On a problem of Hardy and Littlewood in the theory of diophantine approximations.

To professor Ottó Varga on his 50th birthday.

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1. In a paper published recently, we referred to a problem essentially due to G. H. HARDY and J. E. LITTLEWOOD:[1]) obtain sharp estimates for sums of the type $\sum_{1\leq\nu\leq N} B_r(n_\nu x-[n_\nu x])$ and for their quadratic integral means, respectively. Here $B_r(u)$ denotes the BERNOULLI polynomial of degree r, generated by the expansion $we^{wu}/(e^w-1)=1+B_1(u)w+\cdots$ $(|w|<2\pi)$, $n_\nu(\nu=1,2,\ldots,N)$ is a finite sequence of distinct positive integers, $[n_\nu x]$ the integral part of $n_\nu x$; the stress is laid on upper bounds depending upon N only, which are common for *all* N-tuples considered. This difficult question is of importance in the theory of diophantine approximations[2]) and has not been treated yet in full generality. The particularly interesting case of the integral mean of $\sum_{1\leq\nu\leq N} B_1(n_\nu x-[n_\nu x])=\sum_{1\leq\nu\leq N}(n_\nu x-[n_\nu x])-N/2$ has been solved completely by I. S. GÁL,[3]) on the basis of the well-known fundamental identity

$$(1.1)\qquad \int_0^1\left(au-[au]-\frac{1}{2}\right)\left(bu-[bu]-\frac{1}{2}\right)du=\frac{(a,b)}{12\{a,b\}}$$

(where (a,b) signifies, as usual, the greatest common divisor, $\{a,b\}$ the least common multiple, respectively, of the natural numbers a,b); by (1.1), he discussed the sums $\sum_{1\leq k,l\leq N}(n_k,n_l)/\{n_k,n_l\}$ and concluded at the same time an estimate for $\sum_{1\leq\nu\leq N}(n_\nu x-[n_\nu x])$ $(N\to\infty)$, almost everywhere.

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[1]) Cf. [9], [10], [21].

[2]) Cf. [13], Ch. VIII.—IX. and [14]; a question corresponding to the case $r=1$ has been seted as a prize problem for 1947 by the Scientific Society at Amsterdam.

[3]) See [5]. — The problem has been posed to him by P. ERDŐS who gave a little less sharp upper bound.

After having given some time ago — in connection with certain analytic number-theoretical investigations — an extension of (1. 1) for BERNOULLI polynomials,[4]) we proved in [21] a. o. the still more general integral relation

$$\int_0^1 \zeta(1-s, au-[au])\zeta(1-s, bu-[bu])du = \tag{1.2}$$
$$= 2\Gamma(s)^2 \frac{\zeta(2s)}{(2\pi)^{2s}} \left(\frac{(a, b)}{\{a, b\}}\right)^s \qquad \left(\Re(s) > \frac{1}{2}\right)^{5)}$$

(which becomes (1. 1) for $s=1$) and, getting the formula

$$\int_0^1 \left(\sum_{\nu=1}^{N} \zeta(1-s, n_\nu u-[n_\nu u])\right)^2 du = \tag{1.3}$$
$$= 2\Gamma(s)^2(2\pi)^{-2s}\zeta(2s) \sum_{k,l=1}^{N} \left(\frac{(n_k, n_l)}{\{n_k, n_l\}}\right)^s \qquad \left(\Re(s) > \frac{1}{2}\right),$$

raised the question of estimating the right-hand sum.[6]) Since $\zeta(s, u)$, the well-known zeta-function of HURWITZ (arising from $\sum_{m=0}^{\infty}(u+m)^{-s}$, $\Re(s)>1$, by analytic continuation with respect to s) "interpolates" between $B_1(u), B_2(u), \dots$ in view of

$$\zeta(1-r, u) = -(r-1)!\, B_r(u) \qquad (r=1, 2, \dots), \tag{1.4}$$

(1. 3) yields a near extension of the problem above-mentioned.

Now, the present paper has a double intention: to obtain the widest "natural" generalization of the characteristic integral formula (1. 2) and to study the correspondingly generalized problem of HARDY and LITTLEWOOD.

It proves to be very useful to introduce the function

$$\mathfrak{Z}_s(u) = \Gamma(s)^{-1}\bar{\zeta}(1-s, u) \tag{1.5}$$

where $\bar{\zeta}(s, u) = \zeta(s, u)$ $(0 < u \leqq 1)$, having the period 1 with respect to u, i. e. $\bar{\zeta}(s, u) = \zeta(s, u-[u])$ for u not-integer; (1. 5) is, as easily seen, an *entire* function of s and plays an important role in a theory of differentiation and

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[4]) Cf. [16], Lemma 5., p. 106 and [21], p. 45; cf. also [26], f. (21). — Reading the proof-sheets, we get knowledge (from a review) of certain interesting articles of N. P. ROMANOV ([27], [28]), in which the mentioned formula for Bernoulli polynomials is used likewise.

[5]) The last restriction is essential, since the left-integral, as we showed there, does not exist in Lebesgue's sense for $\Re(s) \leqq 1/2$.

[6]) Cf. [21], p. 52. — In consideration of several reflections received by the author, the subject of [21] seems to arouse some interest. We also refer to quite recent related papers of L. J. MORDELL and L. CARLITZ ([24], [1]—[3]); the last has been written, whilst the present work was under press.

integration of complex order, published lately by the author.[7]) — Next we obtain a *"transcendental" functional equation* for $\mathfrak{Z}_s(u)$, involving a resultant[8]) of the form $\mathfrak{Z}_{s_1}(\alpha u) * \mathfrak{Z}_{s_2}(\beta u)|(x)$ (α, β *arbitrary* integers $\neq 0$) and $\Delta = \{|\alpha|, |\beta|\}$ (Theorem I.). This result is a simultaneous extension of (1. 2) as well as of the *"semi-group property"* (cf. [23], (11. 19))

$$(1.6)\qquad \begin{cases} \mathfrak{Z}_{s_1}(u) * \mathfrak{Z}_{s_2}(u)|(x) = \mathfrak{Z}_{s_1+s_2}(x) \\ (0 < x < 1;\ \Re(s_1) > 0, \Re(s_2) > 0) \end{cases}$$

and implies, in particular, the following generalization of (1. 3):

$$(1.7)\qquad \begin{cases} \int_0^1 \Theta_{s_1}^N(\tilde{n}_\nu, u)\Theta_{s_2}^N(\tilde{n}_\nu, u)\,du = \\ = 2(2\pi)^{-(s_1+s_2)} \cos\frac{\pi}{2}(s_1 - s_2)\zeta(s_1+s_2) \sum_{k,l=1}^{N} \frac{n_k^{s_1} \cdot n_l^{s_2}}{\{n_k, n_l\}^{s_1+s_2}}, \end{cases}$$

where $\Re(s_1) + \Re(s_2) > 1$ and

$$(1.8)\qquad \Theta_s^N(\tilde{n}_\nu, u) = \sum_{\nu=1}^{N} \mathfrak{Z}_s(n_\nu u).$$

(Observe that, by (1. 4)—(1. 5), we have $\Theta_r^N(\tilde{n}_\nu, x) = -\sum_{1 \leq \nu \leq N} B_r(n_\nu x - [n_\nu x])$, $r = 1, 2, \ldots$.)

The second part of the article deals with the sums

$$(1.9)\qquad \mathfrak{A}_{\sigma_1, \sigma_2}^{N;\varrho}(\tilde{n}_\nu) = \sum_{k,l=1}^{N} \frac{(n_k, n_l)^\varrho}{n_k^{\sigma_1} n_l^{\sigma_2}} = \sum_{k,l=1}^{N} \frac{n_k^{\varrho+\sigma_1}, n_l^{\varrho+\sigma_2}}{\{n_k, n_l\}^\varrho} \qquad (\varrho \gtreqless 0,\ \sigma_1 \geqq 0,\ \sigma_2 \geqq 0).$$

By means of a new form of (1. 9) (cf. Theorem II.), we obtain general *O-results* of the desired type, namely in any case where this is possible altogether — apart from a limiting one; the simple method permits also the evaluation of certain double series and *asymptotic formulae* when $n_\nu = \nu$ $(\nu = 1, 2, \ldots)$ (Theorems III.—IV.). The last-mentioned applications lead to a generalization of the idea *"average order"* for functions of several variables. — Finally, on the basis of (1. 7)—(1. 8), we deduce an *o-estimate* for $|\Theta_s^N(\tilde{n}_\nu, x)|$, holding

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[7]) See [23]. — We take, of course, $\mathfrak{Z}_0(u) = \lim_{s \to 0} \Gamma(s)^{-1}\zeta(1-s, u) = -1$.

[8]) We use the usual notation for a resultant (convolution) over (0, 1):

$$f_1(u) * f_2(u)|(x) = \int_0^1 f_1(x-t) f_2(t)\,dt = \int_0^1 f_1(t) f_2(x-t)\,dt,$$

where $f_1(u)$, $f_2(u)$ are L-integrable functions, having the period 1.

almost everywhere; here we use a theorem of I. S. GÁL and J. F. KOKSMA on the order of magnitude of summatoric functions (Theorem V.).

2. In what follows, $s=\sigma+i\tau$ denotes a complex variable, while x, t and u are always real. — All integrals are to be taken in LEBESGUE's sense.

