Characterizations of quadratic differences

By B. R. EBANKS (Louisville) and C. T. NG (Waterloo)

Abstract. We characterize two-place functions of the form

$$F(x, y) = f(x + y) + f(x - y) - 2f(x) - 2f(y),$$

which we call a quadratic difference, by means of systems of functional equations. Our functions map groups to groups.

1. Introduction

It is a well-known result of JESSEN, KARPF and THORUP [5] that the Cauchy difference

$$(1.1) \qquad \Delta(x, y) = f(x + y) - f(x) - f(y),$$

for $\Delta : G \times G \to H$ with G and H abelian groups and H divisible, is characterized by the system of functional equations

$$\Delta(x, y) = \Delta(y, x)$$
$$\Delta(x, y) + \Delta(x + y, z) = \Delta(x, y + z) + \Delta(y, z),$$

that is, by symmetry and the cocycle equation. (For an extensive discussion and bibliography, see [4], in which similar results are proved for Cauchy differences of all orders.)

The problem of finding a similar characterization for the *quadratic difference*

$$(1.2) \qquad F(x, y) = f(x + y) + f(x - y) - 2f(x) - 2f(y)$$

was formulated a decade ago [6]. (The name quadratic difference derives from the fact that any solution of $f(x + y) + f(x - y) - 2f(x) - 2f(y) = 0$

is called a quadratic function.) Several necessary conditions are known, one of which is the functional equation

$$(1.3) \quad F(x + y, z) + F(x - y, z) - 2F(y, z)$$
$$= F(x, y + z) + F(x, y - z) - 2F(x, y),$$

introduced by SZÉKELYHIDI [6]. It was demonstrated in [2] that this equation is not sufficient to guarantee (1.2), even when $G = H = R$ (the additive real group). On the other hand some sets of sufficient conditions for (1.2) are known. For instance, (1.3) and boundedness [6], or (1.3) and two times continuous differentiability [3] are sufficient when $F : \mathbb{R}^2 \to \mathbb{R}$. Yet it is clear that analytic conditions are not necessary for (1.2).

The purpose of this paper is to present some sets of conditions which are both necessary and sufficient for $F : G \to H$ to have the representation (1.2). To our knowledge, this has not been done until now. We assume throughout that G is an abelian group and that H is a *uniquely* divisible abelian group. (So H is a vector space over the rationals.) We shall note a few places where the divisibility hypotheses concerning H can be relaxed.

We shall need to make use of the following result on Cauchy differences.

Theorem 1 [4: Theorem 2.2]. *A map $K : G^3 \to H$ is symmetric and satisfies the condition that*

$$(1.4) \quad (x, y) \rightarrowtail K(x, y, w) \text{ is a (1.1) Cauchy difference for each } w \in G,$$

if and only if K is itself a Cauchy difference of order 2, i.e. there exists a map $g : G \to H$ such that

$$K(x, y, z) = g(x+y+z) - g(x+y) - g(x+z) - g(y+z) + g(x) + g(y) + g(z)$$

for all $x, y, z \in G$.

2. Pieces of the solution

In this section, we give characterizations of (1.2) for even f and for odd f, as well as other special solutions of (1.3). These will be combined in the following section to obtain the main results.

Lemma 2.1. *Necessary and sufficient conditions for $F : G \times G \to H$ to have the decomposition (1.2) with odd $f : G \to H$ are*

$$(2.1) \qquad\qquad F(0, -y) = -F(0, y),$$

$$(2.2) \quad F(x, y) = -\tfrac{1}{2} F(0, x + y) - \tfrac{1}{2} F(0, x - y) + F(0, x) + F(0, y).$$

PROOF. If F has the form (1.2) with f odd, then setting $x = 0$ yields

$$(2.3) \qquad\qquad F(0, y) = -2f(y).$$

The oddness of f gives (2.1), and (1.2) with (2.3) translates into (2.2).

Conversely, suppose F satisfies (2.1) and (2.2). Then clearly F has the decomposition (1.2) with f definied by (2.3). Moreover, the oddness of f follows from (2.1). This completes the proof.

Remark. Lemma 2.1 holds when H is any uniquely 2-divisible abelian group. The same is true of the following.

