Filling of a domain by equiareal discs.

To Professor O. Varga on his 50th birthday.

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Let us place in a given domain[1]) D a given number n of non-overlapping convex discs of the same given area $t \leqq D/n$. Suppose that the boundaries of the discs strive to contract in such a way that the total perimeter of the discs should take the least possible value. What shape and arrangement the discs will assume under these conditions?

The asymptotic behaviour of the extremal configuration for great values of n may be described as follows. For small values of t the discs are circles, the arrangement of which is not determined uniquely. For a certain value of $t \sim \frac{\pi}{\sqrt{12}} \frac{D}{n}$ the circles get into close-packing in which arrangement "almost every" circle is touched by six other ones. Increasing t further, the circles blow up to "smooth hexagons" (*Fig.* 1) which, for $t = D/n$, will turn into common regular hexagons.

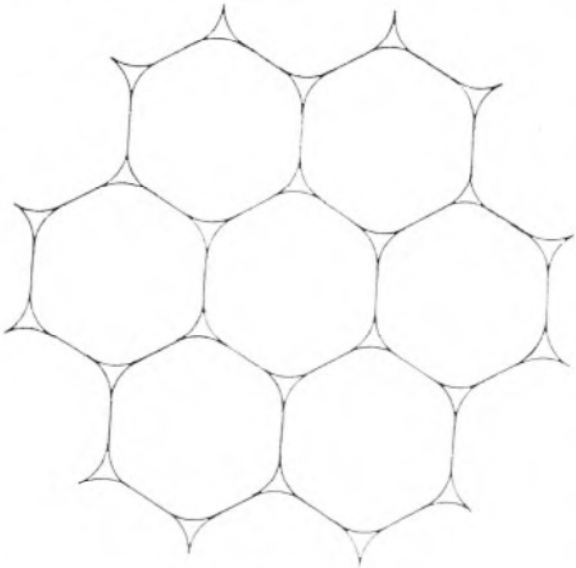

Fig. 1.

Since for great values of n the special shape of D plays no part, we shall restrict ourselves to convex hexagons, i. e. to polygons having at most six sides. This allows us to give an estimation for all values of $n \geqq 1$. Our main result is contained in the following

[1]) We denote a domain and its area by the same symbol.

Theorem. *If p denotes the average perimeter of n convex discs, each of area t, lying in a convex hexagon H without mutual overlapping, then*

$$\frac{p^2}{t} \geqq \begin{cases} \left\{ \sqrt{\sqrt{12}\,\frac{H}{nt}} - \sqrt{(\sqrt{12}-\pi)\left(\frac{H}{nt}-1\right)} \right\}^2 & \text{for } \frac{\pi}{\sqrt{12}} \leqq \frac{nt}{H} \leqq 1 \\ 4\pi & \text{for } \frac{nt}{H} < \frac{\pi}{\sqrt{12}}. \end{cases}$$

Note that nt/H is the packing density of the discs and $\pi/\sqrt{12}$ equals the density of the closest circle-packing.

Denoting the inner and outer parallel-domain of distance ϱ of a convex polygon P by $P_{-\varrho}$ and P_ϱ, a smooth polygon is defined by $(P_{-\varrho})_\varrho$ where ϱ is a positive number less then the inradius of P. It arises by rounding off the corners of P by arcs of a circle of radius ϱ. Considering a circle and a common polygon as degenerated smooth polygons, we can say that our bound for p^2/t equals the "isoperimetric quotient" of a smooth hexagon of area t arising from a regular hexagon of area H/n.

Apart from the case that each disc is a circle, this bound can be attained only if H is a regular hexagon containing only one disc, namely a smooth hexagon belonging to it. But it can be approximated with an arbitrary exactitude for great values of n.

In order to prove our theorem we start by blowing up the discs[2]), preserving their convexity and the property of neither overlapping nor streching out from H. We obtain n convex polygons $P_1, \ldots, P_n$ satisfying the relations

$$P_1 + \cdots + P_n \leqq H$$
$$\nu_1 + \cdots + \nu_n \leqq 6n,$$
$$P_i \supset d_i,$$

ν_i being the number of the sides of P_i and $d_1, \ldots, d_n$ the original discs.

