On the automorphism group of abelian p-groups.

To O. Varga on his 50th birthday.

By L. FUCHS (Budapest).

§ 1. Introduction.

In this note we shall consider the automorphism group[1]) A of abelian primary groups G. The well-ordered descending chain of characteristic subgroups $p^{\alpha}G$ of G (for ordinals $\alpha = 0, 1, \ldots$) gives rise to a well-ordered ascending chain of normal subgroups A_{α} of A. One can intercalate between consecutive terms A_{α} and $\mathsf{A}_{\alpha+1}$ two normal subgroups of A, A_{α}^{*} and A_{α}^{**}, and it is then easy to conclude that $\mathsf{A}_{\alpha}^{*}/\mathsf{A}_{\alpha}$ and $\mathsf{A}_{\alpha}^{**}/\mathsf{A}_{\alpha}^{*}$ are elementary abelian p-groups, while $\mathsf{A}_{\alpha+1}/\mathsf{A}_{\alpha}^{**}$ is isomorphic to a subgroup of the automorphism group of an elementary abelian p-group (thus it is a subgroup of a general linear homogeneous group[2]) over the prime field of characteristic p). The cases when G is countable or contains no element $\neq 0$ of infinite height are of special interest. In these cases we can completely determine the mentioned factor groups and give a necessary criterion for G to have soluble automorphism group.

§ 2. Preliminaries.[3])

Let G be an arbitrary p-group and pG the set of all px with $x \in G$. Define $p^{\alpha}G$ for ordinals α inductively as follows: $p^{0}G = G$; if $\alpha - 1$ exists, then let $p^{\alpha}G = p(p^{\alpha-1}G)$, while if α is a limit ordinal, then $p^{\alpha}G = \bigcap_{\beta<\alpha} p^{\beta}G$. There exists a first ordinal σ, not exceeding the power of G, such that $p^{\sigma+1}G = p^{\sigma}G$. This subgroup $p^{\sigma}G$ is the maximal divisible subgroup D of G.

[1]) For the automorphism group of primary abelian groups see SHODA [9], [10], SPEISER [11] and BAER [1].

[2]) See e. g. JORDAN [4], DICKSON [2], VAN DER WAERDEN [12] etc.

[3]) By a group we shall mean throughout an abelian one. For the concepts and notation not defined here we refer to our book [3].

In this way we obtain a well-ordered descending chain of (fully) characteristic subgroups of G,

$$G \supset pG \supset p^2G \supset \cdots \supset p^\alpha G \supset p^{\alpha+1}G \supset \cdots \supset p^\sigma G = D \supseteq 0.$$

Every element g of G not in D defines a first ordinal γ such that $g \notin p^\gamma G$. This γ is of the form $\alpha+1$, for if g belongs to all $p^\beta G$ with β less than a limit ordinal γ, then $g \in p^\gamma G$ too. If $g \in p^\alpha G$ but $g \notin p^{\alpha+1}G$, then we say g is of height α and write $H(g) = \alpha$. For the elements g of D we put $H(g) = \sigma$.

Let $\mathsf{A}(G)$, or briefly A, be the automorphism group of G. The elements φ of A leaving $p^\alpha G$ elementwise fixed form a subgroup $\mathsf{A}_\alpha = \mathsf{A}_\alpha(G)$ of A. A_α is normal in A, for if $\varphi \in \mathsf{A}_\alpha$ and $\psi \in \mathsf{A}$, then $\psi\varphi\psi^{-1}$ carries every element of $p^\alpha G$ into itself. Thus we obtain a well-ordered ascending chain of normal subgroups of A,

$$\mathsf{E} = \mathsf{A}_0 \subseteq \mathsf{A}_1 \subseteq \mathsf{A}_2 \subseteq \cdots \subseteq \mathsf{A}_\alpha \subseteq \mathsf{A}_{\alpha+1} \subseteq \cdots \subseteq \mathsf{A}_\sigma \subseteq \mathsf{A}$$

where E is the identity subgroup of A.

