On the distribution of zeros of general exponential polynomials.

To O. Varga, to his 50th birthday.

By P. TURÁN (Budapest).

1. Starting from the theory of the zeta-function of Riemann I was led in 1949 to the question to give an upper bound for the number $N(f, a, a+d)$ of the zeros (counted according to their multiplicity) in the real interval

(1. 1) $$a \leqq x \leqq a+d \qquad a>0,\ d>0$$

of the almost-periodical polynomial

$$f(x) = \sum_{\nu=0}^{n} a_\nu \cos \lambda_\nu x$$

with

$$0 = \lambda_0 < \lambda_1 < \cdots < \lambda_n, \qquad a_n > 0,$$

depending only upon λ_n, n, d and a. I have found[1]) by methods treated later in my book[2]) the inequality

(1. 2) $$N(f, a, a+d) < 6d\lambda_n + 6n \log\left(24\frac{a+d}{d}\right).$$

In case of

(1. 3) $$f^*(x) = \sum_{\nu=1}^{n} \frac{\cos(x \log \nu)}{\sqrt{\nu}}$$

(1. 4) $$a^* = 2\pi n^2, \qquad a^* + d^* = 2\pi(n+1)^2$$

which was the starting point of these investigations, (1. 2) gives the estimation

(1. 5) $$N(f^*, a^*, d^*) < (24\pi + 6)n \log n + O(n),$$

while an estimation

$$N(f^*, a^*, d^*) < c\, n \log n + O(n)$$

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[1]) On the distribution of real roots of almost-periodical polynomials, *Publ. Math. Debrecen* 1 (1949), 38—41.

[2]) Eine neue Methode in der Analysis und deren Anwendungen, *Budapest*, 1953.

with $c < 4$ would give by a simple reasoning that the main term of the so-called Riemann-Siegel asymptotical formula for $\zeta\left(\frac{1}{2}+ix\right)$ has in the interval $0 < x \leqq T$ for $T \to +\infty$ at least $c_1 T \log T$ real zeros (c_1 positive numerical constant). Now Mr. T. GANELIUS[3]) kindly informed me about the following theorem of PÓLYA.[4])

If $z = x + iy$

$$\mu_1 < \mu_2 < \cdots < \mu_l$$

and $P_\nu(z)$ is for $\nu = 1, 2, \ldots, l$ a polynomial of degree $\leqq m_\nu - 1$ with

$$m_1 + m_2 + \cdots + m_l = n$$

$$P_1(z) P_l(z) \not\equiv 0,$$

then the number $N_1(g_l, a, a+d)$ of the zeros (according to multiplicity) of the function

$$(1.6) \qquad g_l(z) = \sum_{\nu=1}^{l} P_\nu(z) e^{i\mu_\nu z}$$

in the infinite vertical strip

$$a \leqq x \leqq a + d$$

satisfies the inequality

$$(1.7) \qquad \left| N_1(g_l, a, a+d) - \frac{(\mu_l - \mu_1) d}{2\pi} \right| \leqq n - 1.$$

This estimation supersedes (1.2) in all respects; in the case (1.3)—(1.4) it gives the estimation

$$N_1(f^*, a^*, a^* + d^*) = 4n \log n + O(n),$$

as easy to see. However a theorem like (1.7) cannot be expected to hold for the more general class

$$(1.8) \qquad G_l(z) = \sum_{\nu=1}^{l} P_\nu(z) e^{\omega_\nu z}$$

with arbitrary complex exponents ω_ν or even for the subclass

$$(1.9) \qquad G_l(z) = \sum_{\nu=1}^{l} b_\nu e^{\omega_\nu z}$$

with arbitrary complex coefficients b_ν with

$$b_1 b_2 \ldots b_{l-1} b_l \neq 0.$$

[3]) In a letter from 25. Jan. 1957.

[4]) *Jber. Deutsch. Math. Verein.* **3** (1925) p. 97, Problem 24. Solution in Bd. **37** (1928) p. 83 by N. OBRESCHKOFF.

Nevertheless PÓLYA[5]) proved for the number $N_2(G_l, r)$ of zeros (according to multiplicity) of $G_l(z)$ for $|z| \leqq r$ the formula

$$N_2(G_l, r) = Er + O(\log r)$$

with a positive E, independent upon r and even[6])

$$N_2(G_l, r) = Er + O(1). \tag{1.10}$$

Now we are going to prove with the methods of my book the following theorem on the distribution of the zeros of $G_n(z)$, which is obviously not covered by any of the above named results.