As second step we show that the perimeter p_i of d_i satisfies the inequality $p_i \geqq \Phi(P_i, \nu_i)$, where

$$\Phi(P, \nu) = \begin{cases} 2\sqrt{P\nu \operatorname{tg}\frac{\pi}{\nu}} - 2\sqrt{(P-t)\left(\nu \operatorname{tg}\frac{\pi}{\nu} - \pi\right)} & \text{for } t \leqq P \leqq t\,\frac{\nu}{\pi}\operatorname{tg}\frac{\pi}{\nu},\ \nu \geqq 3 \\ 2\sqrt{\pi t} & \text{for } P > t\,\frac{\nu}{\pi}\operatorname{tg}\frac{\pi}{\nu},\quad \nu \geqq 3 \end{cases}$$

denotes the perimeter of a smooth polygon of area t arising from a regular ν-gon of area P (*Fig.* 2).

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[2]) Cf. L. Fejes Tóth, Filling of a domain by isoperimetric discs, *Publ. Math. Debrecen* 5 (1957) 119—127.

For the following proof we are obliged to Professor H. HADWIGER.

Let P be a convex ν-gon of inradius r and perimeter L and let q be a ν-gon circumscribed about a unit circle the sides of which have the same outer normal directions as the sides of P. Further, let $\bar{P}$ be a regular ν-gon of area $\bar{P}=P$ the corresponding data of which are $\bar{r}\geqq r, \bar{L}\leqq L$ and $\bar{q}\leqq q$ (*Fig.* 3). We consider the non-degenerated smooth polygon $\bar{d}=(\bar{P}_{-\varrho})_\varrho$ $(0<\varrho<\bar{r})$ of perimeter $\bar{p}$ and a convex domain $d\subset P$ of area $d=\bar{d}$ and perimeter p. We have to show that $p\geqq\bar{p}$.

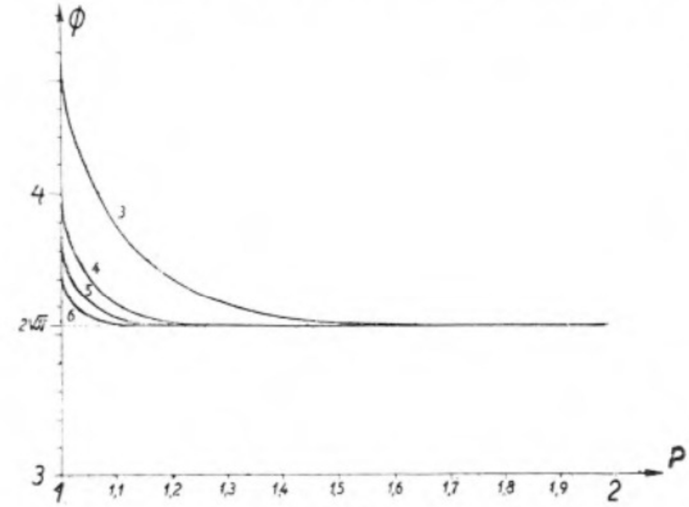

Fig. 2.

Suppose first that $0<\varrho<r$. Denoting the corresponding data of the inner parallel domains of distance ϱ by the index $-\varrho$, we have

$$d=d_{-\varrho}+\int_0^\varrho p_{-t}\,dt.$$

Observing that $d_{-\varrho}\subset P_{-\varrho}$ and $(d_{-t})_t\subset d$ we obtain $d_{-\varrho}\leqq P_{-\varrho}$ and $p_{-t}+2\pi t\leqq p$. Hence

$$d\leqq P_{-\varrho}+p\varrho-\pi\varrho^2.$$

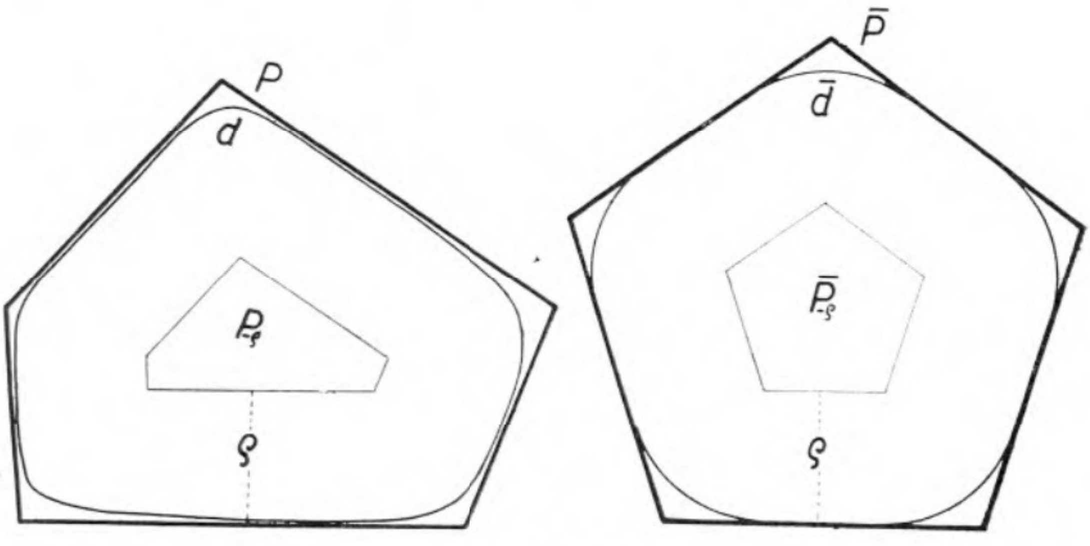

Fig. 3.