Theorem I. *Let $s_1=\sigma_1+i\tau_1$, $s_2=\sigma_2+i\tau_2$ be arbitrary complex numbers with $\sigma_1>0$, $\sigma_2>0$, α and β arbitrary integers $\neq 0$, Λ the least common multiple of $|\alpha|$ and $|\beta|$.*

Then we have for all $x\neq\nu/\Lambda$ $(\nu=0,\pm1,\pm2,\dots)$, in the case $\sigma_1+\sigma_2>1$ even for Λx integer, the relations:

$$\text{(2. 1)}\quad \mathfrak{Z}_{s_1}(\alpha u)*\mathfrak{Z}_{s_2}(\beta u)|(x)=\begin{cases}\dfrac{|\alpha|^{s_1}|\beta|^{s_2}}{\Lambda^{s_1+s_2}}\,\mathfrak{Z}_{s_1+s_2}((\operatorname{sg}\alpha)\Lambda x), & \textit{if}\quad \operatorname{sg}\alpha=\operatorname{sg}\beta;\\[2ex] \dfrac{|\alpha|^{s_1}|\beta|^{s_2}}{\Lambda^{s_1+s_2}\sin\pi(s_1+s_2)}\,[\sin\pi s_1\,\mathfrak{Z}_{s_1+s_2}((\operatorname{sg}\alpha)\Lambda x)+\sin\pi s_2\,\mathfrak{Z}_{s_1+s_2}((\operatorname{sg}\beta)\Lambda x)], & \\ \textit{when}\ \operatorname{sg}\alpha\neq\operatorname{sg}\beta\ \textit{and}\ s_1+s_2\ \textit{is not integer.} & \end{cases}$$

PROOF. Let s_1, s_2, α, β be fixed.

1° We begin with the statement that, by certain properties of $\zeta(s,u)$ published elsewhere (cf. [21], p. 47—50; [20], p. 145—147), $\mathfrak{Z}_s(u)$ is continuous everywhere if $\sigma>1$ and for $u\neq 0,\pm1,\dots$ when $0<\sigma\leqq 1$; in the last case, we have

$$\text{(2. 2)}\qquad \lim_{u\to 1-0}\mathfrak{Z}_s(u)=\Gamma(s)^{-1}\zeta(1-s),$$

but

$$\text{(2. 3)}\qquad \mathfrak{Z}_s(u)=\Gamma(s)^{-1}u^{s-1}+O(1)\qquad (u\to+0),$$

so that $|\mathfrak{Z}_s(u)|$ remains bounded (with $\sigma=1$) or becomes infinite as $u^{\sigma-1}$ (with $\sigma<1$) for $u\to+0$, respectively.

Hence, using also the periodicity of $\mathfrak{Z}_s(u)$, we conclude that 1. $\mathfrak{Z}_{s_1}(\alpha u)$, $\mathfrak{Z}_{s_2}(\beta u)$ are of the class $L(0,1)$; 2. the resultant in (2. 1) exists for almost all x and is L-integrable in $(0,1)$.[9]

2° In order to obtain the Fourier series (over $(0,1)$) of $\mathfrak{Z}_{s_1}(\alpha u)*\mathfrak{Z}_{s_2}(\beta u)|(x)$, we apply the representation

$$\text{(2. 4)}\qquad \mathfrak{Z}_s(u)=\cos\frac{\pi s}{2}\sum_{k=1}^{\infty}\frac{2\cos 2k\pi u}{(2k\pi)^s}+\sin\frac{\pi s}{2}\sum_{k=1}^{\infty}\frac{2\sin 2k\pi u}{(2k\pi)^s}$$
$$(\sigma>1;\ -\infty<u<\infty),$$

[9]) Cf. e. g. [11], p. 10, Th. 4.

resulting immediately from the so-called Hurwitz formula for $\zeta(s, u)$ with $\sigma<0$;[10]) since we showed recently this to hold even for $0\leqq\sigma<1$, $0<u<1$ (cf. [21], p. 49 and [20], p. 148), one finds that (2. 4) remains valid for $0<\sigma\leqq 1$, $u\neq 0, \pm 1, \ldots$ too.

Next we use the well-known fact, that if the Fourier coefficients over (0, 1) of $f_1(u)\in L(0,1)$, $f_2(u)\in L(0,1)$ are $A'_0, 2A'_m, 2B'_m$ $(m=1,2,\ldots)$ and $A''_0, 2A''_m, 2B''_m$ $(m=1,2,\ldots)$, respectively, then[11])

$$(2.5)\qquad \begin{aligned} f_1(u)*f_2(u)|(x)\sim A'_0A''_0+2\sum_{m=1}^{\infty}[(A'_mA''_m-B'_mB''_m)\cos 2m\pi x+\\ +(B'_mA''_m+A'_mB''_m)\sin 2m\pi x]. \end{aligned}$$

Since, by (2. 4) and a theorem of DE LA VALLÉE POUSSIN,[12]) for $f_1(u)=\mathfrak{Z}_{s_1}(\alpha u)$ we get $(m=1,2,\ldots)$:

$$A'_0=0;$$

$$A'_m=\begin{cases}\left(2\frac{m}{|\alpha|}\pi\right)^{-s_1}\cos\frac{\pi s_1}{2} & \text{with } \frac{m}{|\alpha|} \text{ integer,}\\ 0 & \text{otherwise;}\end{cases}$$

$$B'_m=\begin{cases}\left(2\frac{m}{|\alpha|}\pi\right)^{-s_1}\sin\frac{\pi s_1}{2} & \text{with } \frac{m}{|\alpha|} \text{ integer,}\\ 0 & \text{otherwise,}\end{cases}$$

and A''_0, A''_m, B''_m are similarly obtained in case of $f_2(u)=\mathfrak{Z}_{s_2}(\beta u)$, the application of (2. 5) yields

$$(2.6)\quad \left\{\begin{aligned} &\mathfrak{Z}_{s_1}(\alpha u)*\mathfrak{Z}_{s_2}(\beta u)|(x)\sim\\ &\sim 2\frac{|\alpha|^{s_1}|\beta|^{s_2}}{\Lambda^{s_1+s_2}}\sum_{k=1}^{\infty}(2k\pi)^{-(s_1+s_2)}\Big[\cos\frac{\pi}{2}(s_1+\mathrm{sg}(\alpha\beta)s_2)\cos(2k\pi\Lambda x)+\\ &\qquad+\sin\frac{\pi}{2}((\mathrm{sg}\,\alpha)s_1+(\mathrm{sg}\,\beta)s_2)\sin(2k\pi\Lambda x)\Big] \end{aligned}\right.$$

with $\Lambda=\{|\alpha|,|\beta|\}$.

3° Now, we know (cf. (2. 4)), that the sums

$$(2.7)\qquad \left\{\begin{aligned} \mathfrak{P}_s(u)&=\sum_{k=1}^{\infty}\frac{2\cos 2k\pi u}{(2k\pi)^s}\\ \mathfrak{Q}_s(u)&=\sum_{k=1}^{\infty}\frac{2\sin 2k\pi u}{(2k\pi)^s}\end{aligned}\right.\qquad (\sigma\ 0)$$

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[10]) Cf. [30], p. 268.

[11]) Cf. e. g. [11], p. 23, Th. 29. — In our case, it is more convenient to write "real" Fourier series.

[12]) Cf. [11], p. 91, Th. 100.

not only exist but also are continuous for $u \neq 0, \pm 1, \ldots$ and, if $\sigma > 1$, even for all u. Hence the Fourier series in (2. 6) converges to the continuous function

$$\frac{|\alpha|^{s_1}|\beta|^{s_2}}{\Lambda^{s_1+s_2}}\Big[\cos\big((\mathrm{sg}\,\alpha)s_1+(\mathrm{sg}\,\beta)s_2\big)\mathfrak{P}_{s_1+s_2}(\Lambda x)+ \tag{2. 8}$$
$$+\sin\frac{\pi}{2}\big((\mathrm{sg}\,\alpha)s_1+(\mathrm{sg}\,\beta)s_2\big)\mathfrak{Q}_{s_1+s_2}(\Lambda x)\Big],$$

provided that Λx is not an integer i. e. $x \neq \nu/\Lambda$ ($\nu = 0, \pm 1, \pm 2, \ldots$); and this holds also at the points x last-mentioned when $\sigma_1 + \sigma_2 > 1$. At the same time, it follows that (2. 8) must be equal to the convolution (2. 6) for *almost all* x.

On the other hand, we can verify immediately, that $\mathfrak{Z}_{s_1}(\alpha u) * \mathfrak{Z}_{s_2}(\beta u)|(x)$ is continuous at *every* point in question. — For, suppose first that $\sigma_1 + \sigma_2 > 1$. Then $\mathfrak{Z}_{s_1}(\alpha u)$ and $\mathfrak{Z}_{s_2}(\beta u)$ belong to conjugate L-classes, namely for $\sigma_1 \geqq 1$, $\sigma_2 \geqq 1$ because of their boundedness and integrability, otherwise, e. g. if $\sigma_1 < 1$, because of $\mathfrak{Z}_{s_1}(\alpha u) \in L^{1/(1-\sigma_1+\eta)}$, $\mathfrak{Z}_{s_2}(\beta u) \in L^{1/(\sigma_1-\eta)}$ ($\eta > 0$, sufficiently small; cf. (2. 3)). Therefore, by a well-known proposition on resultants,[13]) our assertion is now valid for *all* x. — Secondly, consider the case $\sigma_1 + \sigma_2 \leqq 1$, a point $x \neq 0, \pm 1/\Lambda, \pm 2/\Lambda, \ldots$ being fixed. Then write

$$\int_0^1 \mathfrak{Z}_{s_1}(\alpha t)[\mathfrak{Z}_{s_2}(\beta(x+h-t)) - \mathfrak{Z}_{s_2}(\beta(x-t))]\,dt = \sum_{\nu=1}^{\Lambda} \int_{\frac{\nu-1}{\Lambda}}^{\frac{\nu}{\Lambda}} \tag{2. 9}$$

and split the ν-th integral on the right into three parts

$$\int_{\frac{\nu-1}{\Lambda}}^{\frac{\nu}{\Lambda}} \mathfrak{Z}_{s_1}(\alpha t)[\mathfrak{Z}_{s_2}(\beta(x+h-t)) - \mathfrak{Z}_{s_2}(\beta(x-t))]\,dt = \int_{\frac{\nu-1}{\Lambda}}^{\frac{\nu-1}{\Lambda}+\delta} + \int_{\frac{\nu-1}{\Lambda}+\delta}^{\frac{\nu}{\Lambda}-\delta} + \int_{\frac{\nu}{\Lambda}-\delta}^{\frac{\nu}{\Lambda}} \tag{2. 10}$$

with $0 < \delta < (2\Lambda)^{-1}$. We observe that 1. if t runs over $((\nu-1)/\Lambda, \nu/\Lambda)$, then $\alpha \cdot t$ is between $(\nu-1)/\Lambda_1$ and ν/Λ_1 with $\Lambda_1 = \Lambda/\alpha$ (integer), i. e. in an interval which contains no integer except possibly at the ends; 2. there is an interval J_ν of the type $(p_\nu|\Lambda_2|^{-1}, (p_\nu+1)|\Lambda_2|^{-1})$ with p_ν integer, $\Lambda_2 = \Lambda/\beta$, containing the point $\beta(x-(\nu-1)/\Lambda) = (\Lambda x - \nu + 1)/\Lambda_2$ in its interior, and thus two points: $\xi_{\nu-1} < \xi_\nu$ can be marked in J_ν such that the $|\beta|\delta$-neighbourhood of $\beta(x-(\nu-1)/\Lambda)$ and of $\beta(x+h-(\nu-1)/\Lambda)$ belong also to $(\xi_{\nu-1}, \xi_\nu)$ — if only δ and $|h|$ are chosen sufficiently small.

