On the singularities of a Riemannian manifold.

Dedicated to Professor Ottó Varga, at the occasion of his 50th birthday.

By GY. SZEKERES (Adelaide, S. Australia).

§ 1. Introduction.

Riemannian manifolds, such as the ones which find application in the theory of general relativity, often have regions of singularity where the determinant of the metric tensor becomes zero or the components (or derivatives of components) of the metric tensor become infinite. Well known examples are the singular hypersurfaces which appear in Schwartzschild's and de Sitter's centrosymmetrical solutions of the gravitational field equations. The Schwartzschild field is a Riemannian manifold with the line element

$$\pm ds^2 = \left(1 - \frac{2m}{r}\right)^{-2} dr^2 + r^2 d\omega^2 - \left(1 - \frac{2m}{r}\right) dt^2, \tag{1}$$

$$d\omega^2 = d\theta^2 + \cos^2\theta \, d\varphi^2; \tag{2}$$

it consists of two disjoint regions, $0 < r < 2m$ and $r > 2m$, separated by the singular hypercylinder $r = 2m$. In de Sitter's Universe

$$\pm ds^2 = (1 - r^2/R^2)^{-1} dr^2 + r^2 d\omega^2 - (1 - r^2/R^2) dt^2, \tag{3}$$

there is a singular hypersurface at $r = R$, the "mass horizon" of Eddington.

Neither the Schwartzschild nor the de Sitter hypercylinders are true singularities. In the Schwartzschild case this was first noticed by LEMAÎTRE [2] and worked out in greater detail by SYNGE [4]; in the case of the de Sitter Universe by EDDINGTON ([1], Chapter V). In each case the argument was based on the observation that the singularities can be transformed away in a suitable coordinate system; but an exact definition of what should be regarded as a true singularity of a Riemannian manifold has, to my knowledge, never been proposed.[1])

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[1]) SYNGE ([4], p. 100) gives a definition which however depends on the coordinate system, and refers therefore to singularities of coordinate systems rather than of the manifolds themselves.

Superficially, we may conceive of a singularity as a place in coordinate space where something goes wrong with the metric tensor. But surely this is not a sufficient criterion, since apparent singularities can easily be produced by the simple device of introducing "bad" coordinates in otherweise perfectly well-behaved manifolds. Perhaps the most trivial example of an apparent singularity is the one at $r=0$ of the line element of Euclidean space in polar coordinates,

$$ds^2 = dr^2 + r^2\,d\omega^2. \tag{4}$$

Clearly the singularity is due to the coordinate system, and not to any irregularity of the manifold itself. If we make the substitution $\bar{r} = r - 2m$ in (4), where m is a positive number, we get

$$ds^2 = d\bar{r}^2 + (\bar{r} - 2m)^2\,d\omega^2. \tag{5}$$

Here we have an apparent singularity on the sphere $\bar{r} = 2m$, due to a spreading out of the origin over a sphere of radius $2m$. Since the exterior region $\bar{r} > 2m$ represents the whole of Euclidean space (except the origin), the interior $\bar{r} < 2m$ is entirely disconnected from it and represents a distinct manifold.

In some respects the line elements (4) and (5) are not unlike the Schwartzschild metric (1). This becomes more apparent if we make the substitution

$$\bar{r} = \frac{1}{2m}\int_{2m}^{r}\left(1 - \frac{2m}{r}\right)^{-\frac{1}{2}} dr, \tag{6}$$

valid for $r > 2m$; it transforms (1) into

$$\pm\, ds^2 = 4m^2\,d\bar{r}^2 + r^2\,d\omega^2 - \left(1 - \frac{2m}{r}\right)dt^2 \tag{7}$$

where $r = f(\bar{r})$ is given by (6). In the neighbourhood of $\bar{r} = 0$,

$$r = f(\bar{r}) = 2m\left(1 + \frac{1}{4}\bar{r}^2 - \frac{1}{48}\bar{r}^4 + \dots\right),$$

$$1 - \frac{2m}{r} = \frac{1}{4}\bar{r}^2 - \frac{1}{12}\bar{r}^4 + \dots$$

and the coefficient of dt^2 becomes zero, of the same order as the coefficient of $d\omega^2$ in (4) or (5); in fact the singularity disappears if we make the substitution

$$u = h\cosh\frac{t}{4m}, \qquad v = h\sinh\frac{t}{4m} \tag{8}$$

where $h = h(\bar{r})$ is a solution of

$$d \log h / d\bar{r} = \frac{1}{2}\left(1 - \frac{2m}{r}\right)^{-\frac{1}{2}}, \qquad h'(0) = 1.$$