Now we show that $P_{-\varrho} \leqq \bar{P}_{-\varrho}$. In consequence of

$$P_{-\varrho} = P - \int_0^{\varrho} L_{-t} dt$$

and $\frac{d}{dt} L_{-t} = -2q_{-t}$ we have

$$\frac{d^2}{dt^2} P_{-t} = \frac{d}{dt}(-L_{-t}) = 2q_{-t}$$

and, analogously,

$$\frac{d^2}{dt^2} \bar{P}_{-t} = \frac{d}{dt}(-\bar{L}_{-t}) = 2\bar{q}_{-t} = 2\bar{q},$$

showing that the funktion $f(t) = \bar{P}_{-t} - P_{-t}$ is concave: $f''(t) = 2(\bar{q} - q_{-t}) \leqq 0$. This implies, in view of $f(0) = \bar{P} - P = 0$ and $f(r) = \bar{P}_{-r} \geqq 0$, the desired inequality $f(\varrho) = \bar{P}_{-\varrho} - P_{-\varrho} \geqq 0$.

Thus we find that

$$d \leqq \bar{P}_{-\varrho} + p\varrho - \pi\varrho^2.$$

Comparing this inequality with

$$d = \bar{d} = \bar{P}_{-\varrho} + \bar{p}\varrho - \pi\varrho^2$$

we obtain $p \geqq \bar{p}$.

Assuming now that $r \leqq \varrho < \bar{r}$, we have

$$d = \bar{d} \geqq \bar{p}\varrho - \pi\varrho^2$$

which involves by

$$\frac{d}{dt}(\bar{p}t - \pi t^2) = \bar{p} - 2\pi t > 0, \qquad 0 < t \leqq \bar{r}$$

the inequality

$$\bar{d} \geqq \bar{p}r - \pi r^2.$$

This implies, together with

$$d \leqq pr - \pi r^2$$

again $p \geqq \bar{p}$.

Now we shall show that the function $\Phi(P, \nu)$ is convex.

Since $\Phi(P, \nu)$ is for $P > t\frac{\nu}{\pi}\operatorname{tg}\frac{\pi}{\nu}$, $\nu \geqq 3$ linear and, as a function of P, for any fixed value of $\nu \geqq 3$ convex, we have only to show that for $t \leqq P \leqq t\frac{\nu}{\pi}\operatorname{tg}\frac{\pi}{\nu}$, $\nu \geqq 3$ the inequality

$$\Phi_{PP}\Phi_{\nu\nu} - \Phi_{P\nu}^2 \geqq 0$$

holds.

We introduce the notations

$$x = \frac{P}{t}, \quad y = \frac{\nu}{\pi}\operatorname{tg}\frac{\pi}{\nu}$$

and consider instead of Φ the function

$$z = \frac{1}{2\sqrt{\pi t}}\,\Phi = x^{\frac{1}{2}}\,y^{\frac{1}{2}} - (x-1)^{\frac{1}{2}}(y-1)^{\frac{1}{2}},\quad 1 \leqq x \leqq y \leqq \frac{3\sqrt{3}}{\pi}.$$

By the transformations[3])

$$x = \operatorname{ch}^2 u, \quad y = \operatorname{ch}^2 v$$

this function turns into

$$z = \operatorname{ch}(v-u), \quad 0 \leqq u \leqq v \leqq \operatorname{arch}\sqrt{\frac{3\sqrt{3}}{\pi}}.$$

By some computation we obtain

$$\Phi_{PP}\,\Phi_{\nu\nu} - \Phi_{P\nu}^2 = 4\pi t^{-1}(z_{xx}z_{\nu\nu} - z_{x\nu}^2) =$$
$$= 4\pi t^{-1}\operatorname{sh}(v-u)\,[\operatorname{ch}(v-u)(u_x^2 v_{\nu\nu} - v_\nu^2 u_{xx}) - \operatorname{sh}(v-u)\,u_{xx}v_{\nu\nu}].$$

Therefore, in view of

$$u_{xx} = -\frac{1}{4}x^{-\frac{3}{2}}(x-1)^{-\frac{3}{2}}(2x-1) < 0,$$

the inequality $\Phi_{PP}\,\Phi_{\nu\nu} - \Phi_{P\nu}^2 \geqq 0$ will be proved by showing that $v_{\nu\nu} > 0$.

