{\varrho_\lambda\}$ and $\{\varrho'_\lambda\}$. The equality exposed means that $A_\lambda = A'_{\lambda'}$ holds for any λ where $\lambda \to \lambda'$ is a suitable permutation of Λ. The fundamental idea of the construction is to define the „components" a_λ of an element $a \in A_\lambda$ (any a_λ lies in A_λ), and to form a new element $a\alpha$ $(\in A)$ by these components regarded as the elements of the A'_λ's.

Let us consider the mapping α defined by

$$a\alpha \equiv a_\lambda \pmod{\varrho'_{\lambda'}} \quad \text{for any } \lambda,$$

where the a_λ's are determined by the congruences

(2) $a_\lambda \equiv a \pmod{\varrho_\lambda}$ for the considered λ and $a_\lambda \equiv e \pmod{\varrho_\mu}$ if $(\Lambda \ni)\mu \neq \lambda$,

In order to show that α is an automorphism let us consider the mapping β which assignes to any $a(\in A)$ the system of the a_λ's; β is an isomorphism of A onto the (external) direct product of the A_λ's. The mapping γ which assignes to a the system of its components a'_λ defined by

$$a'_\lambda \equiv a \pmod{\varrho'_\lambda}$$

and

$$a'_\lambda \equiv e \pmod{\varrho'_\mu} \quad \text{for} \quad \mu \neq \lambda,$$

is also an isomorphism onto the same structure. Now, we have $\alpha = \beta\lambda^{-1}$.

B) If $a \in A_\lambda$, then

(3) $\mu \neq \lambda$ implies $a \equiv e \pmod{\varrho_\mu}$, therefore $a_\lambda = a$ holds, and $\mu \neq \lambda$ implies $a_\mu = e$. Thus we have the congruences

$$a\alpha \equiv a \pmod{\varrho'_{\lambda'}}$$

and

$$a\alpha \equiv e \pmod{\varrho'_\mu} \quad \text{if} \quad \mu \neq \lambda'$$

(by the definition of α). The congruences (3) assure that $\mu \neq \lambda'$ implies $a \equiv e \pmod{\varrho'_\mu}$, therefore we have $a\alpha \equiv a \pmod{\varrho'_\nu}$ for every $\nu \in \Lambda$ and this means that $a(\in A_\lambda)$ is fixed by α.

The following statement (ii) is obviously equivalent to those under (i) and (iii) for any pair of elements $a \in A$, $b \in A$ and for an arbitrary λ:

(i) $$a \equiv b \pmod{\varrho_\lambda}$$

(ii) $$a_\lambda \equiv b_\lambda$$

(iii) $$a\alpha \equiv b\alpha \pmod{\varrho'_{\lambda'}}$$

what means that (1) holds.

C) The equality $\{\varrho_\lambda \alpha_1\} = \{\varrho_\lambda \alpha_2\}$ is equivalent to $\{\varrho_\lambda \alpha_1 \alpha_2^{-1}\} = \{\varrho_\lambda\}$. Therefore, the unicity of α will be proved if we show that $\alpha \neq \iota$ implies $\{\varrho_\lambda\} \neq \neq \{\varrho_\lambda \alpha\}$. ($\alpha$ must satisfy the condition of fixing,[2]) ι denotes the identical automorphism of A). We shall give an indirect proof. Let the equivalence of

$$a \equiv b \pmod{\varrho_\lambda}$$
$$\text{and} \quad a\alpha \equiv b\alpha \pmod{\varrho_{\lambda'}}$$