Lemma 2.2. $F : G \times G \to H$ *has decomposition (1.2) with odd $f : G \to H$, if and only if F is an odd solution of (1.3).*

PROOF. Let F be an odd solution of (1.3). Putting $y = z = 0$ in (1.3), we get $F(x, 0) = F(0, 0)$. Since F is odd, $F(0, 0) = 0$, so

$$(2.4) \qquad\qquad F(x, 0) = 0.$$

Putting $y = 0$ in (1.3) and using (2.4), we obtain

$$F(x, z) - 2F(0, z) = F(x, -z).$$

Defining $f : G \to H$ by (cf. (2.3)) $f(x) := -\tfrac{1}{2} F(0, x)$, we have

$$(2.5) \qquad\qquad F(x, z) - F(x, -z) = -4f(z).$$

Also, the oddness of F implies that f is odd.

Now replace (x, z) by $(-x, -z)$ in (1.3) and use the oddness of F to write the result as

$$
\begin{aligned}
-F(x - y, z) &- F(x + y, z) - 2F(y, z) \\
&= -F(x, -y + z) - F(x, -y - z) + 2F(x, -y).
\end{aligned}
$$

Adding this to (1.3) and applying (2.5) several times, we arrive at (1.2), as desired. (This could also be derived from Lemma 2.1.)

Conversely, any F of the form (1.2) satisfies (1.3), and if f is odd, then so is F. This completes the proof.

Lemma 2.3. *Necessary and sufficient conditions for $F : G \times G \to H$ to have a representation (1.2), where f is even and $f(0) = 0$, are*

(2.6)
$$F(x + y, y) + F(x - y, y) - 2F(y, y)$$
$$= F(x, 2y) + F(x, 0) - 2F(x, y),$$

(2.7)
$$F(-x, y) = F(x, y) = F(y, x),$$

(2.8)
$$F(x, 0) = 0,$$

and that the map $K : G^3 \to H$ defined by

(2.9)
$$K(x, y, z) := 2F(y + z, x) + 2F(x + z, y) - 2F(x, y)$$
$$- F(x - z, y) - F(z - y, x) - F(y - x, z)$$

satisfies condition (1.4).

PROOF. Suppose F satisfies (1.2) with f even and $f(0) = 0$. Then it is straightforward to verify that F satisfies (2.6)–(2.8). Moreover, inserting (1.2) into (2.9), we compute that

$$
\begin{aligned}
K(x, y, w) = {}& 2[f(y + w + x) + f(y + w - x) - 2f(y + w) - 2f(x)] \\
& + 2[f(x + w + y) + f(x + w - y) - 2f(x + w) - 2f(y)] \\
& - 2[f(x + y) + f(x - y) - 2f(x) - 2f(y)] \\
& - [f(x - w + y) + f(x - w - y) - 2f(x - w) - 2f(y)] \\
& - [f(w - y + x) + f(w - y - x) - 2f(w - y) - 2f(x)] \\
& - [f(y - x + w) + f(y - x - w) - 2f(y - x) - 2f(w)] \\
= {}& 4[f(x + y + w) - f(x + w) - f(y + w) + f(w)] \\
& - 2[f(x + y - w) - f(x - w) - f(y - w) + f(w)] \\
& - 2[f(x + y) - f(x) - f(y)],
\end{aligned}
$$

using the fact that f is even. The last member of this equation is obviously a Cauchy difference in x and y for each fixed w, since it is the sum of three such Cauchy differences. Hence K satisfies (1.4).

For the converse, let us define $L : G^3 \to H$ by

$$L(x, y, z) := 2[F(x + y, z) - F(x, z) - F(y, z)] + K(x, y, z).$$

Then, by the hypothesis on K, we see that L also satisfies condition (1.4). Moreover, by (2.9) and (2.7) we have

$$
\begin{aligned}
L(x, y, z) = {}& 2[F(x + y, z) + F(y + z, x) + F(z + x, y)] \\
& - [F(x - z, y) + F(z - y, x) + F(y - x, z)] \\
& - 2[F(x, y) + F(y, z) + F(z, x)].
\end{aligned}
$$

(2.10)

That is, L is a symmetric function of its three variables. Hence, by Theorem 1, we obtain

(2.11)
$$
\begin{aligned}
L(x, y, z) = {}& g(x + y + z) \\
& - [g(x + y) + g(x + z) + g(y + z)] + g(x) + g(y) + g(z),
\end{aligned}
$$

for some function $g : G \to H$.

