On algebraically closed modules.

To Professor O. Varga on his 50th birthday.

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§ 1. Introduction.

The concept of algebraically closed module, defined by A. KERTÉSZ [4], coincides with the concept of injective module, which plays an important role in homological algebra.[1]) In [4] KERTÉSZ studies the properties of such modules, and it is apparent from his results that they constitute a natural generalization of algebraically closed (in other terminology divisible) abelian groups. In most cases he finds that the corresponding generalizations of theorems on algebraically closed groups hold true for algebraically closed modules.

However certain results do not retain their validity in the theory of algebraically closed modules over arbitrary rings. For instance, any discrete direct sum of algebraically closed groups is algebraically closed, but as KERTÉSZ shows in [4] this no longer holds for algebraically closed modules over an arbitrary ring. He therefore raises the following problem: what rings R have the property that any disrete direct sum of algebraically closed R-modules is algebraically closed. In § 3 of the present paper we show that this, and also that certain other module-theoretic properties, are characteristic of Noetherian rings.

In § 4 we study the structure of algebraically closed modules over Noetherian rings. First we give another characteristic property of Noetherian rings, proving that any algebraically closed R-modules is a disrete direct sum of minimal algebraically closed R-modules if and only if R is Noetherian; then we characterize the minimal algebraically closed R-modules in terms of the left ideals of R^*. This enables us to give a complete set of invariants for any algebraically closed module over a Noetherian ring.

Finally we give a necessary and sufficient condition that two modules over a Noetherian ring have isomorphic algebraic closures.

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[1]) For terminology and notation see § 2. — In the literature the concept of injective module is mostly defined in the case unitary modules. (See for instance CARTAN—EILENBERG [2].)

§ 2. Preliminaries.

The reader is referred for the most of the module-theoretic concepts used in the sequel, and in particular for that of a system of equations over an R-module, to the paper [4] of KERTÉSZ. Here we give only those concepts and results which are fundamental to our investigations.

A ring R is called Noetherian if it satisfies the ascending chain condition for left ideals, i. e. if every ascending chain of left ideals of R contains only a finite number of different members. It is well known that this is equivalent to the condition that every left ideal in R has a finite number of generators.

The theory of modules over an arbitrary ring R may be reduced in almost every respect to the theory of unitary modules over the Dorroh-extension R^* of R. See KERTÉSZ [3], [4]. The ring R^* consist of all pairs $\langle r, n\rangle$ ($r \in R$; $n \in \mathfrak{G}$, $\mathfrak{G}$ being the ring of the natural numbers) with the following operations:

$$\left.\begin{aligned}\langle r, n\rangle + \langle s, m\rangle &= \langle r+s, n+m\rangle\\ \langle r, n\rangle\langle s, m\rangle &= \langle rs+ns+mr, nm\rangle\end{aligned}\right\} \; rs \in R;\; n, m \in \mathfrak{G}.$$

Thus R^* has $\langle 0, 1\rangle$ as identity; further R is an ideal of R^* and every left ideal of R is also a left ideal of R^*.

It is easy to see that any R-module G can be considered as a unitary R-module with the operation

$$\langle r, n\rangle g = rg + ng \qquad (g \in G).$$

By a direct sum $\Sigma A_i = A_1 + \cdots + A_i + \cdots$ of modules we always mean a discrete direct sum. If $G = A + B$, we say that A (and of course also B) is a direct summand of the module G. If A is a submodule of G such that, for any submodule M of G which is maximal among the submodules that intersect A trivially, $G = A + M$, then A will be called a strictly direct summand of G.

Let Γ be a set of indices. The system of submodules $(\ldots, A_\nu, \ldots)_{\nu\in\Gamma}$ of A is said to be independent if for every finite ordered subset $(\nu_1, \nu_2, \ldots \nu_k)$ of Γ, the condition $A_{\nu_1} \cap \{A_{\nu_2}, A_{\nu_3}, \ldots, A_{\nu_k}\}$ is satisfied, i. e. if their sum in A is direct. A system of elements $(\ldots, a_\nu, \ldots)_{\nu\in\Gamma}$ $(a_\nu \in A)$ is linearly independent if the system of submodules $(\ldots, \{a_\nu\}, \ldots)_{\nu\in\Gamma}$ is independent, where $\{a\}$ denotes the cyclic module generated by a. (We shall always use $\{\cdots\}$ for submodules or left ideals generated by the elements listed inside the brackets.)

Let R be a subring of the ring S, and let L be a left ideal of R or of S. The additiv group of L will often be considered as an R-module, in this case denoted by $L_{(R)}$, the product rl $(r \in R, l \in L)$ being defined as in S.

We write $0(g)$ for the order of the element g $(\in G)$, i. e. for the set of all elements $\langle r, n\rangle$ $(\in R^*)$ with $\langle r, n\rangle g = 0$. Obviously $0(g)$ is a left ideal of R^*. It is easy to show that $R^*_{(R)}/0(g)_{(R)} \cong \{g\}$; in general, if L is an arbitrary left ideal of R^* then $R^*_{(R)}/L_{(R)} \cong \{a\}$, and $0(a) = L$, where $a = \langle 0, 1\rangle + L_{(R)}$ $(\in R^*_{(R)}/L_{(R)})$.

Let H be a submodule of the R-module G, and let

$$\langle r_\nu, n_\nu\rangle x = h_\nu \qquad (\langle r_\nu, n_\nu\rangle \in R^*;\, h_\nu \in H;\, \nu \in \Gamma) \tag{1}$$

be a compatible system of equations in one unknown. H is said to be a pure submodule of G if the solvability in G of the system of equations (1) implies its solvability in H.

If every compatible system of type (1) is solvable in H, then H will be called an algebraically closed module. (The concept of injective module, used mostly in the unitary case, is equivalent to this.)

Set H be a submodule of the R-module G. An element g $(\in G)$ is algebraic over H if there exists an element $\langle r, n\rangle (\in R^*)$ such that $(0 \neq)$ $\langle r, n\rangle g \in H$. In the contrary case g is transcendental over H. The R-module G is algebraic over H if every element g of G is algebraic over H. The following theorems on algebraically closed modules, which can be found in KERTÉSZ [3], [4], will be employed in our considerations without further reference.