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[13]) [11], p. 11., Th. 5.

Let us assume that this last restriction is fulfilled, then after giving an $\varepsilon>0$, we have

$$\left|\int_{\frac{\nu-1}{\Lambda}}^{\frac{\nu-1}{\Lambda}+\delta}\right|<2\max_{(\xi_{\nu-1},\,\xi_\nu)}|\mathfrak{Z}_{s_2}(u)|\cdot\int_{\frac{\nu-1}{\Lambda}}^{\frac{\nu-1}{\Lambda}+\delta}|\mathfrak{Z}_{s_1}(\alpha t)|\,dt<\frac{\varepsilon}{3},$$

$$\left|\int_{\frac{\nu}{\Lambda}-\delta}^{\frac{\nu}{\Lambda}}\right|<2\max_{(\xi_{\nu-1}-\Lambda_2^{-1},\,\xi_\nu-\Lambda_2^{-1})}|\mathfrak{Z}_{s_2}(u)|\cdot\int_{\frac{\nu}{\Lambda}-\delta}^{\frac{\nu}{\Lambda}}|\mathfrak{Z}_{s_1}(\alpha t)|\,dt<\frac{\varepsilon}{3}$$

for δ small enough and (δ fixed)

$$\left|\int_{\frac{\nu-1}{\Lambda}+\delta}^{\frac{\nu}{\Lambda}-\delta}\right|<\max_{\left(\frac{\nu-1}{\Lambda}+\delta,\,\frac{\nu}{\Lambda}-\delta\right)}|\mathfrak{Z}_{s_1}(\alpha t)|\cdot\int_{\frac{\nu-1}{\Lambda}}^{\frac{\nu}{\Lambda}}|\mathfrak{Z}_{s_2}(\beta(x+h-t))-\mathfrak{Z}_{s_2}(\beta(x-t))|\,dt<\frac{\varepsilon}{3},$$

provided that $|h|$ is suitably small. Consequently, the integral (2. 10) remains $<\varepsilon$ in absolut value for $|h|<\vartheta=\vartheta(\varepsilon)$; since this is true alike for $\nu=1,2,\ldots,\Lambda$, (2. 9) yields the desired result.

After all, we get that $\mathfrak{Z}_{s_1}(\alpha u)*\mathfrak{Z}_{s_2}(\beta u)|(x)$ equals the expression (2. 8) for *every* x when $\sigma_1+\sigma_2>1$, and for $x\neq 0, \pm 1/\Lambda, \pm 2/\Lambda, \ldots$ otherwise.

4° We have still merely to verify that (2. 8) may be written in the form (2. 1).

In fact, when $\operatorname{sg}\alpha=\operatorname{sg}\beta$, then considering that (cf. (2. 4))

$$(2.11)\qquad \begin{cases}\cos\dfrac{\pi s}{2}\,\mathfrak{P}_s(u)=\dfrac{1}{2}\,[\mathfrak{Z}_s(u)+\mathfrak{Z}_s(-u)]\\[2ex] \sin\dfrac{\pi s}{2}\,\mathfrak{Q}_s(u)=\dfrac{1}{2}\,[\mathfrak{Z}_s(u)-\mathfrak{Z}_s(-u)]\end{cases}\qquad (\sigma>0),^{14)}$$

the terms in brackets give $\mathfrak{Z}_{s_1+s_2}(\Lambda x)$ for $\alpha>0$ and $\mathfrak{Z}_{s_1+s_2}(-\Lambda x)$ for $\alpha<0$, i. e.

$$\mathfrak{Z}_{s_1}(\alpha u)*\mathfrak{Z}_{s_2}(\beta u)|(x)=$$
$$=\frac{|\alpha|^{s_1}|\beta|^{s_2}}{\Lambda^{s_1+s_2}}\left[\cos\frac{\pi}{2}(s_1+s_2)\,\mathfrak{P}_{s_1+s_2}(\Lambda x)+(\operatorname{sg}\alpha)\sin\frac{\pi}{2}(s_1+s_2)\,\mathfrak{Q}_{s_1+s_2}(\Lambda x)\right]=$$
$$=\frac{|\alpha|^{s_1}|\beta|^{s_2}}{\Lambda^{s_1+s_2}}\,\mathfrak{Z}_{s_1+s_2}((\operatorname{sg}\alpha)\Lambda x).$$

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[14]) Hence it is easily seen, that $\mathfrak{P}_s(u)$ and $\mathfrak{Q}_s(u)$ are also *entire* functions of s, since the poles of $\sec\frac{\pi s}{2}$, $\operatorname{cosec}\frac{\pi s}{2}$ are "cancelled" by the corresponding zeros of $\mathfrak{Z}_s(u)\pm\mathfrak{Z}_s(-u)$.

If $\operatorname{sg}\alpha = -\operatorname{sg}\beta$ and $s_1 + s_2 \neq r$ $(r = 1, 2, \ldots)$, then the use of (2. 11) leads to

$$\mathfrak{Z}_{s_1}(\alpha u) * \mathfrak{Z}_{s_2}(\beta u)|(x) =$$

$$= \frac{|\alpha|^{s_1}|\beta|^{s_2}}{2\Lambda^{s_1+s_2}}\Big\{\mathfrak{Z}_{s_1+s_2}(\Lambda x)\Big[\cos\frac{\pi}{2}(s_1 - s_2)\sec\frac{\pi}{2}(s_1+s_2) + (\operatorname{sg}\alpha)\cdot$$

$$\cdot \sin\frac{\pi}{2}(s_1 - s_2)\operatorname{cosec}\frac{\pi}{2}(s_1+s_2)\Big] + \mathfrak{Z}_{s_1+s_2}(-\Lambda x)\Big[\cos\frac{\pi}{2}(s_1 - s_2)\sec\frac{\pi}{2}(s_1+s_2) -$$

$$-(\operatorname{sg}\alpha)\sin\frac{\pi}{2}(s_1 - s_2)\operatorname{cosec}\frac{\pi}{2}(s_1+s_2)\Big]\Big\};$$

this is by the identity $\left(v_1 + v_2 \neq 0, \pm\frac{\pi}{2}, \pm 2\frac{\pi}{2}, \ldots\right)$

$$\frac{\cos(v_1 - v_2)}{\cos(v_1 + v_2)} + (-1)^{\varkappa}\frac{\sin(v_1 - v_2)}{\sin(v_1 + v_2)} = \begin{cases} \dfrac{2\sin 2v_1}{\sin 2(v_1+v_2)} & \text{with} \quad \varkappa = 0, \\[2ex] \dfrac{2\sin 2v_2}{\sin 2(v_1+v_2)} & \text{with} \quad \varkappa = 1, \end{cases}$$

equivalent to the second part of (2. 1), as an easy survey on the sign possibilities shows. Q. e. d.

3. Consider now the remaining case (*) $\operatorname{sg}\alpha \neq \operatorname{sg}\beta$, $s_1 + s_2 = r$ $(\sigma_1 > 0, \sigma_2 > 0, r > 0$ integer) and a few remarkable *particular cases.*

For (*), we remark first that[15])

$$\mathfrak{P}_r(u) = \sum_{k=1}^{\infty}\frac{2\cos 2k\pi u}{(2k\pi)^r} =$$

$$(3.\ 1) \qquad = \begin{cases} (-1)^{\mu+1}\overline{B}_{2\mu}(u), \quad \text{if} \quad r = 2\mu \ (\mu = 1, 2, \ldots), \\ (-1)^{\mu+1}\displaystyle\int_0^1 [\overline{B}_{2\mu+1}(t) - \overline{B}_{2\mu+1}(u)]\operatorname{ctg}\pi(t-u)\,dt, \\ \qquad\qquad\qquad\qquad \text{if} \quad r = 2\mu+1 \ (\mu = 1, 2, \ldots); \end{cases}$$

$$\mathfrak{Q}_r(u) = \sum_{k=1}^{\infty}\frac{2\sin 2k\pi u}{(2k\pi)^r} =$$

$$(3.\ 2) \qquad = \begin{cases} (-1)^{\mu}\displaystyle\int_0^1 [\overline{B}_{2\mu}(t) - \overline{B}_{2\mu}(u)]\operatorname{ctg}\pi(t-u)\,dt, \\ \qquad\qquad\qquad\qquad \text{if} \quad r = 2\mu \ (\mu = 1, 2, \ldots), \\ (-1)^{\mu+1}\overline{B}_{2\mu+1}(u), \quad \text{if} \quad r = 2\mu+1 \ (\mu = 0, 1, \ldots). \end{cases}$$

[15]) Cf. [25], p. 65—66. — We use another notation of $B_\nu(u)$ (cf. § 1, see Knopp, Infinite series), which differs from that of Nörlund by the factor $1/r!$.