Theorem. *If*

$$\max_j |\omega_j| \leqq M \tag{1.11}$$

and

$$\min_{\mu \neq \nu} |\omega_\mu - \omega_\nu| \geqq \Delta (>0), \tag{1.12}$$

hen the number of zeros of $G_n(z) = \sum_{\nu=1}^{n} b_\nu e^{\omega_\nu z} \not\equiv 0$ *in the square*

$$A \leqq x \leqq A + D$$
$$-\frac{D}{2} \leqq y \leqq \frac{D}{2} \tag{1.13}$$
$$(A > 0,\ D > 0)$$

(according to multiplicity) is not greater than

$$5MD + \log 2n + n \log\left(2 + \frac{2n^3}{\Delta D}\right). \tag{1.14}$$

This upper bound is independent upon the position of the square, upon the coefficients b_ν and as to the exponents it depends only upon M and Δ. As the trivial example $G_2(z) = e^{iMz} - 1$ shows, the constant 5 in (1.14) cannot be replaced by any other $< \frac{1}{2\pi}$, further the example

$$\left(e^{\frac{i}{n-1} x} - 1\right)^{n-1}$$

with a large integer n shows that the dependence upon n, the number of

[5]) Münchener Sitzungsber. (1920) p. 285—290.

[6]) Announced in his paper: Lücken und Singularitäten von Potenzreihen, *Math. Z.* **29** (1929), 549—560; in particular footnote on p. 594.

terms, is also indispensable and the dependence of our upper bound (1. 14) upon n is also not very bad. The only question which remains open is whether or not the dependence upon Δ is superfluous.

2. As to the proof it depends ultimately upon a theorem which played in the German version of my book[7]) a subordinate role but in the Chinese edition[8]) and in the forthcoming completely rewritten English edition its role is essentially increased. We shall state only its most important corollary[9]) in an apparently more general form.

If m is a prescribed nonnegative integer and with a positive δ we have

$$\frac{\min\limits_{\mu\neq\nu}|z_\mu - z_\nu|}{\max\limits_j |z_j|} \geqq \delta, \tag{2.1}$$

then with these z_j's to an arbitrary system $(c_1, c_2, \ldots, c_n)$ of complex numbers there is an integer ν with

$$m+1 \leqq \nu \leqq m+n \tag{2.2}$$

and

$$\frac{|c_1 z_1^\nu + \cdots + c_n z_n^\nu|}{|c_1|\,|z_1|^\nu + \cdots + |c_n|\,|z_n|^\nu} \geqq \frac{1}{2n}\left(\frac{\delta}{2}\right)^{n-1}, \tag{2.3}$$

independently upon m.

We shall transform it to our present aims. Let with $a>0, d>0$ be

$$m = \left[\frac{an}{d}\right], \qquad z_j = e^{\frac{d}{n}w_j} \qquad (j=1,2,\ldots,n)$$

then the above theorem asserts that from

$$\frac{\min\limits_{j\neq l}\left|e^{\frac{d}{n}w_j} - e^{\frac{d}{n}w_l}\right|}{\max\limits_j \left|e^{\frac{d}{n}w_j}\right|} \geqq \delta, \tag{2.4}$$

it follows the existence of an integer ν with

$$a \leqq \frac{d\nu}{n} \leqq a+d \tag{2.5}$$

[7]) L. c. [2]) p. 53.

[8]) *Peking*, 1956.

[9]) See l. c. [2]) p. 56.

such that

$$\frac{\left|\sum_{j=1}^{n} c_j e^{w_j \frac{d\nu}{n}}\right|}{\sum_{j=1}^{n} |c_j| \left|e^{w_j \frac{d\nu}{n}}\right|} \geqq \frac{1}{2n}\left(\frac{\delta}{2}\right)^{n-1},$$

i. e. from (2. 5) a fortiori the inequality

$$\max_{a \leqq t \leqq a+d} \frac{\left|\sum_{j=1}^{n} c_j e^{w_j t}\right|}{\sum_{j=1}^{n} |c_j| \left|e^{w_j t}\right|} \geqq \frac{1}{2n}\left(\frac{\delta}{2}\right)^{n-1} \tag{2. 6}$$

follows. This is what we need in the sequel.

3. Now we turn to the proof of our theorem. Let for

$$1 \leqq j \neq l \leqq n$$

be

$$\omega_j - \omega_l = |\omega_j - \omega_l| e^{i\varphi_{jl}}, \ -\pi < \varphi_{jl} \leqq \pi;$$

this gives $n(n-1)$ vectors so that with each vector also its negative occurs. Hence there is an α with $-\pi < \alpha \leqq \pi$ so, that the open angles with the respective bisectrices $\alpha \pm \frac{\pi}{2}$ and both with the opening $\frac{2\pi}{n(n-1)}$ do not contain any of our vectors. Hence for each (j, l)-pair we have either

$$-\frac{\pi}{2} + \frac{\pi}{n(n-1)} \leqq \varphi_{jl} - \alpha \leqq \frac{\pi}{2} - \frac{\pi}{n(n-1)} \quad \bmod (-\pi, \pi] \tag{3. 1}$$

or

$$\frac{\pi}{2} + \frac{\pi}{n(n-1)} \leqq \varphi_{jl} - \alpha \leqq \frac{3\pi}{2} - \frac{\pi}{n(n-1)} \quad \bmod (-\pi, \pi]. \tag{3. 2}$$