If G is an algebraically closed module, then:

(I) G is strictly direct summand in any of its extensions.

(II) Let L be an arbitrary left ideal of R^*. If φ is a homomorphism of L into G, then there exists an element g of G such that $l\varphi = lg$ for every element l of L.

(III) If φ is a homomorphism of some submodule A of an arbitrary R-module B into G, then φ can be extended to a homomorphism of the whole of B into G.

Two extension G_0 and G_1 of the R-module G will be called equivalent, if it is possible to establish between them an isomorphism under which the elements of G remain fixed.

(IV) Let R be an arbitrary ring.

a) Any R-module G has an algebraically closed extension.

b) For any R-module G_0 the following assertions are equivalent:

b$_1$) G_0 is a maximal algebraic extension of G, b$_2$) G_0 is an algebraically closed algebraic extension of G,

b$_3$) G_0 is a minimal algebraically closed extension of G.

c) Any R-module G has one, and up to equivalence only one extension G_0 having the properties b$_1$, b$_2$, b$_3$.

We call the minimal algebraically closed extension of G the *algebraic closure* of G and denote it by $\mathfrak{A}(G)$. If G is a submodule of an R-module B and A is an algebraic closure of G in B then we write $A=\mathfrak{A}_B(G)$.

§ 3. A characterization of Noetherian rings.

In his paper [4] KERTÉSZ shows that the discrete direct sum of arbitrarily many algebraically closed R-modules is in general not algebraically closed. He raises the problem: to describe the rings R which are such that the discrete direct sum of arbitrarily many algebraically closed R-modules is algebraically closed. In this section we solve this problem, showing that only the Noetherian rings have this property.

Another module-theoretic property for rings can be described as follows.

Property P. For an arbitrary R-module G and for any system of equations

$$r_\nu x = g_\nu \qquad (r_\nu \in R^*;\ g_\nu \in G;\ \nu \in \Gamma) \tag{2}$$

over G the solvability (in G) of every finite subsystem of (2) implies the solvability (in G) of the whole system.

We are now ready to formulate our first result.

Theorem 1. *For an arbitrary ring R, the following conditions are equivalent.*

a) *R is a Noetherian ring,*

b) *R^* is a Noetherian ring,*

c) *R has the Property P,*

d) *in arbitrary R-module G, the union of any ascending chain of pure submodules (of G) is a pure submodule,*

e) *the union of any ascending chain of algebraically closed R-modules is algebraically closed,*

f) *any discrete direct sum of algebraically closed R-modules is algebraically closed.*

PROOF. a) $\Longrightarrow$ b) [2]) Consider any ascending chain

$$A_1 \subseteq A_2 \subseteq \cdots \subseteq A_i \subseteq \cdots$$

[2]) This step is due to L. G. KOVÁCS.

of left ideals is R^* and put $B_i = A_i \cap R$, $C_i = \{A_i, R\}$. Then

$$B_1 \subseteq B_2 \subseteq \cdots \subseteq B_i \subseteq \cdots$$

and

$$C_1 \subseteq C_2 \subseteq \cdots \subseteq C_i \subseteq \cdots$$

are ascending chains of left ideals in the Noetherian ring R and $R^*/R \simeq \mathfrak{E}$ respectively; so, from a certain index n, $B_i = B_{i+1}$ and $C_i = C_{i+1}$ must hold for every $i(>n)$. Take such an i; first $A_{i+1} \subseteq C_{i+1} = C_i = \{A_i, R\}$ means that $A_{i+1} = A_{i+1} \cap \{A_i, R\}$; next, as $A_i \subseteq A_{i+1}$ we can deduce that $A_{i+1} = A_{i+1} \cap \cap \{A_i, R\} = \{A_i, A_{i+1} \cap R\} = \{A_i, B_{i+1}\}$. Finally $B_{i+1} = B_i \subseteq A_i$ gives that $A_{i+1} = \{A_i, B_{i+1}\} = A_i$. Hence our original chain in R^* cannot contain more than n different terms, and as it was an arbitrary chain R^* Noetherian.

b) $\Longrightarrow$ c) Let us suppose that R^* is a Noetherian ring and

$$(3) \qquad r_\nu x = g_\nu \qquad (r_\nu \in R^*;\ g_\nu \in G;\ \nu \in \Gamma)$$

is a system of equations over an arbitrary R-module G such that its finite subsystems are all solvable in G. Denote by L the left ideal of R^* generated by the elements $r_\nu (\nu \in \Gamma)$. Since R^* is Noetherian L is finitely generated, and moreover we may choose a finite set of generators of L from the elements r_ν $(\nu \in \Gamma)$:

$$L = \{r_{\nu_1}, r_{\nu_2}, \ldots, r_{\nu_k}\} \qquad (\nu_i \in \Gamma;\ i = 1, 2, \ldots, k).$$

Consider the following system of equations:

$$(4) \qquad r_{\nu_i} x = g_{\nu_i} \qquad (i = 1, 2, \ldots, k).$$

This is finite subsystem of (3) and thus by our conditions it is solvable in G.

Let $g(\in G)$ be a solution of (4), so that

$$r_{\nu_i} g = g_{\nu_i} \qquad (i = 1, 2, \ldots, k).$$

We shall prove that $g(\in G)$ is a solution of the whole system (3).

Since every finite subsystem of (3) is solvable in G, the mapping $r_\nu \to g_\nu$ induces a homomorphism φ of the left ideal L onto $H = \{\ldots, g_\nu, \ldots\}_{\nu \in \Gamma}$, a submodule of G, and the following equalities are valid:

$$r_{\nu_i} g = r_{\nu_i} \varphi \qquad (i = 1, 2, \ldots, k).$$

Now if

$$r_\nu = \sum_{i=1}^{k} s_i^\nu r_{\nu_i}$$

then for arbitrary $\nu \in \Gamma$,

$$r_\nu g = \left(\sum_{i=1}^{k} s_i^\nu r_{\nu_i}\right) g = \sum_{i=1}^{k} s_i^\nu (r_{\nu_i} g) = \sum_{i=1}^{k} s_i^\nu (r_{\nu_i} \varphi) = \left(\sum_{i=1}^{k} s_i^\nu r_{\nu_i}\right) \varphi = r_\nu \varphi = g_\nu$$

showing that g is a solution of (3).

c) $\Longrightarrow$ d) Let us suppose that R has the Property P and that G is an arbitrary R-module. Consider the following ascending chain:

$$S_1 \subseteq S_2 \subseteq \cdots \subseteq S_\mu \subseteq \cdots \qquad (\mu \in \Delta),$$

where the S_μ $(\mu \in \Delta)$ are pure submodules of G and Δ is a well-ordered (non empty) set of indices. Let S denote the union of S_μ $(\mu \in \Delta)$, $S = \bigcup_{\mu \in \Delta} S_\mu$.