In terms of r,

$$h^2 = 4pe^p, \qquad p = (r - 2m)/2m. \tag{9}$$

In the new coordinate system

$$\pm\, ds^2 = 4m^2\{(1+p)^{-1} e^{-p}(du^2 - dv^2) + (1+p)^2 d\omega^2\} \tag{10}$$

where p is determined from

$$4pe^p = u^2 - v^2. \tag{11}$$

Although the transformations (8) and (9) which carry (1) into (10) are only valid for $u^2 > v^2$, $u > 0$, the line element (10) is regular in a whole neighbourhood of the line $u = v$, $\theta = 0$, $\varphi = 0$, indeed for $u^2 - v^2 > -4/e$ and $-\frac{1}{2}\pi < \theta < \frac{1}{2}\pi$, $-\pi < \varphi < \pi$. This shows that the Schwartzschild "singularity" is just as apparent, brought about by an improper choice of coordinates, as the origin of polar coordinates.[2])

Incidentally, the transformation

$$u = h \sinh \frac{t}{4m}, \qquad v = h \cosh \frac{t}{4m}, \tag{8'}$$

$$h^2 = -4pe^p, \qquad p = \frac{r}{2m} - 1 \tag{9'}$$

carries the region $0 < r < 2m$ of (1) into the region $0 < v^2 - u^2 < 4/e$ of (10), so that the interior of the Schwartzschild hypercylinder is a natural continuation of the exterior.

The manifold (10) differs from (1) in one important respect: each point of (1) is represented exactly twice in (10). Hence in order to obtain a manifold which represents physical reality. it seems to be necessary to identify all pairs of opposite points (u, v) and $(-u, -v)$; this is permissible since the metric at (u, v) is identical whith the metric at $(-u, -v)$. The situation is analogous to the construction of elliptic space from hyperspherical space,[3]) but there is a significant difference. Whereas elliptic space has no singularities, the

[2]) This of course is not an entirely valid conclusion as long as we do not have a precise definition of a singularity; it has the same heuristic character as the similar conclusions of LEMAÎTRE and SYNGE. The transformations (8) and (8') are essentially due to SYNGE.

[3]) [3], p. 7.

Schwartzschild identification introduces an artifical singularity at $u=0$, $v=0$, essentially of the same kind as the singularity at the vertex of a cone obtained by identifying the points (x, y) and $(-x, -y)$ of Euclidean plane. Since (10) is a perfectly well-behaved Riemannian manifold which satisfies the gravitational field equations in the whole region $u^2-v^2>-4/e$, including the origin $(0, 0)$, it seems difficult to find any physical justification (apart from a purely utilitarian one) for this identification process.

The problem of continuation of Riemannian manifolds is fundamental to relativistic cosmology; for on it depends the answer to the question whether a given cosmological frame covers the whole of the Universe, or only part of it. In the case of the de Sitter Universe (3) the answer is well known: if one replaces r, t by new coordinates $\chi, \bar{t}$, given by

$$r = R\cos\chi\cosh\bar{t}, \qquad \tanh(t/R) = \tanh\bar{t}/\sin\chi,$$

the new system extends beyond the mass horizon[4]) and the line element becomes

$$\pm\, ds^2 = R^2\cosh^2\bar{t}\,\{d\chi^2 + \cos^2\chi\,(d\theta^2 + \cos^2\theta\, d\varphi^2)\} - R^2 d\bar{t}^2. \tag{11}$$

This is the metric on a 4-hyperboloid

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 - x_5^2 = R^2$$

in pseudo- Euclidean 5-space with signature $(+ + + + -)$ and there is no way to extend it any further.[5])

The substitution

$$x_1 = \bar{r}\cos\theta\cos\varphi, \qquad x_2 = \bar{r}\cos\theta\sin\varphi, \qquad x_3 = \bar{r}\sin\theta,$$

$$\bar{r} = r(1-r^2/R^2)^{-\frac{1}{2}}\, e^{-t/R}, \qquad \bar{t} = t + \frac{1}{2}R\log(1-r^2/R^2)$$

transforms (3) into the Robertson frame with the line element

$$\pm\, ds^2 = e^{2\bar{t}/R}(dx_1^2 + dx_2^2 + dx_3^2) - d\bar{t}^2. \tag{12}$$

This frame covers the same portion of the Universe as (3) but the metric has no singularities in the finite part of the frame. Since the manifold (12) obviously admits extension, it is clear that for a consistent theory of continuation it will be necessary to consider also singularities which lie at infinity in coordinate space.