Furthermore, by (2.7), (2.8) and (2.10), L vanishes when one variable is equal to 0. It follows then, from (2.11), that

$$
(2.12) \qquad\qquad\qquad g(0) = 0.
$$

Also, the evenness of F yields $L(-x, -y, -z) = L(x, y, z)$, because of (2.10). So (2.11) shows that we may replace g by its even part. Thus, without loss of generality, we may assume that

$$
(2.13) \qquad\qquad\qquad g \text{ is even.}
$$

Finally, consider a cross section of L in (2.10). Using also (2.6)–(2.8), we calculate that

$$
\begin{aligned}
L(x, y, -y) = {}& 2F(x + y, -y) + 2F(-y + x, y) - F(x + y, y) - F(-2y, x) \\
& - F(y - x, -y) - 2F(x, y) - 2F(y, -y) - 2F(-y, x) \\
= {}& F(x + y, y) + F(x - y, y) - 2F(y, y) - F(x, 2y) - 4F(x, y) \\
= {}& -6F(x, y).
\end{aligned}
$$

Therefore F is a cross section of L,

$$
F(x, y) = -\tfrac{1}{6} L(x, y, -y).
$$

By (2.11), we have now

$$
F(x, y) = -\tfrac{1}{6}\{g(x) - g(x + y) - g(x - y) - g(0) + g(x) + g(y) + g(-y)\}.
$$

Defining $f : G \to H$ by

$$
f(x) := \tfrac{1}{6} g(x)
$$

and observing (2.12) and (2.13), this verifies (1.2) and finishes the proof.

Similar to Lemma 2.3 is the following

Lemma 2.4. *Necessary and sufficient conditions for $F : G \times G \to H$ to have representation (1.2) with f even and $f(0) = 0$ are (1.3), (2.7), (2.8), and that the map $K : G^3 \to H$ defined by*

$$(2.14) \quad K(x, y, z) := F(x + y, z) + F(x + z, y) - F(z - y, x) - 2F(y, z)$$

satisfies condition (1.4).

PROOF. If F has representation (1.2) with f even and $f(0) = 0$, then it is clear that F satisfies (1.3), (2.7) and (2.8). In addition, for the map K defined by (2.14), we have

$$
\begin{aligned}
K(x, y, w) &= [f(x + y + w) + f(x + y - w) - 2f(x + y) - 2f(w)] \\
&\quad + [f(x + w + y) + f(x + w - y) - 2f(x + w) - 2f(y)] \\
&\quad - [f(w - y + x) + f(w - y - x) - 2f(w - y) - 2f(x)] \\
&\quad - 2[f(y + w) + f(y - w) - 2f(y) - 2f(w)] \\
&= 2[f(x + y + w) - f(x + w) - f(y + w) + f(w)] \\
&\quad - 2[f(x + y) - f(x) - f(y)],
\end{aligned}
$$

showing that $(x, y) \rightarrowtail K(x, y, w)$ is a Cauchy difference. Thus (1.4) is fulfilled.

Conversely, (2.14) and (2.7) show that $K(x, y, z) = K(x, z, y)$. The following calculation shows that K is also symmetric in its first two arguments. By (1.3), (2.7) and (2.14), we deduce that indeed

$$
\begin{aligned}
K(x, y, z) &= F(x + y, z) + F(x + z, y) - F(z - y, x) - 2F(y, z) \\
&= F(y + x, z) + [F(y, z + x) - F(y - z, x) - 2F(y, z)] \\
&= F(y + x, z) + [F(y + z, x) - F(y, z - x) - 2F(z, x)] \\
&= F(y + x, z) + F(y + z, x) - F(z - x, y) - 2F(x, z) \\
&= K(y, x, z).
\end{aligned}
$$

Hence K is a symmetric function, and by Theorem 1 there exists $h : G \to H$ for which

$$K(x, y, z) = h(x+y+z) - [h(x+y) + h(x+z) + h(y+z)] + h(x) + h(y) + h(z).$$

Because of (2.14), this means that

$$
\begin{aligned}
&F(x + y, w) + F(x + w, y) - F(w - y, x) - 2F(y, w) \\
&= h(x + y + w) - h(x + y) - h(x + w) - h(y + w) + h(x) + h(y) + h(w).
\end{aligned}
$$

With $y = w$, $x = v - w$, this reduces by (2.8) to

$$2F(v, w) - 2F(w, w) = h(v + w) - 2h(v) - h(2w) + h(v - w) + 2h(w),$$

or

(2.15) $2F(v, w) = h(v + w) - 2h(v) + h(v - w) + k(w),$

where $k : G \rightarrow H$ is defined by $k(w) := 2F(w, w) - h(2w) + 2h(w)$.