If

$$r_\nu x = g_\nu \qquad (g_\nu \in S,\ r_\nu \in R^*,\ \nu \in \Gamma) \tag{5}$$

is an arbitrary system of equations over S, solvable in G, then obviously every finite subsystem

$$r_{\nu_i} x = g_{\nu_i} \qquad (i = 1, 2, \ldots, k) \tag{6}$$

of (5) has a solution in G. To every system of equations (6) we can choose an index $\tau \in \Delta$, such that $g_{\nu_i} \in S_\tau$ $(i = 1, 2, \ldots, k)$ and, since S_t is pure in G, the system of equations (6) is solvable in S_τ, hence solvable in S. But R has Property P, so the whole system of equations (5) is solvable in S, thus S is pure in G.

d) $\Longrightarrow$ e) Consider the ascending chain of algebraically closed R-modules:

$$A_1 \subseteq A_2 \subseteq \cdots \subseteq A_\mu \subseteq \cdots \qquad (\mu \in \Delta).$$

Let us denote by A the union of the A_μ's $(\mu \in \Delta)$, $A = \bigcup_{\mu \in \Delta} A_\mu$ and let $\bar{A}$ be an algebraically closed extension of A. Since each A_μ $(\mu \in \Delta)$ is pure in $\bar{A}$, by the condition d) A is pure in $\bar{A}$, hence A is an algebraically closed R-module.

e) $\Longrightarrow$ f) This is clear, since any discrete direct sum of algebraically closed R-modules in the union of an ascending chain of algebraically closed R-modules.

f) $\Longrightarrow$ a) It is sufficient to prove that if R is not a Noetherian ring then we can construct algebraically closed R-modules A_{ν_i} $(i = 1, 2, \ldots)$, such that $A = \sum_{i=1}^{\infty} A_{\nu_i}$ is not algebraically closed.

We show a little more; namely that such A_{ν_i} can be chosen from any set Ω of algebraically closed modules A_ν provided Ω has the following property. Consider all maximal left ideals $K_\mu (\mu \in \Gamma)$ of R^*; put $c_\mu = \langle 0, 1 \rangle + K_{\mu(R)}$ $(\in R^*_{(R)}/K_{\mu(R)})$ and $C_\mu = \mathfrak{A}(\{c_\mu\})$. The property required of Ω is that it contain countably many copies of A_ν, isomorphic to C_μ, for each μ in Γ. This property ensures that to any left ideal K of R^* there is an element a in one of the $A_\nu (\in \Omega)$ such that $K \subseteq O(a)$.

Let us suppose that R does not satisfy the ascending chain condition, in other words that there exist elements:

$$l_0=0,\ l_1,\ l_2,\ \ldots,\ l_n,\ \ldots \qquad (\in R)$$

for which

$$l_{n+1}\notin\{l_0,\ l_1,\ \ldots,\ l_n\}.$$

Set

$$L_n=\{l_0,\ l_1,\ \ldots,\ l_n\} \qquad (n=0,\ 1,\ 2,\ \ldots)$$

$$L=\bigcup_n L_n$$

We choose the A_{ν_i} and define homomorphism φ_i of L_i into $B_i=A_{\nu_1}+ +\cdots+A_{\nu_i}$ such that $L_i\varphi_i \subsetneqq B_{i-1}$ and φ_i is an extension of φ_{i-1}, for every $i=1,2,\ldots,$ using the following induction.

Choose A_{ν_1} (from Ω) such that it contains a nonzero element a_1 for which $O(a_1)\supseteq O(l_1)$ and define $l_1\varphi_1=a_1$. Clearly this defines φ_1 on L_1 and $L_1\varphi_1\subseteq A_{\nu_1}$ but $L_1\varphi_1\subsetneqq B_0=0$. Now assume that $A_{\nu_1},\ \ldots,\ A_{\nu_i};\ \varphi_1,\ \ldots,\ \varphi_i$ are already defined according to the above requirements. The set Ω_i which remains after the removal of $A_{\nu_1},\ \ldots,\ A_{\nu_i}$ from Ω also has the postulated property of Ω.

(i) If $L_i\cap\{l_{i+1}\}=0$ then $L_{i+1}=L_i+\{l_{i+1}\}$ and so φ_{i+1} can be defined by φ_i and, after the choice of a nonzero a_{i+1} in some $A_{\nu_{i+1}}(\in\Omega_i)$ such that $O(a_{i+1})\supseteq O(l_{i+1})$, by $l_{i+1}\varphi_{i+1}=a_{i+1}$.

(ii) If $L_i\cap\{l_{i+1}\}\neq 0$, let K be the left ideal consisting of all elements s of R^* for which $sl_{i+1}\in L_i$ and choose $A_{\nu_{i+1}}(\in\Omega_i)$ such that it has an element a_{i+1} whose order $O(a_{i+1})$ contains K:

$$O(a_{i+1})\supseteq K. \tag{7}$$

The mapping $s\to(sl_{i+1})\varphi_i$ $(s\in K)$ is a homomorphism of K into B_i, for

$$(s_1+s_2)\to[(s_1+s_2)l_{i+1}]\varphi_i=(s_1l_{i+1})\varphi_i+(s_2l_{i+1})\varphi_i$$

and

$$rs\to[(rs)l_{i+1}]\varphi_i=[r(sl_{i+1})]\varphi_i=r[(sl_{i+1})\varphi_i].$$

By (II) (see § 2) there exists an element $a_0(\in B_i)$ such that

$$(sl_{i+1})\varphi_i=sa_0 \tag{8}$$

for any $s\in K$. On account of (7), this is the same as

$$(sl_{i+1})\varphi_i=s(a_0+a_{i+1}). \tag{9}$$

Now we show that (9) can be extended to the definition of a mapping φ_{i+1} with the required properties, namely to