Here $\bar{B}_r(u)$ signifies, as usual, the periodic function with period 1 for which $\bar{B}_r(u)=B_r(u)$ $(0\leqq u<1)$, i. e. $\bar{B}_r(u)=B_r(u-[u])$. Both formulae hold for all u as $r>2$ and for $u\neq 0, \pm 1, \ldots$ if $r=1$; incidentally, in the last case (3. 1) becomes

$$(3.\ 3)\qquad \mathfrak{P}_1(u)=\sum_{k=1}^{\infty}\frac{\cos 2k\pi u}{k\pi}=-\frac{1}{\pi}\log(2\sin\pi u)\qquad (u\neq 0, \pm 1, \ldots).$$

On the basis of the proof of Theorem 1. and of (3. 1)—(3. 2) we can write

$$(3.\ 4)\qquad \frac{\Lambda^r}{|\alpha|^{s_1}|\beta|^{r-s_1}}\,\mathfrak{Z}_{s_1}(\alpha u)*\mathfrak{Z}_{r-s_1}(\beta u)|(x)= \\ =\begin{cases}-\cos\pi s_1\cdot\bar{B}_{2\mu}(\Lambda x)+(\operatorname{sg}\alpha)\sin\pi s_1\cdot\bar{I}_{2\mu}(\Lambda x), & \text{if}\quad r=2\mu,\\ -\sin\pi s_1\cdot\bar{I}_{2\mu+1}(\Lambda x)+(\operatorname{sg}\alpha)\cos\pi s_1\cdot\bar{B}_{2\mu+1}(\Lambda x), & \text{if}\quad r=2\mu+1,\end{cases}$$

where

$$(3.\ 5)\qquad \bar{I}_r(u)=\int_0^1[\bar{B}_r(t)-\bar{B}_r(u)]\operatorname{ctg}\pi(t-u)\,dt.$$

Summing up, we have (cf. (3. 5))

$$(3.\ 6)\qquad \begin{cases}\mathfrak{Z}_{s_1}(\alpha u)*\mathfrak{Z}_{r-s_1}(\beta u)|(x)=\\ =\dfrac{|\alpha|^{s_1}|\beta|^{r-s_1}}{(-\Lambda\operatorname{sg}\alpha)^r}[(\operatorname{sg}\alpha)\sin\pi s_1\cdot\bar{I}_r(\Lambda x)-\cos\pi s_1\cdot\bar{B}_r(\Lambda x)]\\ (\operatorname{sg}\alpha\neq\operatorname{sg}\beta;\ 0<\sigma_1<r;\ r=1,2,\ldots),\end{cases}$$

and this holds for all x if $r\geqq 2$, for $x\neq 0, \pm 1, \ldots$ when $r=1$. — Therefore, in the case in question, the convolution of $\mathfrak{Z}_{s_1}(\alpha u)$ and $\mathfrak{Z}_{s_2}(\beta u)$ can be expressed with the aid of Bernoulli polynomials; it is worthy of notice, that — as the above considerations show — (3. 6) may be regarded as a special (limiting) case of (2. 1), arising for $\sigma_1+\sigma_2\to r$.

If $\alpha=\beta=1$, (2. 1) becomes

$$(3.\ 7)\qquad \mathfrak{Z}_{s_1}(u)*\mathfrak{Z}_{s_2}(u)|(x)=\mathfrak{Z}_{s_1+s_2}(x)\qquad (\sigma_1>0, \sigma_2>0;\ 0<x<1),$$

i. e. the "*semi-group property*" of the functions $\mathfrak{Z}_s(u)$ with respect to the resultant operation, proved and used in [23], § 11. — On the other hand, by putting $\alpha=a>0$, $\beta=-b<0$, $\Lambda=\{a, b\}=ab/(a, b)$, assuming $\sigma_1+\sigma_2>1$

and writing $x=0$ we get[16])

$$(3.8)\quad \begin{cases} \int_0^1 \mathfrak{Z}_{s_1}(at)\mathfrak{Z}_{s_2}(bt)dt = \dfrac{a^{s_1}b^{s_2}}{\{a,b\}^{s_1+s_2}} \dfrac{\sin\pi s_1+\sin\pi s_2}{\sin\pi(s_1+s_2)} \dfrac{\zeta(1-(s_1+s_2))}{\Gamma(s_1+s_2)} = \\ \qquad = \dfrac{(a,b)^{s_1+s_2}}{a^{s_2}b^{s_1}} \dfrac{2}{(2\pi)^{s_1+s_2}} \cos\dfrac{\pi}{2}(s_1-s_2)\zeta(s_1+s_2); \end{cases}$$

in particular, for $s_1=s_2=s=\sigma+i\tau$ and for $s_1=s, s_2=\bar{s}_1=\bar{s}$, respectively, (3. 8) yields the formulae

$$(3.9)\qquad \int_0^1 \mathfrak{Z}_s(at)\mathfrak{Z}_s(bt)dt = \frac{(a,b)^{2s}}{a^s b^s}\frac{2\zeta(2s)}{(2\pi)^{2s}} \qquad \left(\sigma>\frac{1}{2}\right),$$

$$(3.10)\qquad \int_0^1 \mathfrak{Z}_s(at)\overline{\mathfrak{Z}_s(bt)}dt = \frac{(a,b)^{2\sigma}}{a^\sigma b^\sigma}\left(\frac{a}{b}\right)^{i\tau}\frac{2\zeta(2\sigma)\operatorname{ch}\pi\tau}{(2\pi)^{2\sigma}} \qquad \left(\sigma>\frac{1}{2}\right),$$ [17])

which are equivalent to the results given for $\bar{\zeta}(s,u)$ in [21]. — Hence we see, that Theorem 1. represents the simplest common generalization of (3. 7) and (3. 9)—(3. 10) — namely in form of a functional equation.

Another interesting case is: $\alpha=a>0$, $\beta=-b<0$ with $x=\frac{1}{2}$, where we obtain

$$(3.11)\quad \begin{cases} \int_0^1 \mathfrak{Z}_{s_1}(at)\mathfrak{Z}_{s_2}\left(bt+\dfrac{1}{2}\right)dt = \\ = \dfrac{a^{s_1}b^{s_2}}{\{a,b\}^{s_1+s_2}} \dfrac{\sin\pi s_1+\sin\pi s_2}{\sin\pi(s_1+s_2)} \mathfrak{Z}_{s_1+s_2}\left(\dfrac{1}{2}\right) = \qquad (\sigma_1>0, \sigma_2>0) \\ = \dfrac{(a,b)^{s_1+s_2}}{a^{s_2}b^{s_1}} \dfrac{2(2^{1-(s_1+s_2)}-1)}{(2\pi)^{s_1+s_2}} \cos\dfrac{\pi}{2}(s_1-s_2)\zeta(s_1+s_2), \end{cases}$$

since, as we know,

$$\zeta\left(s,\frac{1}{2}\right)=(2^s-1)\zeta(s) \qquad (s\neq 1).$$

[16]) We apply the functional equation of $\zeta(s)$ in the form:

$$\zeta(1-s)=2(2\pi)^{-s}\cos\frac{\pi s}{2}\Gamma(s)\zeta(s).$$

[17]) Concerning (3. 10), we used also the well-known facts, that $\overline{\Gamma(s)}=\Gamma(\bar{s})$ and $\overline{\zeta(s)}=\zeta(\bar{s})$.

We mention, that 1. as we proved quite recently, the functional equations (3. 7) and $\frac{\partial}{\partial u}\mathfrak{Z}_{s+1}(u)=\mathfrak{Z}_s(u)$ substantially suffice, to characterize $\mathfrak{Z}_s(u)$ uniquely (cf. [22]); 2. (3. 8) for $a=b=1$ implies also the *orthogonality relations* given elsewhere for $\zeta(s,u)$ (cf. [20], Th. 3.); 3. all the above formulae furnish certain "integral properties" of the Bernoulli polynomials for s_1, s_2 integers by (cf. (1. 4))

$$\mathfrak{Z}_r(u)=-\bar{B}_r(u) \qquad (u \text{ not-integer};\ r=0,1,\ldots). \tag{3. 12}$$

For example, (3. 7) becomes for $s_1=p$, $s_2=q$ (p, q positive integers)

$$\int_0^1 B_p(t)\bar{B}_q(x-t)\,dt=B_{p+q}(x) \qquad (0<x<1). \tag{3. 13}$$

Furthermore, writing $\alpha=a>0$, $\beta=-b<0$ and putting $s_1=p$, $r=p+q$ in (3. 6) or $s_1=p$, $s_2=q$ $(p,q=1,2,\ldots)$ in (3. 8), it follows

$$\int_0^1 \bar{B}_p(at)\bar{B}_q(bt)\,dt=(-1)^{p-1}\frac{(a,b)^{p+q}}{a^q b^p}\,\frac{B_{p+q}}{(p+q)!}, \tag{3. 14}$$

where $B_{p+q}=(p+q)!\,B_{p+q}(0)$ denotes the corresponding Bernoullian number;[18]) we conclude similarly from (3. 11):

$$\int_0^1 \bar{B}_p(at)\bar{B}_q\left(bt+\frac{1}{2}\right)dt=(-1)^{p-1}(2^{1-(p+q)}-1)\frac{(a,b)^{p+q}}{a^q b^p}\,\frac{B_{p+q}}{(p+q)!}. \tag{3. 11}$$

(3. 13), as well as (3. 14) and (3. 15), are at the same time extensions of certain results of NÖRLUND (cf. [25], p. 31.).

Finally let us notice, that the apparently more general formula

$$\left\{\begin{aligned} &\int_0^1 \mathfrak{Z}_{s_1}(x_1+\mathfrak{a}u)\mathfrak{Z}_{s_2}(x_2+\mathfrak{b}u)\,du= \\ &=\frac{|\mathfrak{a}|^{s_1}|\mathfrak{b}|^{s_2}}{\mathfrak{L}^{s_1+s_2}}\left\{\cos\frac{\pi}{2}[(\mathrm{sg}\,\mathfrak{a})s_1-(\mathrm{sg}\,\mathfrak{b})s_2]\,\mathfrak{P}_{s_1+s_2}\left(\mathfrak{L}\left(\frac{x_1}{\mathfrak{a}}-\frac{x_2}{\mathfrak{b}}\right)\right)+\right. \\ &\qquad\left.+\sin\frac{\pi}{2}[(\mathrm{sg}\,\mathfrak{a})s_1-(\mathrm{sg}\,\mathfrak{b})s_2]\,\mathfrak{Q}_{s_1+s_2}\left(\mathfrak{L}\left(\frac{x_1}{\mathfrak{a}}-\frac{x_2}{\mathfrak{b}}\right)\right)\right\} \end{aligned}\right. \tag{3. 16}$$

— where $\mathfrak{a}$, $\mathfrak{b}$ denote integers $\neq 0$, $\mathfrak{L}=\{|\mathfrak{a}|,|\mathfrak{b}|\}$, $\sigma_1>0$, $\sigma_2>0$ and

[18]) Here take into consideration that $B_3, B_5, \ldots$ are equal to zero and $\zeta(r)=$ $=(-1)^{r/2-1}B_r(2\pi)^r/(2r!)$ if $r=2,4,\ldots$. — For (3. 14), see also [24], (14).