Next, by (2.7) and (2.15), we have

$$h(v + w) - 2h(v) + h(v - w) + k(w) = 2F(v, w)$$
$$= 2F(-v, w) = h(-v + w) - 2h(-v) + h(-v - w) + k(w).$$

That is

$$[h(v + w) - h(-v - w)] + [h(v - w) - h(-v + w)] = 2[h(v) - h(-v)],$$

which means that the odd part of h is additive. So we can replace h by its even part in (2.15), and thus no generality is lost by assuming that

$$h \text{ is even.}$$

Now (2.15) and (2.7) also yield

$$h(v + w) + h(v - w) - 2h(v) + k(w) = 2F(v, w)$$
$$= 2F(w, v) = h(w + v) + h(w - v) - 2h(w) + k(v),$$

from which we get $k(w) + 2h(w) = k(v) + 2h(v) = j$ (constant). Hence (2.15) takes the form

$$2F(v, w) = h(v + w) + h(v - w) - 2h(v) + j - 2h(w).$$

Finally, (2.8) gives $j = 2h(0)$ and so, defining $f : G \rightarrow H$ by

$$f(x) := \tfrac{1}{2}[h(x) - h(0)],$$

we have (1.2) with f even and $f(0) = 0$. This concludes the proof.

Theorem 2.5. *In order for $F : G \times G \rightarrow H$ to have representation (1.2) with even $f : G \rightarrow H$, it is necessary and sufficient that F satisfy one of the following sets of conditions. Either*

(i) *F satisfies (2.6), (2.7), and the map K defined by (2.9) satisfies (1.4); or*

(ii) *F satisfies* (1.3), (2.7), *and the map K defined by* (2.14) *satisfies* (1.4).

PROOF. Given (1.2) with f even, we define $f' : G \to H$ and $F' : G \times G \to H$ by

(2.16)
$$f'(x) := f(x) - f(0),$$
$$F'(x, y) := F(x, y) - F(0, 0).$$

Then F' is represented in the form (1.2) by f', which is even and $f'(0) = 0$. Thus, by Lemmas 2.3 and 2.4, F' satisfies conditions (i) and (ii) (as well as (2.8) $F'(x, 0) = 0$). By (2.16), this means that also F satisfies conditions (i) and (ii).

Conversely, given F satisfying either set (i) or (ii) of conditions, we define F' by $F'(x, y) = F(x, y) - F(0, 0)$ and claim (by Lemma 2.3 or 2.4) the representation (1.2) for F' by means of an even function f' satisfying $f'(0) = 0$.

Defining $f : G \to H$ by

$$f(x) := f'(x) - \tfrac{1}{2} F(0, 0)$$

we have (1.2) for F, with f even. This concludes the proof.

We need a few more preliminary results, and they are collected in the next two lemmas. A map $F : G \times G \to H$ is called *skew-symmetric* (or *anti-symmetric*) if $F(x, y) = -F(y, x)$ for all $x, y \in G$.

Lemma 2.6. *Let* $F : G \times G \to H$ *be skew-symmetric. Then* F *satisfies* (1.3), *if and only if* F *is biquadratic (i.e. quadratic in each variable).*

PROOF. First, suppose F is a skew-symmetric solution of (1.3). By several applications of (1.3) and the skew-symmetry, we compute

$$
\begin{aligned}
F(x + y, z) &+ F(x - y, z) - 2F(y, z) \\
&= F(x, y + z) + F(x, y - z) - 2F(x, y) \\
&= -\{F(y + z, x) + F(y - z, x) - 2F(z, x)\} - 2F(z, x) - 2F(x, y) \\
&= -\{F(y, z + x) + F(y, z - x) - 2F(y, z)\} + 2F(x, z) - 2F(x, y) \\
&= F(z + x, y) + F(z - x, y) - 2F(z, y) + 2F(x, z) - 2F(x, y) \\
&= \{F(z + x, y) + F(z - x, y) - 2F(x, y)\} + 2F(x, z) + 2F(y, z) \\
&= \{F(z, x + y) + F(z, x - y) - 2F(z, x)\} + 2F(x, z) + 2F(y, z) \\
&= -F(x + y, z) - F(x - y, z) + 4F(x, z) + 2F(y, z).
\end{aligned}
$$

Comparing the first and last members, therefore

$$2F(x + y, z) + 2F(x - y, z) = 4F(x, z) + 4F(y, z),$$

which shows that F is quadratic in its first argument. Since F is skew-symmetric, it is also quadratic in the second argument, hence F is bi-quadratic.