$$(z+rl_{i+1})\varphi_{i+1}=z\varphi_i+r(a_0+a_{i+1}) \tag{10}$$

$(z \in L_i,\ r \in R^*)$. First of all we have to ensure that (10) defines a single-valued mapping of L_{i+1}. If an element of L_{i+1} has two different representations in the form used here, say,

$$z_1 + r_1 l_{i+1} = z_2 + r_2 l_{i+1};$$

then $z_1 = z_2 + (r_2 - r_1) l_{i+1}$ (obviously $(r_2 - r_1) \in K$) and, using (9), we see that

$$\begin{aligned} z_1 \varphi_i + r_1(a_0 + a_{i+1}) &= [z_2 + (r_2 - r_1) l_{i+1}] \varphi_i + r_1(a_0 + a_{i+1}) = \\ &= z_2 \varphi_i + [(r_2 - r_1) l_{i+1}] \varphi_i + r_1(a_0 + a_{i+1}) = \\ &= z_2 \varphi_i + (r_2 - r_1)(a_0 + a_{i+1}) + r_1(a_0 + a_{i+1}) = z_2 \varphi_i + r_2(a_0 + a_{i+1}). \end{aligned}$$

This shows that the image obtained by applying (10) to the first form of our element is the same as that obtained from the second form.

The rest is almost obvious; φ_{i+1} preserves the operations, extends φ_i, $L_{i+1}\varphi_{i+1} \subseteq B_{i+1}$; furthermore

$$l_{i+1}\varphi_{i+1} = a_0 + a_{i+1} \notin B_i$$

which gives that $L_{i+1}\varphi_{i+1} \subsetneqq B_i$.

Let φ denote the union of all the φ_i. Naturally φ is a homomorphism of $L = \bigcup_n L_n$ into

$$A = \bigcup_j B_j = \sum_i A_{\nu_i}. \tag{11}$$

Consider the (obviously compatible) system of equations

$$lx = l\varphi \qquad (l \in L) \tag{12}$$

over A. There is no solution of the system (12) in A. Indeed, any element a of A has only a finite number of nonzero components in the direct decomposition (11) and so for some i $a \in B_i$; but then $l_{i+1}a \in B_i$, and $l_{i+1}\varphi \notin B_i$, which show that a cannot be a solution of (12). Thus A is not algebraically closed, and this completes the proof of Theorem 1.

§ 4. Algebraically closed modules over Noetherian rings.

1. To describe the structures of all algebraically closed R-modules over an arbitrary ring R seems to be a rather difficult problem. This is perhaps partly due to the fact that the discrete direct sum of algebraically closed R-modules is not, in general, algebraically closed. This difficulty can be avoided, as we have just seen, only if our considerations are restricted to modules over Noetherian rings. The aim of this section is to solve the problem for this particular case, in other words, to give a description of all algebraically closed modules over Noetherian rings. We close the section by answering a related question.

DEFINITION 1. An R-module $A(\neq 0)$ is a *minimal algebraically closed module* if it is algebraically closed and satisfies one of the following conditions:

a) A has no algebraically closed proper submodules,

b) A is the algebraic closure of any of its nonzero submodules.

The conditions a) and b) are equivalent.

The next simple lemma gives a characterization of the minimal algebraically closed R-modules which will play an important role in the sequel.

Lemma 1. *An algebraically closed R-module is a minimal algebraically closed module if and only if the intersection of any two of its nonzero submodules is likewise nonzero.*

PROOF. If B is a nonzero submodule of the minimal algebraically closed module A then A, being the algebraic closure of B, is algebraic over B. This means that no cyclic submodule and therefore a fortriori no nonzero submodule of A can intersect B in 0.

Conversely, the existence of an algebraically closed proper submodule B in the algebraically closed R-module A would imply that $A=B+C$, $(C\neq 0)$, $B\cap C=0$. This possibility is excluded by the condition in the lemma.

Theorem 2. *A ring R has the property that every algebraically closed R-module is a discrete direct sum of minimal algebraically closed R-modules if and only if R is a Noetherian ring.*

PROOF. Let R be a Noetherian ring. We first show that if $A(\neq 0)$ is an algebraically closed R-module then it has a minimal algebraically closed submodule. Let $B=\mathfrak{A}_A(\{g\})$ where g is a nonzero element of A. If B is not a minimal algebraically closed module then $B=B_1+C_1$; if C_1 is not a minimal algebraically closed module then $C_1=B_2+C_2$, giving $B=B_1+B_2+C_2$. Continuing this process we necessarily arrive at a (finite) index i such that C_i is a minimal algebraically closed module. Otherwise we would obtain decompositions $B=B_1+B_2+\cdots+B_n+C_n$ for each natural number n; putting $D_n=B_1+\cdots+B_n$, we see by Theorem 1 that the union $D=\bigcup\limits_n D_n$ is algebraically closed, so that $B=C+D$, and

$$B=C+B_1+\cdots+B_n+\cdots \tag{13}$$

The decomposition $g=c+b_1+\cdots+b_N$ corresponding to (13) gives $\{g\}\subseteq$

$\subseteq C+D_N$; but $C+D_N$ is an algebraically closed proper submodule of B which is impossible since $B=\mathfrak{A}(\{g\})$.

Consider next a maximal independent set

$$(\ldots, A_\mu, \ldots)_{\mu\in\Gamma} \tag{14}$$

of minimal algebraically closed submodules of A, and let B be their direct sum. From Theorem 1 we conclude that B is algebraically closed, so that $A=B+C$. Since the system (14) is a maximal independent one $C=0$, which means that $A=B=\sum_{\mu\in\Gamma} A_\mu$. This completes the first part of our proof.