Define A_α^* to consist of all automorphism in A leaving $p^{\alpha+1}G$ and $(p^\alpha G)[p]$ elementwise fixed.[4]) Then $\mathsf{A}_\alpha \subseteq \mathsf{A}_\alpha^* \subseteq \mathsf{A}_{\alpha+1}$ and A_α^* is again normal in A. Finally, define A_α^{**} as the subgroup of all $\varphi \in \mathsf{A}_{\alpha+1}$ leaving the cosets of $(p^\alpha G)[p] \mod (p^{\alpha+1}G)[p]$ invariant. Now $\varphi \in \mathsf{A}_\alpha^{**}, \psi \in \mathsf{A}$ and $a \in (p^\alpha G)[p]$ imply

$$a\psi\varphi\psi^{-1} = (a'+b)\varphi\psi^{-1} = (a'+b')\psi^{-1} + b\psi^{-1} = (a'+b)\psi^{-1} + (b'-b)\psi^{-1} = \\ = a + b'' \qquad (b, b', b'' \in (p^{\alpha+1}G)[p])$$

whence A_α^{**} is normal in A. Since obviously $\mathsf{A}_\alpha^* \subseteq \mathsf{A}_\alpha^{**} \subseteq \mathsf{A}_{\alpha+1}$, we arrive at the following well-ordered ascending sequence of normal subgroups of A:

$$(1) \qquad \mathsf{E} = \mathsf{A}_0 \subseteq \mathsf{A}_0^* \subseteq \mathsf{A}_0^{**} \subseteq \mathsf{A}_1 \subseteq \cdots \subseteq \mathsf{A}_\alpha \subseteq \mathsf{A}_\alpha^* \subseteq \mathsf{A}_\alpha^{**} \subseteq \mathsf{A}_{\alpha+1} \subseteq \cdots \subseteq \mathsf{A}_\sigma \subseteq \mathsf{A}.$$

In the sequel this sequence will play a basic role.

§ 3. The factor groups $\mathsf{A}/\mathsf{A}_\alpha$.

Every $\varphi \in \mathsf{A}$ induces an automorphism φ_α of $p^\alpha G$ in the natural way: $a\varphi_\alpha = a\varphi$ for all $a \in p^\alpha G$. If, conversely, we are given an automorphism φ_α of $p^\alpha G$, we may ask whether or not it is induced by some automorphism of G. This situation is considered in the next two lemmas.

Lemma 1. *Any automorphism φ_n of $p^n G$ is induced by some automorphism φ of G where n is a natural integer.*

[4]) For a group H, the symbol $H[p]$ denotes the set of all elements x in H such that $px = 0$.

Clearly, it suffices to verify the statement for $n=1$. Assume that φ_1 is an automorphism of pG, and let $B=\Sigma\{a_\lambda\}$ be a basic subgroup of G. Then the set $[a_\lambda]$ is a basis of $G \bmod pG$, i. e. every $g\in G$ may be written in the form

$$g=k_1a_{\lambda_1}+\cdots+k_ra_{\lambda_r}+pg' \qquad (1\leqq k_i\leqq p-1;\ g'\in G) \tag{2}$$

where the terms $k_1a_{\lambda_1},\ldots,k_ra_{\lambda_r},pg'$ are uniquely determined by g. To each a_λ of order $\geqq p^2$ we determine a $b_\lambda\in G$ such that $pb_\lambda=(pa_\lambda)\varphi_1$ and put $b_\lambda=a_\lambda$ if a_λ is of order p. Writing g in the form (2), we put $g\varphi=k_1b_{\lambda_1}+\cdots+k_rb_{\lambda_r}+(pg')\varphi_1$. The mapping φ thus defined is clearly single-valued and addition-preserving. Suppose that $g\varphi=0$; then $(pg)\varphi=0$ too and so $pg=0$. Thus g may be written in the form (2) with[5]) $O(a_{\lambda_r})=p$. But then $g\varphi=k_1a_{\lambda_1}+\cdots+k_ra_{\lambda_r}+(pg')\varphi_1=0$ whence $k_i=0$, $pg'=0$ and $g=0$. In order to prove that every g in (2) is the image of some $h\in G$ under φ, choose an $x\in G$ with $(px)\varphi_1=pg$. Since $g-x\varphi$ is of order p, $g-x\varphi=$ $=k_1a_{\lambda_1}+\cdots+k_ra_{\lambda_r}+pg'$ with a_{λ_i} of order p. The existence of a $y\in G$ with $(py)\varphi=pg'$ implies that g is the image of $h=x+k_1a_{\lambda_1}+\cdots+k_ra_{\lambda_r}+py$ under φ. Consequently, φ is an automorphism of G inducing φ_1 on pG.

The next lemma is due to ZIPPIN [13].