$(\mathfrak{L}/\mathfrak{a})x_1-(\mathfrak{L}/\mathfrak{b})x_2\neq 0, \pm 1, \ldots$ should be taken — is a consequence of (2. 1) in the (2. 8) form; we have only to write $\alpha=\mathfrak{a}, \beta=-\mathfrak{b}$ and $x=x_1/\mathfrak{a}-x_2/\mathfrak{b}$.

4. Let $n_1<n_2<\cdots<n_N$ denote an arbitrary finite sequence of positive integers. As a natural generalization of $\sum_{\nu=1}^{N}\bar{B}_r(n_\nu x)$ $(r=1,2,\ldots)$, we consider the sum (cf. (3. 12))

$$(4.1)\qquad \Theta_s^N(\tilde{n}_\nu, x)=\sum_{\nu=1}^{N}\mathfrak{Z}_s(n_\nu x)=\Gamma(s)^{-1}\sum_{\nu=1}^{N}\zeta(1-s, n_\nu x-[n_\nu x])\qquad(\sigma>0)$$

and connected integrals of the form $(s_1=\sigma_1+i\tau_1, s_2=\sigma_2+i\tau_2)$

$$(4.2)\qquad G_{s_1,s_2}^N(\tilde{n}_\nu)=\int_0^1\Theta_{s_1}^N(\tilde{n}_\nu,u)\,\Theta_{s_2}^N(\tilde{n}_\nu,u)\,du;$$

by (3. 8), this last may be written $(\sigma_1+\sigma_2>1)$

$$(4.3)\qquad \begin{cases} G_{s_1,s_2}^N(\tilde{n}_\nu)=\sum\limits_{\substack{1\leq\varkappa\leq N\\ 1\leq\lambda\leq N}}\int_0^1\mathfrak{Z}_{s_1}(n_\varkappa u)\mathfrak{Z}_{s_2}(n_\lambda u)\,du=\\ =2(2\pi)^{-(s_1+s_2)}\cos\frac{\pi}{2}(s_1-s_2)\zeta(s_1+s_2)\mathfrak{A}_{s_1,s_2}^N(\tilde{n}_\nu)\end{cases}$$

with

$$(4.4)\qquad \mathfrak{A}_{s_1,s_2}^N(\tilde{n}_\nu)=\sum_{\varkappa,\lambda=1}^{N}\frac{n_\varkappa^{s_1}n_\lambda^{s_2}}{\{n_\varkappa,n_\lambda\}^{s_1+s_2}}=\sum_{\varkappa,\lambda=1}^{N}\frac{(n_\lambda,n_\varkappa)^{s_1+s_2}}{n_\lambda^{s_1}n_\varkappa^{s_2}}.$$

It is clear, that the estimation of $|G_{s_1,s_2}^N(\tilde{n}_\nu)|$ from above depends on upper estimates for $\mathfrak{A}_{\sigma_1,\sigma_2}^N(\tilde{n}_\nu)$, namely we get from (4. 3)

$$(4.5)\qquad \begin{cases} \left|\int_0^1\Theta_{s_1}^N(\tilde{n}_\nu,u)\,\Theta_{s_1}^N(\tilde{n}_\nu,u)\,du\right|\leq\\ \leq 2(2\pi)^{-(\sigma_1+\sigma_2)}\operatorname{ch}\frac{\pi}{2}(\tau_1-\tau_2)\zeta(\sigma_1+\sigma_2)\mathfrak{A}_{\sigma_1,\sigma_2}^N(\tilde{n}_\nu).\end{cases}$$

Hence, in connection with the generalized problem of HARDY—LITTLEWOOD, next we turn our attention to the sums of following type:

$$(4.6)\qquad \mathfrak{A}_{\sigma_1,\sigma_2}^{N;\varrho}(\tilde{n}_\nu)=\sum_{k,l=1}^{N}\frac{(n_k,n_l)^\varrho}{n_k^{\sigma_1}n_l^{\sigma_2}},$$

where $\sigma_1>0$, $\sigma_2>0$ and ϱ arbitrary real.

5. Before we deduce a fundamental identity concerning (4. 6), we need the following extension of Euler's arithmetical function $\varphi(m)$: Let $m = p_1^{\lambda_1} p_2^{\lambda_2} \cdots p_r^{\lambda_r}$ be the standard decomposition of an integer $m > 1$ into prime factors, and ϱ a fixed real number, then we define

$$\varphi_\varrho(m) = m^\varrho (1 - p_1^{-\varrho})(1 - p_2^{-\varrho}) \cdots (1 - p_r^{-\varrho}), \tag{5. 1}$$

furthermore, in addition,

$$\varphi_\varrho(1) = 1. \tag{5. 2}$$

Thus e. g.

$$\varphi_0(m) = \begin{cases} 1 & \text{if } m = 1, \\ 0 & \text{if } m > 1 \end{cases} \tag{5. 3}$$

and

$$\varphi_1(m) = \varphi(m) \tag{5. 4}$$

for all m.

As is at once to see, $\varphi_\varrho(m)$ is *multiplicative* and can be expressed by means of the Möbius function $\mu(m)$ in the following way:

$$\varphi_\varrho(m) = m^\varrho \sum_{d|m} \mu(d) d^{-\varrho} = \sum_{d|m} d^\varrho \mu\left(\frac{m}{d}\right), \tag{5. 5}$$

the summation extending over all positive divisors of m; (5. 5) implies, in view of the Möbius inversion formula[19]):

$$\sum_{d|m} \varphi_\varrho(d) = m^\varrho. \tag{5. 6}$$

(5. 1) shows that $\varphi_\varrho(m) > 0$ if $\varrho > 0$ and $\operatorname{sg} \varphi_\varrho(m) = (-1)^r$ for $\varrho < 0$; we have plainly for all $m > 1$

$$\varphi_\varrho(m) < m^\varrho \qquad (\varrho > 0) \tag{5. 7}$$

and

$$|\varphi_\varrho(m)| < \left(\frac{m}{p_1 p_2 \cdots p_r}\right)^\varrho \leqq 1 \qquad (\varrho < 0). \tag{5. 8}$$

Since, by the well-known multiplication rule of Dirichlet series and by (5. 6), it subsists

$$\left(\sum_{m=1}^{\infty} \frac{\varphi_\varrho(m)}{m^s}\right)\left(\sum_{m=1}^{\infty} \frac{1}{m^s}\right) = \sum_{m=1}^{\infty} \frac{\sum_{d|m} \varphi_\varrho(d)}{m^s} = \sum_{m=1}^{\infty} \frac{1}{m^{s-\varrho}}, \tag{5. 9}$$

19) Cf. e. g. [12], p. 236.

provided that each series here converges absolutely, one finds the *generating function* (cf. (5. 7), (5. 8))

$$\frac{\zeta(s-\varrho)}{\zeta(s)}=\sum_{m=1}^{\infty}\frac{\varphi_\varrho(m)}{m^s}\qquad(\sigma>\max(1,\varrho+1)). \tag{5.10}$$

Note that $\varphi_\varrho(m)$ has been applied incidentally by A. SELBERG ([29]) in connection with the investigation of $\zeta(s)$ on the critical line and, for negative integer ϱ, implicitly in [16], [18] (cf. also [19]); [20]) to the conspicuous relationship of the last-mentioned functions with $\mu(m)$, we will come back elsewhere.

6. Now, it seems to be useful to state explicitly the following general

Lemma. *Let E be given set of pairs a, b of positive integers, $f(a, b)$ an arithmetical function defined over E, then*

$$\sum_{\substack{a,\,b\in E\\(a,\,b)=1}} f(a,b)=\sum_{\substack{d\mid(a,\,b)\\a,\,b\in E}}\mu(d)\sum_{\substack{\alpha=\frac{a}{d},\ \beta=\frac{b}{d}\\ \text{integers}\\ a,\,b\in E}} f(d\alpha,d\beta).\text{[21])} \tag{6.1}$$

In fact, to obtain the sum on the left, we have to omit from

$$\sum_{a,\,b\in E} f(a,b) \tag{6.2}$$

the terms corresponding to a, b which have *at least* one common prime factor, that is to form the difference

$$\sum_{a,\,b\in E} f(a,b)-\sum_{\substack{p\ \text{prime}\\ \alpha=\frac{a}{p},\ \beta=\frac{b}{p}\\ \text{integers}\\ a,\,b\in E}} f(p\alpha,p\beta);$$

but, in this way, we taked into account twice the pairs a, b having two or more distinct common prime factors, hence consider

$$\sum_{a,\,b\in E} f(a,b)-\sum_{\substack{p\ \text{prime}\\ \alpha=\frac{a}{p},\ \beta=\frac{b}{p}\\ \text{integers}\\ a,\,b\in E}} f(p\alpha,p\beta)+\sum_{\substack{p,\,p'\ \text{prime}\\ \alpha=\frac{a}{pp'},\ \beta=\frac{b}{pp'}\\ \text{integers}\\ a,\,b\in E}} f(pp'\alpha,pp'\beta),$$

---

[20]) Doing the proof-reading, we find that $\varphi_\varrho(m)$ has been used quite recently by B. GYIRES ([8]) to give an elegant extension of the determinant theorem of SMITH.

[21]) The left-hand sum extends plainly over all pairs a, b of the set E for which a and b are relatively prime, etc.

etc. — The final step leads clearly to a sum which may be written as the right-hand expression in (6. 1) and, on the other hand, we see that then all terms of (6. 2) have been "sifted" for which $(a, b)>1$.