In the case (3. 1) we have with this α

$$\begin{aligned} \left| e^{\frac{D}{2n} e^{-i\alpha}(\omega_j - \omega_l)} - 1 \right| &= \left| e^{\frac{D}{2n} |\omega_j - \omega_l| e^{i(\varphi_{jl} - \alpha)}} - 1 \right| \geqq \\ &\geqq e^{\frac{D}{2n} |\omega_j - \omega_l| \cos(\varphi_{jl} - \alpha)} - 1 \geqq e^{\frac{D}{2n} \Delta \sin \frac{\pi}{n(n-1)}} - 1 > \\ &> \frac{D\Delta}{2n} \sin \frac{\pi}{n(n-1)} > \frac{D\Delta}{n^3} > \frac{D\Delta}{n^3 + D\Delta}. \end{aligned} \tag{3. 3}$$

In the case (3. 2) we have

$$\left| e^{\frac{D}{2n}e^{-i\alpha}(\omega_j-\omega_l)}-1\right| = \left|1-e^{\frac{D}{2n}|\omega_j-\omega_l|e^{i(\varphi_{jl}-\alpha)}}\right| \geqq$$

$$(3.4) \qquad \geqq 1-e^{-\frac{D}{2n}\Delta\sin\frac{\pi}{n(n-1)}} \geqq \frac{\frac{D}{2n}\Delta\sin\frac{\pi}{n(n-1)}}{1+\frac{D}{2n}\Delta\sin\frac{\pi}{n(n-1)}} \geqq$$

$$\geqq \frac{\frac{D\Delta}{n^3}}{1+\frac{D\Delta}{n^3}} = \frac{D\Delta}{n^3+D\Delta}$$

too. Hence

$$(3.5) \quad \frac{\min\limits_{j\neq l}\left| e^{\frac{D}{2n}e^{-i\alpha}\omega_j}-e^{\frac{D}{2n}e^{-i\alpha}\omega_l}\right|}{\max\limits_j\left| e^{\frac{D}{2n}e^{-i\alpha}\omega_j}\right|} \geqq \frac{D\Delta}{n^3+D\Delta}\cdot\frac{\min\limits_j\left| e^{\frac{D}{2n}\omega_j e^{-i\alpha}}\right|}{\max\limits_j\left| e^{\frac{D}{2n}\omega_j e^{-i\alpha}}\right|} \geqq \frac{D\Delta}{n^3+D\Delta}\cdot e^{-\frac{D}{n}M}.$$

Now we apply (2. 4)—(2. 6) with

$$c_j = b_j e^{\omega_j\left(A+\frac{D}{2}\right)}$$

$$w_j = \omega_j e^{-i\alpha}$$

$$a = 0,\ d = \frac{D}{2}.$$

Then (3. 5) means that (2. 4) is satisfied with

$$(3.6) \qquad \delta = \frac{D\Delta}{n^3+D\Delta}e^{-\frac{DM}{n}}.$$

Hence (2. 6) gives

$$(3.7) \qquad \max_{0\leqq t\leqq\frac{D}{2}} \frac{\left|\sum\limits_{j=1}^{n} b_j e^{\omega_j\left(A+\frac{D}{2}\right)} e^{\omega_j e^{-i\alpha}t}\right|}{\sum\limits_{j=1}^{n}\left| b_j e^{\omega_j\left(A+\frac{D}{2}\right)}\right|\left| e^{\omega_j e^{-i\alpha}t}\right|} \geqq \frac{1}{2n}\left(\frac{D\Delta}{2(n^3+D\Delta)}e^{-\frac{D}{n}M}\right)^{n-1}.$$

If the max on the left is attained for $t = t_0$ and

$$A+\frac{D}{2}+t_0 e^{-i\alpha} = z_0; \qquad 0\leqq t_0\leqq\frac{D}{2},$$

then (3.7) gives

$$\frac{|G_n(z_0)|}{\sum_{j=1}^{n} |b_j|\,|e^{\omega_j z_0}|} \geqq \frac{1}{2n}\left(\frac{D\Delta}{2(n^3+D\Delta)}e^{-\frac{D}{n}M}\right)^{n-1}. \tag{3.8}$$

4. Having once the inequality (3.8) the proof can be easily completed by applying Jensen's estimation to $G_n(z)$ and the circle $|z-z_0|\leqq D\sqrt{2}$ (which obviously covers our square (1.13)). Hence the number of zeros of $G_n(z)$ in the square (1.13) is

$$\leqq \max_{|z-z_0|\leqq De\sqrt{2}} \log\left|\frac{G_n(z)}{G_n(z_0)}\right| = \max_{|w|\leqq De\sqrt{2}} \log\left|\frac{G_n(z_0+w)}{G_n(z_0)}\right|$$

and using (3.8) and (1.11)

$$< \max_{|w|\leqq De\sqrt{2}} \log\left\{\left(\sum_{j=1}^{n}|b_j e^{\omega_j z_0}|\,|e^{\omega_j w}|\right) 2n\left(\frac{2(n^3+D\Delta)}{D\Delta}e^{\frac{D}{n}M}\right)^{n}\frac{1}{\sum_{j=1}^{n}|b_j|\,|e^{\omega_j z_0}|}\right\} \leqq$$

$$\leqq (e\sqrt{2}+1)MD + \log 2n + n\log\left(2+\frac{2n^3}{D\Delta}\right).$$

Q. e. d.

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