Lemma 2. *If G is countable, then every automorphism φ_α of $p^\alpha G$ is induced by some automorphism φ of G.*

φ_α preserves the heights $H(a)$ of the elements $a\in p^\alpha G$, for if a has height β in $p^\alpha G$, then it is of height $\alpha+\beta$ in G. We employ the method of proof of ULM's theorem due to KAPLANSKY and MACKEY [5].[6]) Assume we have a mapping ψ_n with the properties:

(a) ψ_n is a height-preserving isomorphism between the finite subgroups U_n and V_n of G;

(b) ψ_n has the same effect as φ_α on $U_n\cap p^\alpha G$.

When we extend U_n to U_{n+1}, we take an element a in G such that $pa\in U_n$ and a is proper[7]) with respect to U_n, and then proceed to $U_{n+1}=\{U_n,a\}$. In the cited proof it is shown that there exists an element c proper with respect to V_n, $pc\in V_n$, by the aid of which ψ_n can be extended to a height-preserving isomorphism ψ_{n+1} mapping U_{n+1} onto $V_{n+1}=\{V_n,c\}$ such that $a\psi_{n+1}=c$. We intend to show that c can be chosen so that properties (a) and (b) hold for the index $n+1$.

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[5]) $O(x)$ denotes the order of the group element x.

[6]) Cf. also KAPLANSKY [6] or FUCHS [3].

[7]) $a\in G$ is called proper with respect to U if $H(a)\geqq H(a+u)$ for every u in U.

If $H(a) < \alpha$, then $H(a+u) = \min(H(a), H(u)) \leqq H(a) < \alpha$ for all $u \in U_n$, thus $U_{n+1} \cap p^\alpha G = U_n \cap p^\alpha G$, and the same equality holds for V_{n+1}. Consequently, ψ_{n+1} has the desired properties (a), (b).

If $H(a) \geqq \alpha$, then $H(a\varphi_\alpha) = H(a)$ and $a\varphi_\alpha$ must be proper with respect to V_n, since if we had $H(a\varphi_\alpha + v) > H(a\varphi_\alpha)$ for some $v \in V_n$, then surely $H(v) = H(a\varphi_\alpha) \geqq \alpha$, $a\varphi_\alpha + v \in p^\alpha G$, whence $H(a\varphi_\alpha + v) = H(a + v\varphi_\alpha^{-1}) > H(a)$ $(v\varphi_\alpha^{-1} \in U_n)$ would be a contradiction. We have further $(pa)\psi_n = (pa)\varphi_\alpha = p(a\varphi_\alpha)$, and so we can take $a\varphi_\alpha$ for c in the proof of ULM's theorem.

Now using the same inference as in the proof of ULM's theorem, we conclude that the sequence $\psi_n (n = 1, 2, \ldots)$ defines a unique automorphism φ of G inducing φ_α on $p^\alpha G$. Q. e. d.

By making use of these lemmas we can prove:

Theorem 1. *If G is a p-group, then for every natural integer n we have*

$$\mathsf{A}(G)/\mathsf{A}_n(G) \simeq \mathsf{A}(p^n G).$$

If G is countable, then for every ordinal α

$$\mathsf{A}(G)/\mathsf{A}_\alpha(G) \simeq \mathsf{A}(p^\alpha G).$$

If $\varphi \in \mathsf{A}(G)$ induces $\varphi_\alpha \in \mathsf{A}(p^\alpha G)$, then put $\varphi \to \varphi_\alpha$. This correspondence is a homomorphism of $\mathsf{A}(G)$ into $\mathsf{A}(p^\alpha G)$ whose kernel is obviously $\mathsf{A}_\alpha(G)$. It is onto $\mathsf{A}(p^\alpha G)$ whenever α is a natural integer (by Lemma 1) or G is countable (by Lemma 2).

§ 4. The factor groups $\mathsf{A}_\alpha^*/\mathsf{A}_\alpha$.

Now we turn our attention to the groups A_α^* and want to determine the structure of $\mathsf{A}_\alpha^*/\mathsf{A}_\alpha$.