Next we establish

Theorem II. *For every sequence* n_ν $(\nu=1, 2, \ldots, N)$ *of distinct positive integers and for arbitrary* $\varrho, \sigma_1, \sigma_2$, *it holds the formula* (cf. (4. 6))

$$\mathfrak{A}^{N;\varrho}_{\sigma_1,\sigma_2}(\tilde{n}_\nu)=\sum_{\substack{d|n_\nu\\ 1\leqq\nu\leqq N}}{}' \frac{\varphi_\varrho(d)}{d^{\sigma_1+\sigma_2}} Z^N_{\sigma_1}(\tilde{n}_\nu; d) Z^N_{\sigma_2}(\tilde{n}_\nu; d) \tag{6. 3}$$

with

$$Z^N_\sigma(\tilde{n}_\nu; d)=\sum_{\substack{q_\nu=\frac{n_\nu}{d}\text{ int.}\\ 1\leqq\nu\leqq N}} q_\nu^{-\sigma}. \tag{6. 4}$$

The summation in (6. 3) *is extended over all different divisors of the N-tuple* (n_ν),[22]) *while* (6. 4) *is to form in considering all the quotients* $n_1/d, \ldots, n_N/d$ *which are integers.*[23])

Proof. Applying (6. 1), we can write

$$\left.\begin{aligned}
\mathfrak{A}^{N;\varrho}_{\sigma_1,\sigma_2}(\tilde{n}_\nu) &= \sum_{\substack{1\leqq k\leqq N\\ 1\leqq l\leqq N}} \frac{(n_k, n_l)^\varrho}{n_k^{\sigma_1} n_l^{\sigma_2}} = \sum_{\substack{t|n_\nu\\ 1\leqq\nu\leqq N}}{}' \sum_{\substack{\delta_j=\frac{n_j}{t}\text{ int.}\\ (\delta_k,\delta_l)=1\\ 1\leqq k,\, l\leqq N}} \frac{t^\varrho}{(t\delta_k)^{\sigma_1}(t\delta_l)^{\sigma_2}} = \\
&= \sum_{\substack{t|n_\nu\\ 1\leqq\nu\leqq N}}{}' t^{\varrho-(\sigma_1+\sigma_2)} \sum_{\substack{d\left|\frac{n_\nu}{t}\text{ int.}\right.\\ 1\leqq\nu\leqq N}}{}' \mu(d) \sum_{\substack{q_j=\frac{n_j}{td}\text{ int.}\\ 1\leqq k,\, l\leqq N}} (dq_k)^{-\sigma_1}(dq_l)^{-\sigma_2} = \\
&= \sum_{\substack{t|n_\nu\\ 1\leqq\nu\leqq N}}{}' t^{\varrho-(\sigma_1+\sigma_2)} \sum_{\substack{d\left|\frac{n_\nu}{t}\text{ int.}\right.\\ 1\leqq\nu\leqq N}}{}' \frac{\mu(d)}{d^{\sigma_1+\sigma_2}} Z^N_{\sigma_1}(\tilde{n}_\nu; td) Z^N_{\sigma_2}(\tilde{n}_\nu; td).
\end{aligned}\right\} \tag{6. 5}$$

---

[22]) Therefore the dash on the summation sign indicates that eventual common divisors of the n_ν-s are to be taken only once.

[23]) We remark, that (6. 3) may be state also in a little more general form: in case of $\sum\limits_{1\leqq k\leqq N_1} \sum\limits_{1\leqq l\leqq N_2} (n_k, n_l)^\varrho / n_k^{\sigma_1} n_l^{\sigma_2}$ $(N_1\leqq N_2)$ we have only to write N_2 in (6. 3) under the summation sign and $Z^{N_1}_{\sigma_1}$, $Z^{N_2}_{\sigma_2}$ for $Z^N_{\sigma_1}$, $Z^N_{\sigma_2}$, respectively.

Hence, by further appropriate rearrangements, we get (cf. (5. 5))

$$
(6.6)\quad \begin{cases}
\mathfrak{A}^{N;\varrho}_{\sigma_1,\sigma_2}(\tilde{n}_\nu) = \sum\limits'_{\substack{d|n_\nu \\ 1\leqq\nu\leqq N}} \dfrac{\mu(d)}{d^{\sigma_1+\sigma_2}} \sum\limits'_{\substack{t\left|\frac{n_\nu}{d}\right.\text{int.} \\ 1\leqq\nu\leqq N}} t^{\varrho-(\sigma_1+\sigma_2)} Z^N_{\sigma_1}(\tilde{n}_\nu; dt) Z^N_{\sigma_2}(\tilde{n}_\nu; dt) = \\
= \sum\limits'_{\substack{\tau|n_\nu \\ 1\leqq\nu\leqq N}} \sum\limits_{d|\tau} \mu(d) d^{-\varrho}\cdot\tau^{\varrho-(\sigma_1+\sigma_2)} Z^N_{\sigma_1}(\tilde{n}_\nu;\tau) Z^N_{\sigma_2}(\tilde{n}_\nu;\tau) = \\
= \sum\limits'_{\substack{\tau|n_\nu \\ 1\leqq\nu\leqq N}} \dfrac{\tau^\varrho}{\tau^{\sigma_1+\sigma_2}} \left(\sum\limits_{d|\tau} \dfrac{\mu(d)}{d^\varrho}\right) Z^N_{\sigma_1}(\tilde{n}_\nu;\tau) Z^N_{\sigma_2}(\tilde{n}_\nu;\tau) = \\
= \sum\limits'_{\substack{\tau|n_\nu \\ 1\leqq\nu\leqq N}} \dfrac{\varphi_\varrho(\tau)}{\tau^{\sigma_1+\sigma_2}} Z^N_{\sigma_1}(\tilde{n}_\nu;\tau) Z^N_{\sigma_2}(\tilde{n}_\nu;\tau).
\end{cases}
$$

Q. e. d.

7. We are coming to the question: what conditions must be fulfilled by the parameters ϱ and $\sigma_1>0$, $\sigma_2>0$, in order that $\mathfrak{A}^{N;\varrho}_{\sigma_1,\sigma_2}(\tilde{n}_\nu)$ should have an upper bound *depending* (apart from $\varrho, \sigma_1, \sigma_2$) *on* N only; in other words: if N is given, when will $\mathfrak{A}^{N;\varrho}_{\sigma_1,\sigma_2}(\tilde{n}_\nu)$ remain under a constant $P=P(N)= =P(N;\varrho,\sigma_1,\sigma_2)$ simultaneously for *all* N-tuples (n_ν) of distinct positive integers?

Since $(n_k, n_l)=n_k$ if $n_k=n_l$, it subsists the trivial inequality

$$(7.1)\qquad \mathfrak{A}^{N;\varrho}_{\sigma_1,\sigma_2}(\tilde{n}_\nu) \geqq \sum_{k=1}^{N} n_k^{\varrho-(\sigma_1+\sigma_2)},$$

whence we see at once, that

$$(7.2)\qquad \varrho \leqq \sigma_1+\sigma_2$$

is a *necessary* condition. — Assuming (7. 2) and putting $(n_k, n_l)=t_{kl}$, $n_k= =t_{kl}\cdot\delta_k$, $n_l=t_{kl}\cdot\delta_l$, we find

$$(7.3)\qquad \mathfrak{A}^{N;\varrho}_{\sigma_1,\sigma_2}(\tilde{n}_\nu) = \sum_{k,l=1}^{N} t_{kl}^{\varrho-(\sigma_1+\sigma_2)}\delta_k^{-\sigma_1}\delta_l^{-\sigma_2} < N^2,$$

whatever be (n_ν), so that (7. 2) turns out to be also *sufficient*.

Our next purpose is to sharpen (7. 3) in a possibly simple way, but persisting in the most general case. — To fix our ideas, *we suppose throughout* $(0<)n_1<n_2<\cdots<n_N$ and $(0<)\sigma_1\leqq\sigma_2$.

Theorem III. 1. *Let* (*) $\varrho \leqq 0$, *or* (**) $0<\varrho<\sigma_1+\sigma_2-1$. *Then, independently of the choice of* (n_ν):

$$\mathfrak{A}_{\sigma_1,\sigma_2}^{N;\varrho}(\tilde{n}_\nu) \leqq \begin{cases} c\cdot\zeta(\sigma_1)\zeta(\sigma_2), \text{ if } \sigma_1>1, \sigma_2>1; \\ c\cdot\zeta(\sigma_2)(\log N+1), \text{ if } \sigma_1=1, \sigma_2>1; \\ c\cdot(\log N+1)^2 \text{ for } \sigma_1=\sigma_2=1; \\ c\cdot\zeta(\sigma_2)(1-\sigma_2)^{-1}(N^{1-\sigma_2}-\sigma_2), \text{ if } \sigma_1<1, \sigma_2>1; \\ c\cdot(1-\sigma_1)^{-1}(N^{1-\sigma_1}-\sigma_1)(\log N+1), \text{ if } \sigma_1<1, \sigma_2=1; \\ c\cdot(1-\sigma_1)^{-1}(1-\sigma_2)^{-1}(N^{1-\sigma_1}-\sigma_1)(N^{1-\sigma_2}-\sigma_2), \text{ if } \sigma_1<1, \sigma_2<1; \end{cases}$$

where c *means* 1 *or* $\zeta(\sigma_1+\sigma_2-\varrho)/\zeta(\sigma_1+\sigma_2)$, *according to the case* (*) *and* (**), *respectively.*

2. *Let* $\varrho>0$ *and* $\sigma_1+\sigma_2-1\leqq\varrho<\sigma_1+\sigma_2$. *Then there exist constants* $C_j=C_j(\varrho,\sigma_1,\sigma_2)$ $(j=1,2,3)$ *such that, simultaneously for every N-tuple* (n_ν), *we have*

$$\mathfrak{A}_{\sigma_1,\sigma_2}^{N;\varrho}(\tilde{n}_\nu) < \begin{cases} C_1 N, \text{ if } \sigma_2>1; \\ C_2 N\log N, \text{ if } \sigma_2=1; \\ C_3 N^{2-\sigma_2}, \text{ if } \sigma_2<1. \end{cases}$$

3. *If* $\varrho=\sigma_1+\sigma_2$ *and* $\sigma_2>1$, *then the last inequalities keep their validity with* σ_1 *for* σ_2.