be true for a suitable permutation $\lambda \to \lambda'$ of Λ. We are going to show that $\alpha = \iota$. By the condition of fixing, if $a \not\equiv e \pmod{\varrho_\lambda}$, and $u \neq \lambda$ implies $a \equiv e \pmod{\varrho_\mu}$, then $a = a\alpha \not\equiv e \pmod{\varrho_{\lambda'}}$, and $\mu \neq \lambda'$ implies $a = a\alpha \equiv e \pmod{\varrho_\mu}$. This means that $a \neq e$, $a \in A_\lambda$ imply that $a \in A_{\lambda'}$, thus $\lambda \to \lambda'$ is the identical permutation of Λ. Let a be an arbitrary element of A. For any index λ the congruence $a \equiv a_\lambda \pmod{\varrho_\lambda}$ holds (where a_λ is constructed by the rule (2)). Therefore we have $a\alpha \equiv a_\lambda \alpha \pmod{\varrho_\lambda}$. But $a_\lambda \alpha = a_\lambda$, so $a\alpha \equiv a \pmod{\varrho_\lambda}$. This is true for every λ, thus $a\alpha = a$.

D) If $a \in A_\lambda$, then $\mu \neq \lambda$ implies $a \equiv e \pmod{\varrho_\mu}$, therefore $a \equiv e \pmod{\varrho_\mu \alpha}$, thus $\nu \neq \lambda'$ implies $a \equiv e \pmod{\varrho'_\nu}$, what means $a \in A'_{\lambda'}$. The proof of $A_\lambda \subseteq A'_{\lambda'}$ is completed by the argument that λ' is the same index for each element of A_λ. The inverse inclusion $A_\lambda \supseteq A'_{\lambda'}$ can be verified by a similar inference, starting with the form $\{\varrho_\lambda\} = \{\varrho'_\lambda \alpha^{-1}\}$ of (1).

Theorem 2. *Let us consider the mapping which assignes to any ($\mathfrak{SC}$)-system $\{\varepsilon_\lambda\}$ of endomorphisms of A the ($\mathfrak{SC}$)-system of congruence relations induced by the ε_λ's. The domain of values of this mapping is the collection of every ($\mathfrak{SC}$)-system of congruence relations. If we restrict our attention only to the ($\mathfrak{IOSC}$)-systems of endomorphisms, then the mapping mentioned above becomes an one-to-one correspondence, the domain of values remains unchanged.*

PROOF. Let $\{\varrho_\lambda\}$ be an ($\mathfrak{SC}$)-system of congruence relations. Let us define the system of endomorphisms $\{\varepsilon_\lambda\}$ by what follows:

$$a\varepsilon_\lambda \equiv a \pmod{\varrho_\lambda}, \quad \text{and} \quad \mu \neq \lambda \quad \text{implies} \quad a\varepsilon_\lambda \equiv e \pmod{\varrho_\mu}.$$

One can easily prove that $\{\varepsilon_\lambda\}$ is an ($\mathfrak{IO}$)-system which induces $\{\varrho_\lambda\}$.

Thus we have got that every ($\mathfrak{SC}$)-system of congruence relations is induced by at least one ($\mathfrak{IOSC}$)-system of endomorphisms. We are going to prove that two distinct ($\mathfrak{IOSC}$)-systems cannot induce identical congruence relations. This can be done by elaborating the following principle: for any ($\mathfrak{IOSC}$)-system $\{\varepsilon_\lambda\}$, if we regard the induced ($\mathfrak{SC}$)-system $\{\varrho[\varepsilon_\lambda]\}$ of congruence relations, and construct an ($\mathfrak{IOSC}$)-system $\{\varepsilon'_\lambda\}$ by the procedure seen

[2]) I. e. α leaves fixed any element of each A_λ.

in the beginning of the proof, then $\{\varepsilon_\lambda\}=\{\varepsilon'_\lambda\}$. In fact, for arbitrary $a\in A$ and $\lambda\in\Lambda$ we have $(a\varepsilon_\lambda)\varepsilon_\lambda=a\varepsilon_\lambda$, therefore

$$a\varepsilon_\lambda\equiv a\equiv a\varepsilon'_\lambda\ (\mathrm{mod}\ \varrho[\varepsilon_\lambda])$$

and $\lambda\neq\mu\in\Lambda$ implies $(a\varepsilon_\lambda)\varepsilon_\mu=a(\varepsilon_\lambda\varepsilon_\mu)=e$, therefore

$$a\varepsilon_\lambda\equiv e\equiv a\varepsilon'_\lambda\ (\mathrm{mod}\ \varrho[\varepsilon_\mu]);$$

thus $a\varepsilon_\lambda=a\varepsilon'_\lambda$.