Let B^α be a basic subgroup of $p^\alpha G$ and write $B^\alpha = \sum_{n=1}^{\infty} B_n^\alpha$ where B_n^α is a direct sum of cyclic groups of the same order p^n. Any $\varphi \in \mathsf{A}_{\alpha+1}$ is completely determined on $p^\alpha G$ by its effect on the elements of some basis of B^α. If $\varphi \in \mathsf{A}_\alpha^*$, then φ acts identically on B_1^α. Let a_λ be a basis element of $B_2^\alpha + \cdots + B_n^\alpha + \cdots$ and $\varphi \in \mathsf{A}_\alpha^*$; then $a_\lambda\varphi = a_\lambda + g_\lambda$ with $g_\lambda \in (p^\alpha G)[p]$, since $pa_\lambda = (pa_\lambda)\varphi = pa_\lambda + pg_\lambda$ and φ is height-preserving. Therefore every $\varphi \in \mathsf{A}_\alpha^*$ defines an element $\langle \ldots, g_\lambda, \ldots \rangle$ of the complete direct sum of $r(B_2^\alpha + \cdots + B_n^\alpha + \cdots) = r(pB^\alpha/p^2B^\alpha) = r(p^{\alpha+1}G/p^{\alpha+2}G)$ copies[8]) of $(p^\alpha G)[p]$. If $\psi \in \mathsf{A}_\alpha^*$ defines $\langle \ldots, h_\alpha, \ldots \rangle$ similarly, then $\langle \ldots, g_\lambda, \ldots \rangle = \langle \ldots, h_\lambda, \ldots \rangle$ if and only if φ and ψ have the same effect on $p^\alpha G$, that is, $\varphi\psi^{-1} \in \mathsf{A}_\alpha$.

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[8]) See e. g. my book [3].

Because of $a_\lambda\varphi\psi = a_\lambda\psi + g_\lambda\psi = a_\lambda + h_\lambda + g_\lambda$ the automorphism $\varphi\psi \in \mathsf{A}^*_\alpha$ defines the sum $\langle \ldots, g_\lambda + h_\lambda, \ldots \rangle$. We are led to the first half of

Theorem 2.[9]) *The factor group $\mathsf{A}^*_\alpha/\mathsf{A}_\alpha$ is isomorphic to a subgroup of the complete direct sum C_α of $r(p^{\alpha+1}G/p^{\alpha+2}G)$ copies of $(p^\alpha G)[p]$. If α is a non-negative integer or G is countable, then $\mathsf{A}^*_\alpha/\mathsf{A}_\alpha \simeq \mathsf{C}_\alpha$.*

The second part will be proved if we can show that in the mentioned cases the elements g_λ can be chosen arbitrarily in $(p^\alpha G)[p]$. Let $\langle \ldots, g_\lambda, \ldots \rangle$ be an arbitrary element of C_α and define $\overline{\varphi}$ to act identically on $\{p^{\alpha+1}G, (p^\alpha G)[p]\}$ and $a_\lambda\overline{\varphi} = a_\lambda + g_\lambda$. It is easily seen that $\overline{\varphi}$ is an automorphism of $p^\alpha G$; in view of Lemma 1 resp. 2 it is induced by some automorphism φ of G. Evidently, $\varphi \in \mathsf{A}^*_\alpha$.

§ 5. The factor groups $\mathsf{A}^{**}_\alpha/\mathsf{A}^*_\alpha$.

Let $[b_\mu]$ be a basis of $(p^\alpha G)[p] \bmod (p^{\alpha+1}G)[p]$. Then $B^\alpha_1 = \Sigma\{b_\mu\}$ is a direct summand of a basic subgroup of $p^\alpha G$ and hence of $p^\alpha G$; $B^\alpha_1 \simeq (p^\alpha G)[p]/(p^{\alpha+1}G)[p]$. By definition, every $\varphi \in \mathsf{A}^{**}_\alpha$ maps b_μ into some $b_\mu + g_\mu$ with $g_\mu \in (p^{\alpha+1}G)[p]$. If $\psi \in \mathsf{A}^{**}_\alpha$ carries b_μ into $b_\mu + h_\mu$, then $\langle \ldots, g_\mu, \ldots \rangle = \langle \ldots, h_\mu, \ldots \rangle$ if and only if φ and ψ agree on $(p^\alpha G)[p]$, that is, $\varphi\psi^{-1} \in \mathsf{A}^*_\alpha$. Since $b_\mu\varphi\psi = (b_\mu + g_\mu)\psi = b_\mu + h_\mu + g_\mu$ implies $\varphi\psi \to \langle \ldots, g_\mu + h_\mu, \ldots \rangle$, therefore the correspondence $\varphi \to \langle \ldots, g_\mu, \ldots \rangle$ is an isomorphism of the factor group $\mathsf{A}^{**}_\alpha/\mathsf{A}^*_\alpha$ into the complete direct sum of $r(B^\alpha_1) = r((p^\alpha G)[p]/(p^{\alpha+1}G)[p])$ copies of $(p^{\alpha+1}G)[p]$. This isomorphism is easily seen to be onto if α is a non-negative integer or G is countable. Hence we infer:

Theorem 3. *The factor group $\mathsf{A}^{**}_\alpha/\mathsf{A}^*_\alpha$ is isomorphic to a subgroup of the complete direct sum D_α of $\mathfrak{v}_\alpha = r((p^\alpha G)[p]/(p^{\alpha+1}G)[p])$ copies[10]) of $(p^{\alpha+1}G)[p]$. If α is a non-negative integer or if G is countable, then $\mathsf{A}^{**}_\alpha/\mathsf{A}^*_\alpha \simeq \mathsf{D}_\alpha$.*

§ 6. The factor groups $\mathsf{A}_{\alpha+1}/\mathsf{A}^{**}_\alpha$.

Let B^α_1 have the same meaning as in § 5. Then $p^\alpha G = B^\alpha_1 + H^\alpha$. Now if $\varphi \in \mathsf{A}_{\alpha+1}$, then $b_\mu\varphi = c_\mu + g_\mu$ $(c_\mu \in B^\alpha_1, g_\mu \in H^\alpha)$ and the mapping $b_\mu \to c_\mu$ is easily seen to induce an automorphism $\overline{\psi}$ of B^α_1. If $\varphi_1 \in \mathsf{A}_{\alpha+1}$ defines similarly $\overline{\psi}_1 \in \mathsf{A}(B^\alpha_1)$, then $\varphi\varphi_1 \to \overline{\psi}\overline{\psi}_1$, and clearly $\overline{\psi} = \overline{\psi}_1$ if and only if $\varphi\varphi_1^{-1}$ leaves

[9]) The first part of this theorem has been proved in [3], but its formulation is not quite correct.

[10]) $\mathfrak{v}_\alpha$ is the αth Ulm invariant of G; see Kaplansky [6].

every b_μ in the same coset mod $(p^{\alpha+1}G)[p]$, i. e. $\varphi\varphi_1^{-1} \in \mathsf{A}_\alpha^{**}$. We obtain an isomorphism between the factor group $\mathsf{A}_{\alpha+1}/\mathsf{A}_\alpha^{**}$ and a subgroup of $\mathsf{A}(B_1^\alpha)$. This subgroup coincides with $\mathsf{A}(B_1^\alpha)$ whenever α is a non-negative integer or G is countable.

The automorphism group of B_1^α depends only on the rank $r(B_1^\alpha) = \mathfrak{p}_\alpha$. $\mathsf{A}(B_1^\alpha)$ is known to be isomorphic to the so-called general linear homogeneous group $GL(\mathfrak{p}_\alpha, F_p)$ on the vector space of dimension $\mathfrak{p}_\alpha$ over the prime field F_p of characteristic p.

Theorem 4. *The factor group $\mathsf{A}_{\alpha+1}/\mathsf{A}_\alpha^{**}$ is isomorphic to a subgroup of $GL(\mathfrak{p}_\alpha, F_p)$ where $\mathfrak{p}_\alpha$ is the αth Ulm invariant of G. If α is a non-negative integer or G is countable, it is isomorphic to $GL(\mathfrak{p}_\alpha, F_p)$ itself.*

Observe that Theorems 2—4 yield a full description of the factor groups in (1) in the following two important cases: 1. G is countable and reduced, 2. G contains no elements $\neq 0$ of infinite height.

§ 7. Remarks.

For infinite groups KUROŠ and ČERNIKOV [7][11]) defined different types of soluble groups (which are equivalent in the case of finite groups). Recall the following definitions:

A chain Γ of subgroups of a group A is a normal system if (i) Γ contains the trivial subgroups of A, (ii) Γ contains the union and intersection of any set of members of Γ, and (iii) $\mathsf{A}' \in \Gamma$ is a normal subgroup of $\mathsf{A}'' \in \Gamma$ whenever $\mathsf{A}' \subset \mathsf{A}''$ and Γ contains no B with $\mathsf{A}' \subset \mathsf{B} \subset \mathsf{A}''$. If all members of Γ are normal in the whole group A, Γ is then an invariant system. [If Γ is well-ordered with respect to inclusion, then it is called a normal (invariant) series.] Γ is a soluble normal system if all factor groups A''/A' with $\mathsf{A}', \mathsf{A}''$ as in (iii) are abelian. — Now the group A is said to be an RN-group if it has a soluble normal system and an $\overline{\text{RN}}$-group if every normal system of it can be refined to a soluble normal system.