Writing $\frac{\pi}{\nu} = \omega$ we can represent v_ν in the form

$$v_\nu = -\frac{1}{2\pi}ABC,$$

where

$$A = (y\cos^4\omega)^{-\frac{1}{2}},\ B = \left(\frac{y-1}{\omega^4}\right)^{-\frac{1}{2}},\ C = \frac{1}{\omega}\left(1 - \frac{\sin 2\omega}{2\omega}\right),\quad 0 < \omega \leqq \frac{\pi}{3}.$$

Hence

$$v_{\nu\nu} = -\frac{\pi}{\nu^2}v_{\nu\omega} = \frac{1}{2\nu^2}(A_\omega BC + AB_\omega C + ABC_\omega).$$

But A, B and C being positive, it suffices to show that also $A_\omega > 0, B_\omega > 0$ and $C_\omega > 0$.

Since in the interval $\left(0, \frac{\pi}{3}\right)$

$$A^{-2} = y\cos^4\omega = \frac{\sin 2\omega}{2\omega}\cos^2\omega$$

is a decreasing function of ω, A is an increasing function and thus $A_\omega > 0$.

We proceed to prove that

$$B_\omega = -\frac{1}{2}B^3\frac{dB^{-2}}{d\omega} = \frac{5B^3}{2\omega^5\cos^2\omega}\left(\frac{\sin 2\omega}{2\omega} + \frac{4}{5}\sin^2\omega - 1\right) > 0.$$

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[3]) We are indebted to Dr. A. Békéssy for suggesting these transformations.

For $\omega \leqq 1$ we have

$$\frac{\sin 2\omega}{2\omega}+\frac{4}{5}\sin^2\omega>\frac{1}{2\omega}\left(2\omega-\frac{8\omega^3}{3!}+\frac{32\omega^5}{5!}-\frac{128\omega^7}{7!}\right)+\frac{4}{5}\left(\omega-\frac{\omega^3}{3!}\right)^2=$$

$$=1+\frac{2}{15}(1-\omega^2)\omega^2+\frac{\omega^6}{105}>1.$$

On the other hand, for $1\leqq\omega\leqq\frac{\pi}{3}$ $\frac{\sin 2\omega}{2\omega}+\frac{4}{5}\sin^2\omega$ is, in virtue of

$$10\omega^2\left(\frac{\sin 2\omega}{2\omega}+\frac{4}{5}\sin^2\omega\right)'=10\,\omega\cos 2\omega+(8\omega^2-5)\sin 2\omega<$$

$$<10\cos 2+8\frac{\pi^2}{9}-5=-0,38\ldots<0,$$

a decreasing function. Hence

$$\frac{\sin 2\omega}{2\omega}+\frac{4}{5}\sin^2\omega\geqq\frac{\sin\frac{2\pi}{3}}{\frac{2\pi}{3}}+\frac{4}{5}\sin^2\frac{\pi}{3}=1{,}013\ldots>1,$$

on account of which in the whole interval $\left(0,\frac{\pi}{3}\right)$ $B_\omega>0$ holds.

Finally we have

$$C_\omega=\frac{2\cos^2\omega}{\omega^3}(\operatorname{tg}\omega-\omega)>0.$$

This completes the proof of the convexity of $\Phi(P,\nu)$.

At last we remark that, in view of $1\leqq x\leqq y$,

$$2z_x=x^{-\frac{1}{2}}y^{\frac{1}{2}}-(x-1)^{-\frac{1}{2}}(y-1)^{\frac{1}{2}}\leqq 0,$$

$$2z_y=x^{\frac{1}{2}}y^{-\frac{1}{2}}-(x-1)^{\frac{1}{2}}(y-1)^{-\frac{1}{2}}\geqq 0,$$

from wich we conclude, by $y_\nu<0$, that $\Phi(P,\nu)$ is in both variables a non-increasing function.

Now we hawe, in virtue of the above remarks and JENSEN's inequality,

$$\sum_{i=1}^{n}p_i\geqq\sum_{i=1}^{n}\Phi(P_i,\nu_i)\geqq n\,\Phi\left(\frac{P_1+\cdots+P_n}{n},\frac{\nu_1+\cdots+\nu_n}{n}\right)\geqq n\,\Phi\left(\frac{H}{n},6\right).$$

This is just the inequality to be proved.

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