Corollary. *For any ($\mathfrak{S}\mathfrak{C}$)-system $\{\varepsilon_\lambda\}$ of endomorphisms there is exactly one ($\mathfrak{J}\mathfrak{O}\mathfrak{S}\mathfrak{C}$)-system $\{\varepsilon'_\lambda\}$ of endomorphisms such that the two systems induce identical congruence relations. The ε'_λ's can be characterized by the equalities*

$$(a\varepsilon'_\lambda)\varepsilon_\lambda=a\varepsilon_\lambda,\quad \textit{and}\quad (a\varepsilon'_\lambda)\varepsilon_\mu=e\quad \textit{for any}\quad \mu\neq\lambda.$$

§ 4.

In this § we do not suppose the existence of a unit element.

Let A be the direct product of its factor algebras A/ϱ_λ. We shall define the following two subgroups $G_1\supseteq G_2$ of the group G of the automorphisms of A. The automorphism α belongs to G_1 if and only if for each $\lambda\in\Lambda$ there exist indices $\mu,\nu\in\Lambda$ such that $\varrho_\lambda\alpha=\varrho_\mu$ and $\varrho_\lambda\alpha^{-1}=\varrho_\nu$; $\alpha\in G_2$ if and only if $\varrho_\lambda\alpha=\varrho_\lambda$ for any $\lambda\in\Lambda$.

Theorem 3. *G_1 is a splitting Schreierian extension of G_2 by an (unrestricted) direct product of symmetric groups. G_2 is isomorphic to the (unrestricted) direct product of the automorphism groups of the algebras A/ϱ_λ.*

PROOF. If the automorphisms α_λ $(\lambda\in\Lambda)$ of A are such that $\varrho_\lambda\alpha_\lambda=\varrho_{\lambda'}$ where $\lambda\to\lambda'$ is a permutation of Λ, then the mapping $a\to a\alpha$ defined by $a\alpha\equiv a\alpha_\lambda\ (\mathrm{mod}\ \varrho_{\lambda'})$ (for any λ) is an automorphism of A.[3])

Any automorphism $\alpha\in G_1$ of A induces a permutation $\lambda\to\lambda'$ of the index set Λ defined by $\varrho_\lambda\alpha=\varrho_{\lambda'}$. Λ splits into equivalence classes (λ and λ' are equivalent if and only if A/ϱ_λ and $A/\varrho_{\lambda'}$ are isomorphic). Any permutation $\bar\alpha$ of Λ such that for any $\lambda\in\Lambda$, $\lambda\bar\alpha$ and λ belong to the same class, is induced by some α (by the first sentence of the proof). The mapping $\alpha\to\bar\alpha$ is a homomorphism of G_1 with kernel G_2 onto the group $\bar G_1$ formed by the induced permutations of Λ. $\bar G_1$ is obviously the direct product of the sym-

[3]) E. g. for the operation $a\cdot b$ of two variables we have

$$(a\cdot b)\,\alpha\equiv(a\cdot b)\,\alpha_\lambda=(a\alpha_\lambda)\cdot(b\alpha_\lambda)\equiv(a\alpha)\cdot(b\alpha)\ (\mathrm{mod}\ \varrho_{\lambda'})$$

for any λ. The inverse mapping canbe given by $a\alpha^{-1}\equiv a\alpha_\lambda^{-1}\ (\mathrm{mod}\ \varrho_\lambda)$. Indeed, this definition of α^{-1} implies $a\alpha^{-1}\alpha_\lambda\equiv a\ (\mathrm{mod}\ \varrho_{\lambda'})$, so $a\alpha^{-1}\alpha\equiv a\ (\mathrm{mod}\ \varrho_{\lambda'})$ for each λ.