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[4]) For a full discussion see Schrödinger [3], Chapter I.

[5]) Apart from a trivial extension to the 2-hyperboloid $x_1=0$, $x_2=0$, $x_3^2+x_4^2-x_5^2=R^2$ which is not covered by the system (11).

§ 2. Definition of a singularity.

In analogy with the theory of analytic functions, we may define a singularity of a Riemannian manifold as a boundary point into which the manifold cannot be continued in a reasonable manner. The definition can be made precise if we know what a boundary point is and what we should mean by a reasonable continuation. In the case of the Riemann surface of an analytic function the problem is fairly trivial as the surface is superimposed upon the number sphere and a "reasonable" continuation can be accomplished by means of power series. In Riemannian manifolds the analytic structure is carried by the Riemannian metric, and it is in terms of this metric that we have to formulate the process of continuation.

Let us briefly review the conventions that we are going to use. An n-dimensional manifold is a connected Hausdorff space R with the property that each point p of R has a neighbourhood U which is homeomorphic to an open set V in real Euclidean n-space E_n. Let φ be a homeomorphism of U onto V, $\varphi(p) = \mathbf{x} = \{x_\mu;\ \mu = 1, \ldots, n\}$. U and φ determine a local coordinate system or coordinate frame X and we say that p is covered by X. In a Riemannian manifold there is also given a symmetrical metric tensor $g_{\mu\nu}(\mathbf{x})$ which obeys the usual transformation laws. In the non-definite case we agree to choose the sign of the line element

$$ds^2 = \pm \sum_{\mu,\nu} g_{\mu\nu} dx_\mu dx_\nu \tag{13}$$

so that always $ds \geqq 0$. We also assume that R is of class ∞, that is, it can be covered by a set of "admissible" frames so that in these frames the $g_{\mu\nu}(\mathbf{x})$ have continuous partial derivatives of any order. Our standpoint is that a point which cannot be covered by such a frame is a singularity.

A further restriction on admissible frames is that they cover a connected domain in which the determinant of the metric tensor does not vanish. Thus in the example of (1), the exterior $r > 2m$ and interior $r < 2m$ are covered by distinct frames, even if it is the same analytical formula which expresses the metric tensor in both parts. If $\bar{R}$ is a submanifold of R in the relative topology of R, of the same dimension as R and relatively to the same set of admissible frames, then R will be called an *extension* of $\bar{R}$. In particular, we shall call R *complete* if it cannot be immersed in any proper extension.

An open set D of R will be called a *domain* if and only if it can be covered by a single admissible frame. Our first purpose is to define boundary points of a domain, and this will be done by means of geodesic arcs emanating from that boundary point.

Let D be a domain, φ the coordinate mapping of a frame X which covers D. By a *geodesic arc in D* we shall always mean a semi-open geodesic arc[6])

$$L: p(s) = \varphi^{-1}(\mathbf{x}(s)), \qquad p(s) \in D, \qquad 0 < s \leqq b \tag{14}$$

where $0 < b < \infty$ and s is a distinguished parameter in which the geodesic equations have the form

$$\frac{d^2 x_\mu}{ds^2} + \left\{ {\mu \atop \varrho\,\sigma} \right\} \frac{dx_\varrho}{ds} \frac{dx_\sigma}{ds} = 0, \qquad \mu = 1, \ldots, n. \tag{15}$$

Here $\mathbf{x}(s) = \{x_\mu(s)\}$ and $\left\{ {\mu \atop \varrho\,\sigma} \right\}$ is the Christoffel affinity derived from $g_{\mu\nu}$. The parameter s is uniquely determined apart from a constant factor; if L is not a null geodesic, we make the parameter unique by requiring that $b - s$ be equal to the length of arc measured from the endpoint $p(b)$ of L.

A geodesic arc will be called an *interior arc* of D if $\mathbf{y} = \lim\limits_{s \to 0} \mathbf{x}(s)$ exists and $\mathbf{y} \in \varphi(D)$. The point $q = \varphi^{-1}(\mathbf{y})$ is called the *origin* of L. If $\lim\limits_{s \to 0} \mathbf{x}(s)$ does not exist, or it does exist but is not in $\varphi(D)$, then L will be called a *boundary arc*. Note that a boundary arc has always a finite length.