The subgroups $\mathsf{A}_\alpha, \mathsf{A}_\alpha^*, \mathsf{A}_\alpha^{**}$ of A do not form an invariant system, since they do not satisfy (ii). For limit ordinals γ we define the groups $\mathsf{A}^{(\gamma)}$ as the union of all A_β with $\beta < \gamma$ to obtain the system

$$(3) \qquad \mathsf{E} = \mathsf{A}_0 \subseteq \mathsf{A}_0^* \subseteq \mathsf{A}_0^{**} \subseteq \mathsf{A}_1 \subseteq \cdots \subseteq \mathsf{A}^{(\gamma)} \subseteq \mathsf{A}_\gamma \subseteq \mathsf{A}_\gamma^* \subseteq \mathsf{A}_\gamma^{**} \subseteq \mathsf{A}_{\gamma+1} \subseteq \cdots \qquad \subseteq \mathsf{A}_\sigma \subseteq \mathsf{A}$$

which is obviously an invariant system for A. Clearly, A will be an $\overline{\text{RN}}$-group only if in (3) the factor groups $\mathsf{A}_\alpha^*/\mathsf{A}_\alpha, \mathsf{A}_\alpha^{**}/\mathsf{A}_\alpha^*, \mathsf{A}_{\alpha+1}/\mathsf{A}_\alpha^{**}$ (for every α) and $\mathsf{A}_\gamma/\mathsf{A}^{(\gamma)}$ (for every limit γ) are all $\overline{\text{RN}}$-groups. The first two types of factor

[11]) Cf. also KUROŠ [8].

groups are always abelian, thus A *is an* $\overline{\mathrm{RN}}$*-group only if the factor groups* $\mathsf{A}_{\alpha+1}/\mathsf{A}_\alpha^*$ *and* $\mathsf{A}_\gamma/\mathsf{A}^{(\gamma)}$ *are* $\overline{\mathrm{RN}}$*-groups.*

Assume that G is a countable reduced p-group or a p-group without elements of infinite height, and $\mathsf{A}(G)$ is an $\overline{\mathrm{RN}}$-group. Then all the factor groups $\mathsf{A}_{\alpha+1}/\mathsf{A}_\alpha^{**}$ must again be $\overline{\mathrm{RN}}$-groups, and a fortiori RN-groups. Since subgroups of RN-groups are again RN-groups, in view of Theorem 4 we conclude that all groups $GL(r, F_p)$ with natural integers $r \leqq \mathfrak{p}_\alpha$ have to be RN-groups, i. e. soluble in the sense generally used for finite groups. JORDAN [4][12]) established a composition series of $GL(r, F_p)$, by showing that it contains a simple non-abelian group with the exception of the following cases: $r=1$ for every p and $r=2$ for $p=2, 3$ in which cases $GL(r, F_p)$ is soluble. Hence we obtain as a necessary condition the inequalities: $\mathfrak{p}_\alpha \leqq 2$ for $p=2, 3$ and $\mathfrak{p}_\alpha \leqq 1$ for $p \geqq 5$. If this condition is satisfied, then $\mathsf{A}_{\alpha+1}/\mathsf{A}_\alpha^{**}$ are finite soluble groups, and if in case $\mathfrak{p}_\alpha = 2$ we insert between A_α^{**} and $\mathsf{A}_{\alpha+1}$ subgroups corresponding to the commutator series of $\mathsf{A}_{\alpha+1}/\mathsf{A}_\alpha^{**}$, we obtain from (3) a finer series which shows that in the considered case $\mathsf{A}(G)$ is an $\overline{\mathrm{RN}}$-group only if the factor groups $\mathsf{A}_\gamma/\mathsf{A}^{(\gamma)}$ are $\overline{\mathrm{RN}}$-groups.

Since we have no criterion for $\mathsf{A}_\gamma/\mathsf{A}^{(\gamma)}$ to be an $\overline{\mathrm{RN}}$-group, a complete result can be stated only if G is a bounded group. In this case we have: *If* G *is a bounded p-group, then* $\mathsf{A}(G)$ *is an* $\overline{\mathrm{RN}}$*-group if and only if* G *is finite and its invariants are* $\leqq 2$ *in case* $p=2$ *or* 3, *and are* $\leqq 1$ *in case* $p \geqq 5$. *Then* $\mathsf{A}(G)$ *is a finite soluble group.*

Note added in proof, July 10, 1960. Professor K. A. HIRSCH has kindly called my attention to the fact that one of his pupils, MRS. FREEDMAN, has also investigated the automorphism groups of primary abelian groups.

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