Conversely any biquadratic function F satisfies (1.3), for

$$F(x + y, z) + F(x - y, z) - 2F(y, z) = 2F(x, z)$$
$$= F(x, y + z) + F(x, y - z) - 2F(x, y).$$

This completes the proof.

Remarks. (i) It was shown in [2] that biquadratic skew-symmetric functions are not in general of the form (1.2). In particular, not every solution of (1.3) has the form (1.2).

(ii) As in Lemmas 2.1 and 2.2, it is only necessary that H be uniquely 2-divisible in Lemma 2.6.

For the next and final lemma of this section, for a given $F : G \times G \to H$ let F_o and F_e be the canoncial odd and even parts of F. That is, define F_o, $F_e : G \times G \to H$ by

(2.17)
$$F_o(x, y) = \tfrac{1}{2}[F(x, y) - F(-x, -y)],$$
$$F_e(x, y) = \tfrac{1}{2}[F(x, y) + F(-x, -y)].$$

Furthermore, we decompose F_e into its symmetric and skew-symmetric parts F_{e+} and F_{e-}, viz.

(2.18)
$$F_{e+}(x, y) = \tfrac{1}{2}[F_e(x, y) + F_e(y, x)],$$
$$F_{e-}(x, y) = \tfrac{1}{2}[F_e(x, y) - F_e(y, x)].$$

Lemma 2.7. *Let* $F : G \times G \to H$ *satisfy* (1.3). *Then:*

    (i) F_o *and* F_e *satisfy* (1.3);
    (ii) F_e *satisfies*

$$F_e(x, y) = F_e(-x, y) = F_e(x, -y);$$

   (iii) F_{e+} *and* F_{e-} *satisfy* (1.3); *and*
   (iv) F_{e+} *satisfies* (2.7).

PROOF. Let F satisfy (1.3). Then part (i) clearly follows, by definition of F_e and F_o and the structure of (1.3).

Now, suppose F_e satisfies (1.3), and put $x = 0$ to get

(2.19) $F_e(-y, z) - F_e(y, z) = F_e(0, y + z) + F_e(0, y - z) - 2F_e(0, y).$

Applying the evenness of F_e to the right hand side of this equation, since $F_e(0, t) = F_e(0, -t)$ we get

$$F_e(-y, z) - F_e(y, z) = -F_e(0, -y - z) + F_e(0, -y + z) - 2F_e(0, -y).$$

By (2.19), the right hand side of this equation is equal to $F_e(y, z) - F_e(-y, z)$, hence we have

$$2F_e(-y, z) = 2F_e(y, z).$$

Since H is uniquely divisible by 2, we see that F_e is even in its first variable. But since F_e is even, we have

$$F_e(x, y) = F_e(-x, -y) = F_e(x, -y).$$

Thus F_e is even in each variable separately, which proves part (ii).

As for part (iii), assume that F satisfies (1.3), and define $T{:}G \times G {\to} H$ by $T(x, y) = F_e(y, x)$. Then, by parts (i) and (ii) of the proof, we calculate that

$$\begin{aligned}
T(x + y, z) &+ T(x - y, z) - 2T(y, z) \\
&= F_e(z, x + y) + F_e(z, x - y) - 2F_e(z, y) \\
&= F_e(z, y + x) + F_e(z, y - x) - 2F_e(z, y) \\
&= F_e(z + y, x) + F_e(z - y, x) - 2F_e(y, x) \\
&= F_e(y + z, x) + F_e(y - z, x) - 2F_e(y, x) \\
&= T(x, y + z) + T(x, y - z) - 2T(x, y).
\end{aligned}$$

That is, T satisfies (1.3). Therefore $F_{e+} = \frac{1}{2}(F_e + T)$ and $F_e = \frac{1}{2}(F_{e-} - T)$ also satify (1.3).

Finally, F_{e+} is by definition symmetric, and by part (ii) of this proof

$$\begin{aligned}
F_{e+}(-x, y) &= \tfrac{1}{2}[F_e(-x, y) + F_e(y, -x)] \\
&= \tfrac{1}{2}[F_e(x, y) + F_e(y, x)] = F_{e+}(x, y).
\end{aligned}$$

Hence F_{e+} satisfies (2.7).