Conversely, suppose that R is not a Noetherian ring, but that every algebraically closed R-modules is a discrete direct sum of minimal algebraically closed R-modules. Consider the set of all maximal left ideals $K_\mu(\mu\in\Gamma)$ of R^* and put $\mathfrak{m}_\mu$ for the cardinal number of the algebraically closed module $C_\mu=\mathfrak{A}(\{c_\mu\})$ where $c_\mu=\langle 0,1\rangle+K_{\mu(R)}$ $(\in R^*_{(R)}/K_{\mu(R)})$. First we prove that C_μ is a minimal algebraically closed module. Let $B(\neq 0)$ be a submodule of C_μ. Since the elements of B are algebraic over $\{c_\mu\}$, and since from the maximality of K_μ it follows that $\{c_\mu\}$ is a minimal R-module, we see that $\{c_\mu\}\subseteq B$. This shows that the intersection of any two nonzero submodules of C_μ is likewise nonzero; hence by Lemma 1 we infer that C_μ is a minimal algebraically closed R-module. Let now.

$$C=\mathfrak{A}\Big(\sum_{\mu\in\Gamma}\sum_{\mathfrak{n}_\mu}\{c_\mu\}\Big), \tag{15}$$

where $\sum_{\mathfrak{n}_\mu}\{c_\mu\}$, denotes the discrete direct sum of $\mathfrak{n}_\mu$ copies of $\{c_\mu\}$, the infinite cardinal numbers $\mathfrak{n}_\mu$ satisfying the proper inequality $\mathfrak{n}_\mu>\mathfrak{m}_\mu$ for each $\mu\in\Gamma$.

By our hypothesis, C is a discrete direct sum of minimal algebraically closed R-modules $A_\nu(\nu\in\Delta)$,

$$C=\sum_{\nu\in\Delta} A_\nu. \tag{16}$$

We show that for each index μ $(\in\Gamma)$ there exists an infinity of direct summands A_{ν_μ} $\big(\nu_\mu\in\Delta_\mu(\subseteq\Delta)\big)$ in the direct decomposition (16) of C, which are such that $C_\mu\simeq A_{\nu_\mu}$. Take c_μ from any of the $\mathfrak{n}_\mu$ factors of (15) that correspond this μ, and consider its decomposition according to (16):

$$c_\mu=b_{\nu_1}+\cdots+b_{\nu_k} \qquad (b_{\nu_i}\in A_{\nu_i}).$$

It is easy to see that $O(c_\mu)=O(b_{\nu_1})\cap\cdots\cap O(b_{\nu_k})$. Since $0(c_\mu)$ is a maximal left ideal of R^*, there follow the equalities $O(c_\mu)=O\big(b_{\nu_1}\big)=\cdots=O(b_{\nu_k})$.

Then the isomorphismus $C_\mu \cong \mathfrak{A}(R^*_{(R)}/O(b_{\nu_i})_{(R)})$ $(i=1,2,\ldots,k)$ show that there exists at least one $A_{\nu_\mu}(\nu_\mu \in \Delta_\mu)$ with the required property.

Let us consider the direct sum of all $A_{\nu_\mu}(\nu_\mu \in \Delta_\nu)$ isomorphic to C_μ, $A^\mu = \sum_{\nu_\mu \in \Delta_\mu} A_{\nu_\mu}$. Then

$$C = A^\mu + \sum_{\nu \notin \Delta_\mu} A_\nu. \tag{17}$$

We show that if the order of an element c $(\in C)$ is the maximal left ideal K_μ then $c \in A^\mu$. Indeed, according to (17),

$$c = b^\mu + b'_{\nu_1} + \cdots + b'_{\nu_k} \qquad (b'_{\nu_i} \in A_{\nu_i})$$

where necessarily $K_\mu \subseteq O(b'_{\nu_i})$ $(i=1,2,\ldots,k)$ and as by our condition $\nu_i \notin \Delta_\mu$ implies $A_{\nu_i} \not\cong C_\mu$, we conclude, that $O(b'_{\nu_k}) = R^*$ and therefore $b'_{\nu_k} = 0$ $(i=1,2,\ldots,k)$.

Now as we have $\mathfrak{n}_\mu$ distinct copies of c_μ, whose order is K_μ, the cardinality of A^μ is at least $\mathfrak{n}_\mu$. On the other hand, the cardinality of $A^\mu = \sum_{\nu_\mu \in \Delta_\mu} A_{\nu_\mu}$ is $\mathfrak{m}_\mu$ times the cardinality of Δ_μ. As $\mathfrak{n}_\mu$ is infinite and greater than $\mathfrak{m}_\mu$ it follows that the cardinality of Δ_μ is at least $\mathfrak{n}_\mu$ and so a fortiori infinite. So we see that this set of the $A_\nu(\nu \in \Delta)$ has the property of the set Ω on p. 316 and so, by the method given there, one can select a subset Δ' of Δ such that $A = \sum_{\nu \in \Delta'} A_\nu$ is not algebraically closed. But $C = A + \sum_{\nu \in \Delta, \notin \Delta'} A_\nu$, so A is a direct summand of the algebraically closed module C. This is clearly a contradiction, completing the proof of Theorem 2.

Applying Theorem 2 to the case $A = \mathfrak{A}(\{g\})$ we obtain the following Corollary.

Corollary. *If $A = \mathfrak{A}(\{g\})$ is a module over a Noetherian ring, then $A = A_1 + A_2 + \cdots + A_n$, where A_i $(i=1,2,\ldots,n)$ are minimal algebraically closed modules.*

Our aim in the following will be to describe the minimal algebraically closed R-modules in terms of the left ideals of R^*.

DEFINITION 2. A left ideal L of R is *reducible* if there exist left ideals $M(\neq L)$, $N(\neq L)$ in R such that $L = M \cap N$. In the contrary case L is *irreducible*.

DEFINITION 3. A decomposition of L

$$L = L_1 \cap L_2 \cap \cdots \cap L_n \tag{18}$$

is *irredundant* if, for every index i, L_i is irreducible and

$$L_i \not\supseteq (L_1 \cap \cdots \cap L_{i-1} \cap L_{i+1} \cap \cdots \cap L_n).$$

It is easy to prove (see for instance McCoy [5] p. 199) the following lemma.

Lemma 2. *If R is a Noetherian ring then every left ideal of R has an irredundant decomposition of the form* (18).