Proof. We premise the elementary estimation formulae $(x\geqq 1)$:

$$(7.4)\quad Z_\sigma(x)=\sum_{\nu=1}^{[x]}\nu^{-\sigma}\leqq 1+\int_1^x u^{-\sigma}du=\begin{cases}(1-\sigma)^{-1}(x^{1-\sigma}-\sigma), \text{ if } \sigma\geqq 0, \sigma\neq 1; \\ \log x+1, \text{ if } \sigma=1.\end{cases}$$

1. Suppose first $\varrho\leqq 0$. — Then obviously

$$(7.5)\quad \mathfrak{A}_{\sigma_1,\sigma_2}^{N;\varrho}(\tilde{n}_\nu)\leqq\sum_{k,l=1}^{N}\frac{1}{n_k^{\sigma_1}n_l^{\sigma_2}}=\Big(\sum_{1\leqq k\leqq N}n_k^{-\sigma_1}\Big)\Big(\sum_{1\leqq l\leqq N}n_l^{-\sigma_2}\Big)\leqq Z_{\sigma_1}(N)Z_{\sigma_2}(N),$$

and the inequalities III. 1, with $c=1$ follow by (7. 4) at once.

Secondly, let $0<\varrho<\sigma_1+\sigma_2-1$. Then, using the convergence and sum of the series with *positive* terms $\sum_{d=1}^{\infty}\varphi_\varrho(d)d^{-(\sigma_1+\sigma_2)}$ (cf. (5. 10)), we obtain by (6. 3)

$$(7.6)\quad \begin{cases}\mathfrak{A}_{\sigma_1,\sigma_2}^{N;\varrho}(\tilde{n}_\nu)\leqq\sum_{d=1}^{n_N}\frac{\varphi_\varrho(d)}{d^{\sigma_1+\sigma_2}}\Big(\sum_{1\leqq q\leqq N}q^{-\sigma_1}\Big)\Big(\sum_{1\leqq q\leqq N}q^{-\sigma_2}\Big)< \\ \qquad <\frac{\zeta(\sigma_1+\sigma_2-\varrho)}{\zeta(\sigma_1+\sigma_2)}Z_{\sigma_1}(N)Z_{\sigma_2}(N),\end{cases}$$

so that (7. 4) implies again the assertion with $c=\zeta(\sigma_1+\sigma_2-\varrho)/\zeta(\sigma_1+\sigma_2)$.

2. In the case $\varrho>0$, $\sigma_1+\sigma_2-1\leqq\varrho<\sigma_1+\sigma_2$, the result lies deeper a little. Now write our sum in the form

$$(7.7)\quad \begin{cases} \mathfrak{A}_{\sigma_1,\sigma_2}^{N;\varrho}(\tilde{n}_\nu)=\sum'_{\substack{d|n_\nu\\ 1\leqq\nu\leqq N}}\dfrac{\varphi_\varrho(d)}{d^{\sigma_2}}Z_{\sigma_2}^N(\tilde{n}_\nu;d)\sum_{\substack{\nu=1\\ d|n_\nu}}^{N} n_\nu^{-\sigma_1}= \quad ^{24)}\\ \qquad =\sum_{1\leqq\nu\leqq N} n_\nu^{-\sigma_1}\sum_{d|n_\nu}\dfrac{\varphi_\varrho(d)}{d^{\sigma_2}}Z_{\sigma_2}^N(\tilde{n}_\nu;d), \end{cases}$$

and consider the estimation involved here (cf. (5. 7)):

$$(7.8)\quad \mathfrak{A}_{\sigma_1,\sigma_2}^{N;\varrho}(\tilde{n}_\nu)\leqq\Big(\sum_{1\leqq\nu\leqq N} n_\nu^{-\sigma_1}\sum_{d|n_\nu} d^{\varrho-\sigma_2}\Big)Z_{\sigma_2}(N).$$

By certain well-known investigations of WIGERT and GRONWALL (cf. e. g. [7] and [12]), we have for the "divisor functions"

$$(7.9)\quad T_\omega(m)=\sum_{d|m} d^{-\omega}\qquad (\omega \text{ arbitrary real})$$

the inequalities ($\varepsilon>0$, arbitrarily small)

$$(7.10)\quad T_\omega(m)<K_{\omega,\varepsilon}m^{\omega+\varepsilon}\qquad (\omega\geqq0;\ m=1,2,\ldots),$$

$$(7.11)\quad T_\omega(m)=m^\omega T_{-\omega}(m)<K_{-\omega,\varepsilon}m^\varepsilon\qquad (\omega<0;\ m=1,2,\ldots),$$

where $K_{\omega,\varepsilon}$ denotes a suitable constant, depending on α and ε only. — For $|\omega|>1$, one can put besides $\varepsilon=0$ and $K_{|\omega|,\varepsilon}=\zeta(|\omega|)$.

If $\sigma_1+\sigma_2-1\leqq\varrho<\sigma_1+\sigma_2$, choose ε such that $\sigma_1\geqq\varepsilon$; on the other hand, if $\max(\sigma_2,\sigma_1+\sigma_2-1)\leqq\varrho<\sigma_1+\sigma_1$, let us fix ε in accordance with $\varrho\leqq\sigma_1+$ $+\sigma_2-\varepsilon$. Then (7. 11)—(7. 12) yield

$$(7.12)\quad n_\nu^{-\sigma_1}T_{\varrho-\sigma_2}(n_\nu)<\begin{cases} K_{\sigma_2-\varrho,\varepsilon}n_\nu^{\varepsilon-\sigma_1}\leqq K_{\sigma_2-\varrho,\varepsilon} \text{ when } \varrho<\sigma_2,\\ K_{\varrho-\sigma_2,\varepsilon}n_\nu^{\varrho-(\sigma_1+\sigma_2)+\varepsilon}\leqq K_{\varrho-\sigma_2,\varepsilon} \text{ otherwise};\end{cases}$$

consequently, applying (7. 8), we get for any case in question

$$(7.13)\quad \mathfrak{A}_{\sigma_1,\sigma_2}^{N;\varrho}(\tilde{n}_\nu)<K_{|\varrho-\sigma_2|,\varepsilon}\cdot NZ_{\sigma_2}(N).$$

(7. 13), together with (7. 4), leads to the estimates III. 2; the above considerations hold, of course, when the role of σ_1 and σ_2 is interchanged, but the result is weaker by $\sigma_1\leqq\sigma_2$.

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[24]) Therefore, the last sum extends over all terms of the sequence $n_1<n_2<\cdots<n_N$, which are divisible by d.

3. Finally, take $\varrho = \sigma_1 + \sigma_2$ and $\sigma_2 > 1$.

Then we obtain, as before (cf. (7. 8)),

$$\mathfrak{A}^{N;\,\sigma_1+\sigma_2}_{\sigma_1,\,\sigma_2}(\tilde{n}_\nu) \leqq \left(\sum_{\nu=1}^{N} n_\nu^{-\sigma_2} T_{\sigma_2}(n_\nu)\right) \cdot Z_{\sigma_1}(N) \tag{7. 14}$$

and hence, in view of

$$m^{-\omega} T_\omega(m) < \zeta(\omega) \qquad (\omega > 1;\ m = 1, 2, \ldots) \tag{7. 15}$$

the inequality

$$\mathfrak{A}^{N;\,\sigma_1+\sigma_2}_{\sigma_1,\,\sigma_2}(\tilde{n}_\nu) < \zeta(\sigma_2) N Z_{\sigma_1}(N). \tag{7. 16}$$

The use of (7. 4) clearly completes the proof.

Let us remark, that Theorem III. does not imply GÁL's main result in [5] ($C > 0$, constant)

$$\mathfrak{A}^{N;\,2}_{1,\,1}(\tilde{n}_\nu) = \sum_{k,\,l=1}^{N} \frac{(n_k, n_l)^2}{n_k \cdot n_l} < CN(\log\log N)^2, \tag{7. 17}$$

since III. 2. concerns the limiting case $\varrho = \sigma_1 + \sigma_2$ merely with the supplementary condition $\sigma_2 > 1$. — The "nearest" particular estimation included in III. 2. is (cf. (7. 16) with $\sigma_1 = 1$, $\sigma_2 = 1 + \varepsilon$)

$$\mathfrak{A}^{N;\,2+\varepsilon}_{1,\,1+\varepsilon}(\tilde{n}_\nu) = \sum_{k,\,l=1}^{N} \frac{(n_k, n_l)^{2+\varepsilon}}{n_k \cdot n_l^{1+\varepsilon}} < \zeta(1+\varepsilon) N \sum_{\nu=1}^{N} \frac{1}{\nu} = O(N \log N), \tag{7. 18}$$

where $\varepsilon > 0$ is arbitrarily small.

It seems, however, probable that an appropriate refinement of the method applied above would yield a. o. a simple verification of (7. 17), which is got in [5], as well-known, by a quite complicated argument of combinatory character. We hope to treat this problem on another occasion.