3. Main results

Using the results of the previous section, we can construct several characterizations of quadratic differences. The characterizations are carried out by decompositions. The decompositions used are accomplished by the standard definitions of odd, even, symmetric and skew-symmetric parts of a function. Explicitly these are given by (2.17) and (2.18).

Theorem 3.1. *Let $F : G \times G \to H$ be given, and decompose F into the sum*

$$(3.1) \qquad\qquad F = F_o + F_e$$

using definition (2.17). In order for F to have the representation (1.2) for some $f : G \to H$, it is necessary and sufficient that

(a) F_o satisfies either (1.3) or the pair of conditions (2.1), (2.2); and
(b) F_e satisfies either (i) or (ii) of Theorem 2.5.

PROOF. First, we establish sufficiency. Suppose that F_o and F_e fulfill statements (a) and (b) respectively. Then, by applying Lemma 2.1 or 2.2 (according to (a)) to F_o, we deduce that F_o has the representation

$$(3.2) \qquad F_o(x, y) = f_o(x + y) + f_o(x - y) - 2f_o(x) - 2f_o(y),$$

with odd $f_o : G \to H$. Furthermore, for F_e, Theorem 2.5 gives

$$(3.3) \qquad F_e(x, y) = f_e(x + y) + f_e(x - y) - 2f_e(x) - 2f_e(y),$$

with even $f_e : G \to H$. Hence, by (3.1), F has the form (1.2) with $f = f_e + f_o$.

Conversely, suppose F is of the form (1.2). Then F_o and F_e have the forms (3.2) and (3.3), respectively, where f_o and f_e are the odd and even parts (resp.) of f. By Lemmas 2.1 and 2.2, F_o satisfies statement (a) of the theorem. For F_e, we get statement (b) from Theorem 2.5. This completes the proof.

We characterize a more general form in the next theorem, in which we do not assume that the even part of F is symmetric.

Theorem 3.2. *A map* $F : G \times G \to H$ *has the form*

(3.4) $\qquad F(x, y) = f(x + y) + f(x - y) - 2f(x) - 2f(y) + \Sigma(x, y),$

with $\Sigma : G \times G \to H$ *skew-symmetric and biquadratic, and with arbitrary* $f : G \to H$, *if and only if* F *fulfills* (1.3) *and the map* K_{e+} *defined by means of* F_{e+} *through either* (2.9) *or* (2.14) *satisfies* (1.4). *[Note: Then* $\Sigma = F_{e-}$.]

PROOF. Suppose that F fulfills (1.3) and that F_{e+} fulfills one of the stated hypotheses. By Lemma 2.7, (1.3) carries over also to F_{e+}, to F_o, and F_{e-}. We write F in the form

$$F = F_o + F_{e+} + F_{e-}.$$

By Lemma 2.2, we have (3.2) with f_o odd. Also, by Lemma 2.6, we conclude that F_{e-} is biquadratic, and we put $\Sigma = F_{e-}$. It only remains to consider F_{e+}, and (3.4) will be established as soon as we show that F_{e+} has the form (3.3) with even f_e.

To this end, let us observe that F_{e+} satisfies (2.7), by Lemma 2.7. Thus (since (2.6) is a special case of (1.3)) F_{e+} satisfies either (i) or (ii) of Theorem 2.5, hence F_{e+} has the form (3.3) with even f_e.

For the converse, we observe first that any F of the form (3.4), with Σ biquadratic, satisfies (1.3). Next, we calculate F_{e+}. Since any quadratic function is even, we have

$$F_e(x, y) = f_e(x + y) + f_e(x - y) - 2f_e(x) - 2f_e(y) + \Sigma(x, y),$$

where f_e is the even part of f. Then

$$F_{e+}(x, y) = \tfrac{1}{2}\{F_e(x, y) + F_e(y, x)\}$$
$$= \tfrac{1}{2}\{2f_e(x + y) + 2f_e(x - y) - 4f_e(x) - 4f_e(y) + \Sigma(x, y) + \Sigma(y, x)\}.$$

Since Σ is skew-symmetric, this reduces to a representation of the form (3.3) for F_{e+}. Therefore, by Theorem 2.5, the map K_{e+} defined in terms of F_{e+}, through either (2.9) or (2.14), satisfies (1.4). This concludes the proof of the theorem.

Theorem 3.2 permits us to state a further characterization of quadratic differences, which is the following.