Lemma 3. *The order of any element of a minimal algebraically closed R-module is an irreducible left ideal.*

Proof. Let a be an arbitrary element of a minimal algebraically closed R-module A. By definition $A = \mathfrak{A}_A(\{a\})$ and suppose that

$$O(a) = L_1 \cap L_2 \qquad L_i \neq O(a) \qquad i = 1, 2. \tag{19}$$

Write $a_i = \langle 0, 1 \rangle + L_{i(R)} (\in R^*_{(R)}/L_{i(R)};\ i = 1, 2)$, and consider the element $x = a_1 + a_2$ of the R-module $C = \{a_1\} + \{a_2\}$. Then $O(x) = O(a_1) \cap O(a_2) = L_1 \cap L_2 = O(a)$; hence $\mathfrak{A}(\{x\}) \simeq \mathfrak{A}(\{a\}) = A$ and so $A' = \mathfrak{A}(\{x\})$ is a minimal algebraically closed R-module.

By the condition (19) there exist elements $l_1 \in L_1$, $l_2 \in L_2$ such that $l_1 a_2 \neq 0$, $l_2 a_1 \neq 0$ and therefore $0 \neq l_1 a_2 = l_1 x\ (\in A')$, $0 \neq l_2 a_1 = l_2 x\ (\in A')$. According to Lemma 1 the intersection of the R-modules $\{l_2 a_1\}$, $\{l_1 a_2\}$ is not equal to zero, contrary to the fact that $C = \{a_1\} + \{a_2\}$ is a direct sum.

Theorem 3. *A module A over a Noetherian ring R is a minimal algebraically closed module if and only if $A = \mathfrak{A}(\{a\})$, where $O(a)$ is an irreducible left ideal of R^*.*

Proof. Let A be a minimal algebraically closed R-module. Then for an arbitrary element $(0 \neq) a \in A$, $A = \mathfrak{A}_A(\{a\})$, and by Lemma 3, $O(a)$ is an irreducible left ideal.

On the other hand, let $A = \mathfrak{A}(\{a\})$ and $O(a)$ irreducible. Then by the Corollary to Theorem 2,

$$A = A_1 + A_2 + \cdots + A_n, \tag{20}$$

where A_i $(i = 1, 2, \ldots, n)$ are minimal algebraically closed modules. In this decomposition $a = a_1 + a_2 + \cdots + a_n$, whence $O(a) = O(a_1) \cap \cdots \cap O(a_n)$. Since $O(a)$ is irreducible either $O(a) = O(a_1)$ or $O(a) = \bigcap_{i=2}^{n} O(a_i)$. By induction on the number of the direct summands of the decomposition (20) one can easily see that for some index i, $O(a) = O(a_i)$, hence that $A \simeq A_i$ where $A_i \simeq \mathfrak{A}_{A_i}(\{a_i\})$. So it follows that A is a minimal algebraically closed R-module.

Lemma 4. *If R is a Noetherian ring, A is an algebraically closed R-module and the system of elements $(\ldots, a_\nu, \ldots)_{\nu \in \Gamma}$ is a maximal linearly independent system in A, then $A = \sum_{\nu \in \Gamma} A_\nu$ where $A_\nu = \mathfrak{A}_A(\{a_\nu\})$.*

PROOF. To prove that for every finite ordered subset

$$(\nu_1, \ldots, \nu_k)(\subseteqq \Gamma), \quad A_{\nu_1} \cap \{A_{\nu_2}, \ldots, A_{\nu_k}\} = 0,$$

we show that an equality

$$b_{\nu_1} + b_{\nu_2} + \cdots + b_{\nu_k} = 0 \qquad (b_{\nu_i} \in A_{\nu_i}) \tag{21}$$

implies $b_{\nu_i} = 0$ $(i = 1, 2, \ldots, k)$. Suppose on the contrary that $b_{\nu_1} \neq 0$. Since A_{ν_1} is algebraic over $\{a_{\nu_1}\}$ there exist elements $r, s (\in R^*)$ such that $r b_{\nu_1} = s a_{\nu_1} \neq 0$, and thus it follows from (21) that

$$s a_{\nu_1} + r b_{\nu_2} + \cdots + r b_{\nu_k} = 0$$

with $s a_{\nu_1} \neq 0$. Continuing this process with the next element not equal to zero, we finally arrive at the equality:

$$s_1 a_{\nu_1} + \cdots + s_k a_{\nu_k} = 0$$

where for at least two indices i_1, i_2

$$s_{i_1} a_{\nu_{i_1}} \neq 0, \quad s_{i_2} a_{\nu_{i_2}} \neq 0.$$

This contradicts the independence of the system $(\ldots, a_\nu, \ldots)_{\nu \in \Gamma}$.

Let $A' = \sum_{\nu \in \Gamma} A_\nu$. Then by Theorem 1 A' is algebraically closed, so that $A = A' + A''$. By the maximality of $(\ldots, a_\nu, \ldots)_{\nu \in \Gamma}$ the only possibility is that $A'' = 0$, so $A = \sum_{\nu \in \Gamma} A_\nu$, as required.

2. Let now R be a Noetherian ring and L any irreducible left ideal of R. We can associate with L a minimal algebraically closed R-module, namely $A = \mathfrak{A}(R^*_{(R)}/L_{(R)})$. We shall say that A the *minimal algebraically closed R-module corresponding to L.*

Call the irreducible left ideals L_1 and L_2 of R^* *equivalent* if there exists a submodule $(\neq 0)$ of $R^*_{(R)}/L_{1(R)}$ isomorphic to a submodule of $R^*_{(R)}/L_{2(R)}$. To prove that this relation is an equivalence it suffices to show that it is transitive.

Let L_1, L_2 and L_3 be arbitrary irreducible left ideals of R^*, L_1 equivalent to L_2 and L_2 equivalent to L_3. By definition there exist submodules $H_1 \subseteqq R^*_{(R)}/L_{1(R)}$, $H_2 \subseteqq R^*_{(R)}/L_{2(R)}$, $H_2' \subseteqq R^*_{(R)}/L_{2(R)}$ and $H_3 \subseteqq R^*_{(R)}/L_{3(R)}$ such that $H_1 \cong H_2$ and $H_2' \cong H_3$. Since L_2 is an irreducible left ideal, the module $R^*_{(R)}/L_{2(R)}$ is a submodule of a minimal algebraically closed R-module, $R^*_{(R)}/L_{2(R)} \subseteqq \mathfrak{A}(R^*_{(R)}/L_{2(R)})$. Lemma 1 now gives $H_2' \cap H_2 = H \neq 0$. Since $H_1 \cong H_2$ and $H_2' \cong H_3$ there exist submodules $H \cong H_1'(\subseteqq H_1)$ and $H \cong H_3'(\subseteqq H_3)$, so $H_1' \cong H_3'$, showing that L_1 is equivalent to L_3.