8. In the most important particular case: $n_\nu = \nu$ $(\nu = 1, 2, \ldots)$, (6. 3) becomes (cf. (7. 4)):

$$\mathfrak{A}^{\varrho}_{\sigma_1,\,\sigma_2}(N) = \sum_{k,\,l=1}^{N} \frac{(k, l)^\varrho}{k^{\sigma_1} l^{\sigma_2}} = \sum_{d=1}^{N} \frac{\varphi_\varrho(d)}{d^{\sigma_1+\sigma_2}} Z_{\sigma_1}\left(\frac{N}{d}\right) Z_{\sigma_2}\left(\frac{N}{d}\right); \tag{8. 1}$$

it is easy to see, that (8. 1) may also be written e. g. in the form (cf. (6. 6)):

$$\mathfrak{A}^{\varrho}_{\sigma_1,\,\sigma_2}(N) = \sum_{d=1}^{N} \frac{\mu(d)}{d^{\sigma_1+\sigma_2}} \sum_{t=1}^{\left[\frac{N}{d}\right]} t^{\varrho-(\sigma_1+\sigma_2)} Z_{\sigma_1}\left(\frac{N}{dt}\right) Z_{\sigma_2}\left(\frac{N}{dt}\right). \tag{8. 2}$$

By (8.1)—(8.2), one can plainly determine (with several systems of values $\varrho, \sigma_1, \sigma_2$) the actual asymptotic behaviour of $\mathfrak{A}^{\varrho}_{\sigma_1, \sigma_2}(N)$ as $N \to \infty$, our previous O-results being thus improved for the special sequence in question.[25])

As an illustration, next we give the very simple

Theorem IV. *Suppose* $\sigma_2 \geqq \sigma_1 > 1$ *and* $0 \leqq \varrho < \sigma_1 + \sigma_2 - 1$. *Then holds*

$$\mathfrak{A}^{\varrho}_{\sigma_1, \sigma_2}(N) = \frac{\zeta(\sigma_1)\zeta(\sigma_2)}{\zeta(\sigma_1+\sigma_2)}\zeta(\sigma_1+\sigma_2-\varrho) + O(N^{Q}\log N), \tag{8.3}$$

where $Q = \max(\varrho + 1 - \sigma_1 - \sigma_2, 1 - \sigma_1)$; *in particular, for* $N \to \infty$ *we have*

$$\sum_{k,l=1}^{\infty} \frac{(k,l)^{\varrho}}{k^{\sigma_1} l^{\sigma_2}} = \frac{\zeta(\sigma_1)\zeta(\sigma_2)}{\zeta(\sigma_1+\sigma_2)}\zeta(\sigma_1+\sigma_2-\varrho).\text{[26])} \tag{8.4}$$

Proof. Since

$$Z_{\sigma}(x) = \zeta(\sigma) + O(x^{1-\sigma}) \qquad (\sigma > 1), \tag{8.5}$$

(8.1) yields (cf. (5.7), (5.10))

$$\mathfrak{A}^{\varrho}_{\sigma_1, \sigma_2}(N) = \sum_{d=1}^{N} \frac{\varphi_{\varrho}(d)}{d^{\sigma_1+\sigma_2}} \left[\zeta(\sigma_1)\zeta(\sigma_2) + O\left(\frac{N^{1-\sigma_1}}{d^{1-\sigma_1}}\right) + \right.$$

$$\left. + O\left(\frac{N^{1-\sigma_2}}{d^{1-\sigma_2}}\right) + O\left(\frac{N^{2-\sigma_1-\sigma_2}}{d^{2-\sigma_1-\sigma_2}}\right)\right] =$$

$$= \zeta(\sigma_1)\zeta(\sigma_2)\frac{\zeta(\sigma_1+\sigma_2-\varrho)}{\zeta(\sigma_1+\sigma_2)} + O(N^{\varrho+1-(\sigma_1+\sigma_2)}) +$$

$$+ O\left(N^{1-\sigma_1}\sum_{d=1}^{N} d^{\varrho-(\sigma_2+1)}\right) + O\left(N^{1-\sigma_2}\sum_{d=1}^{N} d^{\varrho-(\sigma_1+1)}\right) + O\left(N^{2-(\sigma_1+\sigma_2)}\sum_{d=1}^{N} d^{\varrho-2}\right).$$

(7.4) shows that, in any case, the remainder is $O(N^{Q}\log N)$, and (8.3) follows. — For (8.4), we have only to observe that, by hypothesis, Q is negative.

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[25]) Sums of type $\mathfrak{A}^{\varrho}_{\sigma_1, \sigma_2}(N)$ occur in connection with various investigations; cf. e.g. [15], p. 175—177; [4], [16], [17].

[26]) If e.g. $\sigma_2 > \varrho + 1$, $\sigma_1 > 1$, then the double series (8.4) converges also in the more restricted sense of Pringsheim. (Cf. Bromwich, Infinite series, 2. ed., 1926, p. 85.) — For $\varrho = 0$ arises the well-known formula: $\sum_{k,l=1}^{\infty} k^{-\sigma_1} l^{-\sigma_2} = \zeta(\sigma_1)\zeta(\sigma_2)$.

Further interesting *examples* are:

$$\mathfrak{A}_{0,0}^{-1}(N)=\sum_{k,l=1}^{N}\frac{1}{(k,l)}=\frac{6\zeta(3)}{\pi^2}N^2+O(N\log N), \tag{8.6}$$

$$\mathfrak{A}_{1,1}^{1}(N)=\sum_{k,l=1}^{N}\frac{1}{\{k,l\}}=\frac{2}{\pi^2}\log^3 N+O(\log^2 N), \tag{8.7}$$

$$\begin{cases}\mathfrak{A}_{1,1}^{2}(N)=\sum\limits_{k,l=1}^{N}\dfrac{(k,l)}{\{k,l\}}=D\cdot N+O(\log^2 N),\\ \qquad D=\dfrac{2}{\zeta(3)}\sum\limits_{d=1}^{\infty}d^{-2}\sum\limits_{\nu=1}^{d}\nu^{-1};\end{cases} \tag{8.8}$$

(8. 6) can be deduced from (8. 1), (8. 7) from (8. 2), while (8. 8) requires the identity

$$\sum_{k,l=1}^{N}\frac{(k,l)}{\{k,l\}}=2\sum_{d=1}^{N}\frac{\mu(d)}{d^2}\sum_{t=1}^{\left[\frac{N}{d}\right]}\left(\frac{1}{t}\sum_{\nu=1}^{t}\frac{1}{\nu}\right)\left[\frac{N}{dt}\right]-N. \tag{8.9}$$

We mention, that our asymptotic relations suggest an extension of the usual idea of *average order* for functions of several variables, namely the following

DEFINITION. *Let* $f(\nu_1,\nu_2,\ldots,\nu_p)\geqq 0$ *be an arithmetical function defined for all systems of* p *positive integers. If there exists a positive function* $\mathfrak{M}_p(f;N)$ *such that*

$$N^{-p}\sum_{\substack{\nu_j\leqq N\\(j=1,2,\ldots,p)}}f(\nu_1,\nu_2,\ldots,\nu_p)\sim\mathfrak{M}_p(f;N) \tag{8.10}$$

as $N\to\infty$, *then we say that* $f(\nu_1,\nu_2,\ldots,\nu_p)$ *is of the mean order* $\mathfrak{M}_p(f;N)$ *for* $\nu_j\leqq N$ $(j=1,2,\ldots,p)$, $N\to\infty$.[27])

Thus, in virtue of (8. 3)—(8. 4) and (8. 6)—(8. 8), the *mean order of* $(k,l)^{\varrho}/k^{\sigma_1}l^{\sigma_2}$ for $k,l\leqq N$ $(N\to\infty)$ in the case $\sigma_2\geqq\sigma_1>1$, $0\leqq\varrho<\sigma_1+\sigma_2-1$ is given by $\zeta(\sigma_1)\zeta(\sigma_2)\zeta(\sigma_1+\sigma_2-\varrho)\zeta(\sigma_1+\sigma_2)^{-1}N^{-2}$, furthermore $(k,l)^{-1}$, $\{k,l\}^{-1}$, $(k,l)/\{k,l\}$ have the mean order $6\zeta(3)/\pi^2$, $(2/\pi^2)(\log^3 N/N^2)$ and D/N, respectively; results which may be interpreted in rather instructive manner.

9. In view of the connections found in § 4, it is natural to ask, how estimates for sums of type (4. 1) can be got with the aid of the inequalities for $\mathfrak{A}_{\sigma_1,\sigma_2}^{N;\varrho}(\tilde{n}_\nu)$, proved in Theorem III.

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[27]) For $p=1$, $\mathfrak{M}_p(f;N)$ does not cover exactly the average order in the sense of [12], Ch. 13., but the difference is unimportant.

Now, we formulate still an immediate corollary of III. 2. and of a theorem of I. S. GÁL and J. F. KOKSMA:[28]) let $g_\nu(x) \in L^2(0,1)$ $(\nu = 1, 2, \ldots)$ and suppose

$$\int_0^1 |g_{M+1}(x) + \cdots + g_{M+N}(x)|^2 dx = O(N \log^\theta N) \qquad (\theta \geqq 0)$$

uniformly with respect to M, then the limit relation

$$|g_1(x) + \cdots + g_N(x)| = o\left(N^{\frac{1}{2}} \log^{\frac{3}{2}+\frac{\theta}{2}+\varepsilon} N\right) \qquad (\varepsilon > 0, \text{ arbitrary})$$

holds at almost every point $x \in (0,1)$.

Theorem V. *If $\sigma > 1$, we have in case of any sequence $n_1 < n_2 < n_3 < \cdots$ of positive integers, for almost all x, the relation*

$$(9.1) \qquad \Theta_s^N(\tilde{n}_\nu; x) = \sum_{\nu=1}^{N} \mathfrak{Z}_s(n_\nu x) = o\left(N^{\frac{1}{2}} \log^{\frac{3}{2}+\varepsilon} N\right),$$

where ε denotes an arbitrarily small positive number.

In fact, putting $s_1 = \bar{s}_2 = s = \sigma + i\tau$ $(\sigma > 1)$ in (4. 5) and combining it with (7. 16), we obtain

$$(9.2) \qquad \int_0^1 \left| \sum_{\nu=M+1}^{M+N} \mathfrak{Z}_s(n_\nu x) \right|^2 dx < 2(2\pi)^{-2\sigma} \operatorname{ch} \pi\tau \zeta(2\sigma) \zeta(\sigma)^2 N \qquad (M = 1, 2, \ldots);$$

by the above-cited result, (9. 2) implies the assertion at once.

Note that for $s = r$ $(r > 1$, fixed integer), (9. 1) yields p. p.

$$(9.3) \qquad \sum_{\nu=1}^{N} B_r(n_\nu x - [n_\nu x]) = o\left(N^{\frac{1}{2}} \log^{\frac{3}{2}+\varepsilon} N\right)$$

(cf. (3. 12)), while trivial is only the estimate $O(N)$, holding everywhere.

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[28]) Cf. [6]. p. 203, Théorème 6. — This is a special case of a quite more general proposition of the authors.

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