metric groups of the equivalence classes of Λ. Our following aim is to construct a subgroup H of G_1 for which $G_1 = G_2H$ and $G_2 \cap H = 1$ hold. Such a subgroup can be given by defining a transitive system of isomorphisms $\varphi_{\lambda_1, \lambda_2}$ among the A/ϱ_λ's in any equivalence class of Λ. For any element $\bar{\alpha}$ of $\bar{G}_1$, we define an automorphism α^* of A by the inclusions

$$a\alpha^* \in \bar{a}^\lambda \varphi_{\lambda, \lambda\alpha} \quad (\lambda \text{ runs through } \Lambda)$$

where $\bar{a}^\lambda$ denotes the congruence class containing a modulo ϱ_λ. The α^*'s form a subgroup H with the required properties.

Every element of G_2 induces an automorphism in any A/ϱ_λ. Thus we have a homomorphism of G_2 into the direct product $\times G_\lambda$ of the automorphism groups of the A/ϱ_λ's. If α and β are distinct elements of G_2, then we have $a\alpha \not\equiv a\beta \pmod{\varrho_\lambda}$ for some $a \in A$ and $\lambda \in \Lambda$, therefore the mentioned homomorphism is an isomorphism. Our last aim is to prove that this isomorphism maps G_2 *onto* $\times G_\lambda$. Let α_λ be an element of G_λ for each λ, then for any $a \in A$ the system of inclusions

$$a\alpha \in \bar{a}^\lambda \alpha_\lambda \qquad (\lambda \text{ runs through } \Lambda)$$

has exactly one solution in A. The mapping $a \to a\alpha$ is an automorphism of A inducing the prescribed α_λ's.

§ 5.

G. BIRKHOFF states on the page 87[4]) of his book [2] that *the representations of an algebra A as a direct union $A = A_1 \times \ldots \times A_r$ correspond one-one with the sets of permutable congruence relations $\theta_1, \ldots, \theta_r$ on A satisfying $\theta_1 \cap \ldots \cap \theta_r = 0$ and $(\theta_1 \cap \ldots \cap \theta_{i-1}) \cup \theta_i = I$ $(i = 2, \ldots, r)$.* The proof of this theorem contains an incorrect step. One can apply the lemma (exposed on the same page) only in the case if $\theta_1 \cap \cdots \cap \theta_{i-1}$ is permutable with θ_i. This assumption is, however, not fulfilled in general, because it is not implied by the congruence relations $\theta_1, \theta_2, \cdots, \theta_r$ being pair-wise permutable.

Professor L. FUCHS and Dr. G. SZÁSZ have kindly communicated their counter-example disproving the theorem of BIRKHOFF. Let A be a semigroup containing 7 elements denoted by $(0,0,0)$, $(0,0,1)$, $(0,1,0)$, $(0,1,2)$, $(1,0,0)$, $(1,0,2)$, $(1,1,1)$. The operation is defined by the rule $xy = x$ for any pair $x \in A$, $y \in A$. So every equivalence relation in A is a congruence relation. Let the congruence relations θ_1, θ_2, θ_3 defined by what follows: $x \equiv y \pmod{\theta_i}$

[4]) Page 131 of the Russian edition.

if and only if the i-th component of x is equal to the i-th component of y ($x \in A$, $y \in A$, i can be 1, 2 or 3). Then we have

$$\theta_j \theta_k = \theta_j \cup \theta_k = \theta_k \theta_j = \mathrm{I} \quad \text{if} \quad j \neq k,$$

and $(\theta_j \cap \theta_k) \cup \theta_l = \mathrm{I}$ if j, k, l are distinct. But A is obviously not isomorphic to the direct product of its factor semigroups A/θ_1, A/θ_2 and A/θ_3.

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