Denote by $\mathfrak{C}$ the class of irreducible left ideals of R^* equivalent to L. We show that the correspondence between such equivalency classes and the minimal algebraically closed R-modules, induced by the above association of left ideals with minimal algebraically closed R-modules, is one-to-one. Let L_1, L_2 be equivalent left ideals of R^* and A_1, A_2 the corresponding minimal algebraically closed modules, respectively. Then there exist elements a_1 $(\in R^*_{(R)}/L_{1(R)})$ and a_2 $(\in R^*_{(R)}/L_{2(R)})$ with $O(a_1)=O(a_2)$, since by definition $R^*_{(R)}/L_{1(R)}$ has a submodule isomorphic to a submodule of $R^*_{(R)}/L_{2(R)}$. The following relations

$$A_1=\mathfrak{A}_{A_1}(\{a_1\})\simeq\mathfrak{A}(R^*_{(R)}/O(a_1)_{(R)})=\mathfrak{A}(R^*_{(R)}/O(a_2)_{(R)})\simeq\mathfrak{A}_{A_2}(\{a_2\})=A_2$$

establish the validity of the first part of our statement.

On the other hand, it is easy to see that if A is a minimal algebraically closed R-module then the left ideals L which correspond to A belong to only one class $\mathfrak{C}$; so we are entitled to speak about A corresponding to $\mathfrak{C}$ or $\mathfrak{C}$ corresponding to A.

Let us consider all the classes $\mathfrak{C}_\vartheta$ $(\vartheta\in\theta)$ of irreducible left ideals of R^*. With each class $\mathfrak{C}_\vartheta$ we associate a cardinal number $\mathfrak{m}_\vartheta$, and denote by S the system $[\mathfrak{C}_\vartheta, \mathfrak{m}_\vartheta]_{\vartheta\in\theta}$. S determines an algebraically closed R-module

$$G=\sum_{\vartheta\in\theta}\sum_{\mathfrak{m}_\vartheta} A_\vartheta,$$

where $\sum_{\mathfrak{m}_\vartheta} A_\vartheta$ denotes the discrete direct sum of $\mathfrak{m}_\vartheta$ copies of A_ϑ, A_ϑ being the minimal algebraically closed module corresponding to the class $\mathfrak{C}_\vartheta$.

Conversely, let G be any algebraically closed module over a Noetherian ring. Then by Theorem 2

$$G=\sum_{\vartheta\in\theta}\sum_{\mathfrak{m}_\vartheta} A_\vartheta,$$

where the A_ϑ are minimal algebraically closed modules. Let $\mathfrak{C}_\vartheta$ be the class corresponding to A_ϑ; then we associate with G the system $S=[\mathfrak{C}_\vartheta, \mathfrak{m}_\vartheta]_{\vartheta\in\theta}$.

The following theorem shows that this correspondence is one-to-one.

Theorem 4. *If G is an algebraically closed module over a Noetherian ring then any two decomposition*

$$G=\sum_{\vartheta\in\theta}\sum_{\mathfrak{m}_\vartheta} A_\vartheta=\sum_{\vartheta\in\theta}\sum_{\mathfrak{n}_\vartheta} A_\vartheta$$

of G into discrete direct sums of minimal algebraically closed modules are isomorphic. ($\mathfrak{n}_\vartheta=\mathfrak{m}_\vartheta$ *for every* $\vartheta\in\theta$.)

This theorem is an immediate consequence of a general result of G. Azumaya ([1] Theorem 1.).

If abelian group are considered as modules over the ring consisting of one element (i. e. as unitary modules over $\{0\}^*\simeq\mathfrak{G}$) the reader may deduce the following well-known result.

Corollary. *An arbitrary algebraically closed abelian group G has a unique decomposition of the form:*

$$G = \sum_{\mathfrak{m}} \mathfrak{R} + \sum_{p} \sum_{\mathfrak{n}_p} C_{(p^\infty)}$$

where $\mathfrak{R}$ is the additive group of the rationals, $C(p^\infty)$ is Prüfer's group of type p^∞, and p runs over all prime numbers.

The next theorem establishes a duality between the decomposition of the left ideals of R^* and those the corresponding algebraically closed R-modules.

Theorem 5. *Let L be an arbitrary left ideal in R^* and $A = \mathfrak{A}(R^*_{(R)}/L_{(R)})$. Then to each irredundant decomposition of L there exists a decomposition $A = A_1 + \cdots + A_n$, where $A_i \simeq \mathfrak{A}(R^*_{(R)}/L_{i(R)})$, and conversely.*

PROOF. Let $A = \mathfrak{A}(\{a\})$, where a is the element $\langle 0, 1\rangle + L_{(R)}$ of $R^*_{(R)}/L_{(R)}$, and let $L = L_1 \cap \cdots \cap L_n$ be an irredundant decomposition of type (18) of L. We use the following notation: $a_i = \langle 0, 1\rangle + L_{i(R)} (\in R^*_{(R)}/L_{i(R)})$; for $i = 1, 2, \ldots, n$, $A'_i = \mathfrak{A}\{a_i\})$, and $X = A'_1 + \cdots + A'_n$. If $x = a_1 + \cdots + a_n$, then

$$O(x) = O(a_1) \cap \cdots \cap O(a_n) = L_1 \cap \cdots \cap L_n = L = O(a)$$

and therefore $B = \mathfrak{A}_X(\{x\}) \simeq A$. Since the decomposition of L is irredundant, there exist elements $l_i \in R^*$ such that $O \neq l_i x = l_i a_i = a^i (\in B \cap A'_i)$. Suppose that $X = B + B'$ and $(0 \neq) b' \in B'$, then

$$b' = a'_1 + \cdots + a'_n \qquad (a'_i \in A'_i;\ i = 1, 2, \ldots, n).$$

By the method used in the first part of the proof of Lemma 4 we obtain easily

$$0 \neq tb' = (s_1 a^1 + \cdots + s_n a^n) \in B$$

which contradict the fact that $B \cap B' = 0$. Thus $A \simeq B = X = A'_1 + \cdots + A'_n$, showing that there exists a decomposition $A = A_1 + \cdots + A_n$ with $A_i \simeq A'_i$ $(i = 1, 2, \ldots, n)$ as desired.