Theorem 3.3. *In order for $F : G \times G \to H$ to have quadratic decomposition (1.2) in terms of an arbitrary $f : G \to H$, it is necessary and sufficient that*
(a) F satisfies (1.3),
(b) the map K_{e+} defined by means of F_{e+} through either (2.9) or (2.14) satisfies (1.4), and
(c) F satisfies

$$(3.5) \qquad F(x + y, x - y) + 2F(x, y) = F(x, x) + F(y, y) + F(0, 0).$$

PROOF. Suppose hypotheses (a), (b) and (c) are fulfilled. By Theorem 3.2, F has the form (3.4) with arbitrary $f : G \to H$ and with skew-symmetric, biquadratic $\Sigma : G \times G \to H$. For such F, (3.5) is satisfied if and only if

$$(3.6) \qquad \Sigma(x + y, x - y) + 2\Sigma(x, y) = 0,$$

since $\Sigma(x, x) = 0$ by skew-symmetry. We have to show that $\Sigma = 0$. Now we employ the representation (see [1]) for the biquadratic Σ in the form

$$(3.7) \qquad \Sigma(x, y) = A(x, x; y, y),$$

where $A : G^4 \to H$ is 4-additive and has the partial symmetries

$$A(x, y; u, v) = A(y, x; u, v) = A(x, y; v, u).$$

Moreover, the skew-symmetry of Σ implies

$$A(x, x; y, y) = -A(y, y; x, x),$$

and in particular $A(x, x; x, x) = 0$. Inserting (3.7) into (3.6), expanding by additivity, and using all properties of A we arrive at

$$2[A(x, y; x, x) - A(x, y; x, y) + A(x, y; y, y)] + A(x, x; y, y) = 0.$$

Considering the terms which are even in x, we find that

$$A(x, x; y, y) = 2A(x, y; x, y).$$

Here the left hand side is skew-symmetric as a function of x and y, whereas the right hand side is symmetric. Hence both sides are zero. Therefore $\Sigma = 0$ by (3.7), and (3.4) reduces to (1.2).

Conversely, any map of the form (1.2) satisfies (a) and (b) by Theorem 3.2, and (c) is verified easily by direct substitution.

Finally, Theorem 3.2 has another consequence which offers a slight improvement over part of Theorem 3.1. Namely, if F_e satisfies part (ii) of Theorem 2.5, then we can remove the assumption (which is a part of (2.7)) that $F_e(-x, y) = F_e(x, y)$. As shown in Lemma 2.7, this is a consequence of (1.3) for F_e.

Corollary 3.4. *In order for $F : G \times G \to H$ to be of the form* (1.2) *with arbitrary $f : G \to H$, it is necessary and sufficient that*
(a) F satisfies (1.3),
(b) the map K_e defined in terms of F_e by (2.14) *satisfies* (1.4), *and*
(c) F_e is symmetric.

PROOF. Condition (c) means that $F_e = F_{e+}$, so $F_{e-} = 0$. The rest follows from Theorem 3.2.

Open problem. It is still not known if (3.4), with Σ biquadratic and skew-symmetric, is the general solution of (1.3).

References

[1] J. ACZÉL, The general solution of two functional equations by reduction to functions additive in two variables and with the aid of Hamel bases, *Glasnik Mat.–Fiz. Astron.* **20** (1965), 65–72.

[2] B. R. EBANKS, (in) Report of the 22nd International Symposium on Functional Equations, *Aequationes Math.* **29** (1985), 91–92 and 105–106.

[3] B. R. EBANKS, Differentiable solutions of a functional equation of Székelyhidi, *Utilitas Math.* **36** (1989), 197–199.

[4] B. R. EBANKS, K. J. HEUVERS and C. T. NG, On Cauchy differences of all orders, *Aequationes Math.* **42** (1991), 137–153.

[5] B. JESSEN, J. KARPF and A. THORUP, Some functional equations in groups and rings, *Math. Scandinavica* **22** (1968), 257–265.

[6] L. SZÉKELYHIDI, (in) Report of the 21st International Symposium on Functional Equations, *Aequationes Math.* **26** (1983), 284.

B. R. EBANKS
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF LOUISVILLE
LOUISVILLE, KY 40292
U.S.A.

C. T. NG
DEPARTMENT OF PURE MATHEMATICS
UNIVERSITY OF WATERLOO
WATERLOO, ONTARIO
CANADA N2L3G1