Conversely let $A = A_1 + \cdots + A_n$, where $A = \mathfrak{A}(\{a\})$, $a = \langle 0, 1\rangle + L_{(R)}$ $(\in R^*_{(R)}/L_{(R)})$ and $A_i (i = 1, 2, \ldots, n)$ are minimal algebraically closed modules. Then $a = a_1 + \cdots + a_n$ so that $L = O(a) = O(a_1) \cap \cdots \cap O(a_n) = L_1 \cap \cdots \cap L_n$. This is an irredundant decomposition. For in the contrary case we may suppose without loss of generality that $L = L_1 \cap \cdots \cap L_m$ $(m < n)$ is an irredundant decomposition of L. Then using our hypothesis and the statement proved above, we obtain that $A = A'_1 + \cdots + A'_m = A_1 + \cdots + A_n$ ($m < n$, and the A_i, A'_j $(i=1,2,\dots,n;\ j=1,2,\dots,m)$ are minimal algebraically closed modules) in contradiction to Theorem 4. This completes the proof of Theorem 5.

Let R be a Noetherian ring. We say that the irredundant decompositions $L=L_1\cap\dots\cap L_k$, $N=N_1\cap\dots\cap N_m$ of two left ideals L and N of R^* are *similar* if there exists a one-to-one correspondence between the $L_i (i=1,2,\dots,k)$ and the N_j $(j=1,2,\dots,m)$ such that the corresponding irreducible left ideals are equivalent. As immediate consequences of Theorem 4 and Theorem 5 we obtain the following corollaries.

Corollary 1. *Any two irredundant decompositions of a left ideal L of R^* are similar.*

Corollary 2. *Let $\{a\}$ and $\{b\}$ be two arbitrary cyclic R-modules. Then $\mathfrak{A}(\{a\})\cong\mathfrak{A}(\{b\})$ if and only if $O(a)$ and $O(b)$ have similar irredundant decompositions.*

The following question: under which conditions are the algebraic closures of two modules (over the same Noetherian ring) isomorphic, is strongly suggested by Corollary 2, and indeed an answer follows from our results. We conclude our paper by expounding this.

Consider any module G over a Noetherian ring R, and in particular the set N of elements of irreducible order in G.

First we show that for any nonzero element g of G, $\{g\}\cap N\neq 0$. Theorem 2 and Lemma 3 implies that $A=\mathfrak{A}(\{g\})$ has at least one element $a\,(\neq 0)$ whose order is irreducible. As a is algebraic over $\{g\}$, some nontrivial multiple $rg (r\in R^*)$ of g belongs to $\{a\}\subseteqq\mathfrak{A}_A(\{a\})$. Theorem 3 shows at once that $O(rg)$ is irreducible.

From this it follows that $N\neq 0$ and so N contains linearly independent subsets. Take any maximal linearly independent subset M of N and denote by M_ϑ the set of all elements of M such that their order belong to the class $\mathfrak{C}_\vartheta$ (See p. 324). Put $\mathfrak{m}_M(\vartheta)$ for the cardinality of M_ϑ; $\mathfrak{m}_M(\vartheta)$ is a function defined on θ.

Lemma 5. *If M and M' are any two maximal independent subset of N then $\mathfrak{m}_M(\vartheta)=\mathfrak{m}_{M'}(\vartheta)$ for every $\vartheta\in\theta$.*

Proof. It is easy to see that M is a maximal linearly independent set in $A=\mathfrak{A}(G)$. Indeed, any nonzero element a of A is algebraic over G and so has a nontrivial multiple $ra (r\in R^*)$ in G; this ra, as any nonzero element of G, has a nontrivial multiple $s(ra)$ $(s\in R^*)$ in N; so $\{a\}\cap\{M\}=0$ would imply $\{sra\}\cap M=0$ with $0\neq sra\in N$, contradicting the maximality of M in N. Now Lemma 4 gives that $A=\sum_{m\in M}\mathfrak{A}_A(\{m\})$ and this we can write as

$A = \sum_{\vartheta \in \theta} \sum_{m \in M_\vartheta} \mathfrak{A}_A(\{m\}) \simeq \sum_{\vartheta \in \theta} \sum_{\mathfrak{m}_M(\vartheta)} A_\vartheta$. Similarly

$$A = \sum_{m' \in M'} \mathfrak{A}_A(\{m'\}) = \sum_{\vartheta \in \theta} \sum_{m' \in M'_\vartheta} \mathfrak{A}_A(\{m'\}) \simeq \sum_{\vartheta \in \theta} \sum_{\mathfrak{m}_{M'}(\vartheta)} A_\vartheta.$$

So Theorem 4 proves that $\mathfrak{m}_M(\vartheta) = \mathfrak{m}_{M'}(\vartheta)$ for all $\vartheta \in \theta$.

The meaning of Lemma 5 is that $\mathfrak{m}_M(\vartheta)$ does not depend on the particular choice of M, but is an invariant of the module G. While proving Lemma 5 we had to see also that $\mathfrak{A}(G)$ is completely determined by $\mathfrak{m}_G(\vartheta)$ — now we may write this instead of $\mathfrak{m}_M(\vartheta)$ — so we have the following answer to our question.

Theorem 6. *The algebraic closures of two modules G, H over a Noetherian ring are isomorphic if and only if $\mathfrak{m}_G(\vartheta) = \mathfrak{m}_H(\vartheta)$ for every $\vartheta \in \theta$.*

Added in proof: The first part of Theorem 2 and Theorem 3 has also been obtained by EBEN MATLIS, *Injective modules over Noetherian rings, Pacific J. Math.* 8 (1958) 511—528.

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