Clearly the definition of an interior arc or boundary arc is independent of the frame which covers D. If L is a boundary arc, we say that it determines a *boundary point*[7]) of D; if also $\mathbf{y} = \lim\limits_{s \to 0} \mathbf{x}(s)$ exists in a certain coordinate frame, we say that the boundary point (or the origin of L) lies at $\mathbf{y}$ (possibly at infinity). We agree that every *subarc* of L, i. e. every boundary arc of the form

$$L': p(s) = \varphi^{-1}(x(s)), \qquad 0 < s \leqq b_1 < b$$

determines the same boundary point as L. A more general concept of equivalence of boundary arcs will be needed and established later on.

It may happen that L is a boundary arc in a certain domain D but an interior arc in a domain D^*. We say that L is a boundary arc of R if the following is true: (i) L is a boundary arc in some domain D of R; (ii) no subarc of L is an interior arc of any domain D^* of R. The boundary point defined by L is then a boundary point of R; and it is called a *singularity* if it remains a boundary point in every extension of R. This at the moment is a rather loose definition; the final form must take account of the equivalence of boundary arcs. Note that if a proper extension of R exists at all

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[6]) By definition, L is a bicontinuous map of the interval $0 < s \leqq b$.

[7]) We disregard possible boundary points which are not accessible by a geodesic arc. They would not contribute anything new to the extension problem.

then R must necessarily have non-singular boundary points; and R is complete if and only if all its boundary points are singularities. Thus every R which has no boundary arcs is complete.

In order to arrive at a useful theory of equivalence of boundary arcs, we introduce the concept of *normal coordinates* relatively to a geodesic arc L. These are combinations of Fermi and Riemann coordinates, and are defined as follows. Suppose first that L is a non-null geodesic. Then normal coordinates relatively to L are coordinates y_μ with the following properties:

(i) The metric tensor takes Galilean values $g_{\mu\nu} = \pm\delta_{\mu\nu}$ at all points of L.

(ii) $\partial g_{\mu\nu}/\partial y_\varrho = 0$ for every μ, ν, ϱ at all points of L.

(iii) The equation of L is $y_\mu = \delta_{\mu n} s$ $(0 < s \leqq b)$.

(iv) The equation of any geodesic G which radiates from a point $p(\sigma)$, $0 < \sigma \leqq b$ of L, perpendicularly to L, is

$$y_\mu = \gamma_\mu s \quad \text{for} \quad 1 \leqq \mu < n, \qquad y_n = \sigma,$$

where the γ_μ are constants.

To construct coordinates with these properties, select $n-1$ mutually perpendicular non-null unit vectors $\mathbf{a}^1, \ldots, \mathbf{a}^{n-1}$, each perpendicular to L, at the endpoint $p(b)$. Through Fermi propagation of these vectors (i. e. congruent displacement along L) we obtain a set of coordinate directions $\mathbf{a}^\nu(\sigma)$ at each $p(\sigma)$. For the y_n-coordinate along L choose $y_n = s = b - \lambda$ where b is the total length of L (hence finite) and λ is the length of arc measured from $p(b)$; for the coordinate plane $y_n = \sigma$ take the hyperplane $P(\sigma)$ generated by all the geodesics running from $p(\sigma)$ in the directions spanned by the vectors $\mathbf{a}^\nu(\sigma)$. Finally, in $P(\sigma)$ itself we choose Riemannian coordinates corresponding to the $n-1$ predetermined coordinate directions $\mathbf{a}^\nu(\sigma)$. Clearly these coordinates will have the required properties (i)—(iv), the only freedom in their choice being a rotation of the axes of y_μ, $\mu < n$, about the axis of y_n.

If L is a null arc, some modification is necessary. For if $\mathbf{a}^2, \ldots, \mathbf{a}^{n-1}$ are mutually perpendicular non-null vectors at $p(b)$, each perpendicular to L, then $\mathbf{a}^1$ cannot have this same property. However, we can choose $\mathbf{a}^1$ to be a null-vector perpendicular to $\mathbf{a}^2, \ldots, \mathbf{a}^{n-1}$ and not in the direction of L. This condition specifies the direction of $\mathbf{a}^1$ uniquely and yields an admissible set of coordinate directions. The rest of the construction is same as above, except that s is not the length of arc of L but a distinguished parameter. By adjusting the magnitude of $\mathbf{a}^1$ if necessary, we obtain normal coordinates with the following properties:

(i) $g_{1n} = 1$, $g_{11} = g_{nn} = 0$, $g_{\mu\nu} = \pm\delta_{\mu\nu}$ $(2 \leqq \mu \leqq \nu \leqq n-1)$ at all points of L;

(ii) $\partial g_{\mu\nu}/\partial y_\varrho = 0$ for every μ, ν, ϱ at all points of L;

(iii) the equation of L is $y_\mu = s\delta_{\mu\nu}$ $(0 < s \leqq b)$;

(iv) the equation of any geodesic G which radiates from $p(\sigma)$ on L in any direction in the coordinate hyperplane of $y_1, \ldots, y_{n-1}$ is

$$y_\mu = \gamma_\mu s \text{ for } 1 \leqq \mu < n, \qquad y_n = \sigma,$$

where the γ_μ are constants.

Again these coordinates are uniquely determined, apart from a constant rotation of the coordinate axes of y_μ, $1 < \mu < n$, in the plane of $y_2, \ldots, y_n$.

Among all coordinate systems which cover L, normal coordinates are undoubtedly the best behaved in the neighbourhood of L. Two things in particular should be noted about normal coordinates. First, their construction is entirely independent of the original frame which covered L. From any other frame we would have arrived to exactly the same family of normal systems. Secondly, the construction does not require any knowledge of the behaviour of the manifold at the origin of L itself. This property is extremely useful from the point of view of detecting singularities. Suppose that L is an interior arc; then there exists a geodesic arc L^* with the property that (i) L is contained in L^*, (ii) $q = \lim_{s\to 0} p(s)$ is an interior point of L^*. Let Y be a normal frame relatively to L. We can introduce normal coordinates y^*_μ relatively to L with the property that $y^*_\mu = y_\mu$ $(\mu = 1, \ldots, n-1)$, $y_n = y^*_n + c$ for a positive constant c in a neighbourhood of L. In other words, we can continue the normal frame of L so that it should also cover the origin of L. It follows therefore that if $g^{(k)}_{\mu\nu}(\mathbf{y}(\sigma))$ denotes any partial derivative of $g_{\mu\nu}$ of order $k \geqq 1$ in the system Y at the point $p(\sigma) = \varphi^{-1}(y(\sigma))$ of L and $g^{(k)}_{\mu\nu}(\mathbf{y}^*(\sigma + \mathrm{c}))$ the corresponding derivative in the system Y^*, we have $g^{(k)}_{\mu\nu}(\mathbf{y}(\sigma)) = g^{(k)}_{\mu\nu}(\mathbf{y}^*(\sigma + c))$ and in particular

$$g^{(k)}_{\mu\nu}(y^*(c)) = \lim_{\sigma\to 0} g^{(k)}_{\mu\nu}(\mathbf{y}(\sigma)). \tag{16}$$

Hence if for some derivative $g^{(k)}_{\mu\nu}$,

$$\lim_{\sigma\to 0} g^{(k)}_{\mu\nu}(\mathbf{y}(\sigma)) \tag{17}$$

does not exist (or is infinite) then L is certainly a boundary arc of R and defines a singularity. Thus normal coordinates provide us with a convenient method to detect singularities.

We shall call a singularity determined by L *ordinary* if there exists a derivative $g^{(k)}_{\mu\nu}$ for which the limit (17) does not exist (or is infinite). There exist non-ordinary singularities for which all limits (17) exist and are finite for every $k > 0$. Such is the case for instance with the origin $\mathbf{O}: u=0, v=0$ of the Schwartzschild manifold, obtained from (10) by identifying opposite points. Any geodesic arc whose origin is at $\mathbf{O}$ determines a non-ordinary singularity; this is obvious from the fact that in the complete manifold (10), $\mathbf{O}$ is an interior point.

In this last example it is clearly undesirable to have a separate singularity for each boundary arc which originates at $\mathbf{O}$. This leads us immediately to the problem of equivalence of boundary arcs. What we need is an equivalence relation of geodesic arcs, sufficiently strong to identify all interior arcs with the same origin, but sufficiently weak to keep interior and boundary arcs apart.

Let Y be a normal coordinate system relatively to the geodesic arc L, and D the domain covered by Y. Let L^* be a second geodesic arc in D, not necessarily distinct from L, Y^* a normal system relatively to L^* and suppose that Y^* covers a sub-arc L' of L. We say that L^* is *associated directly* with L if the following is true:

(i) If $y_\mu(\sigma^*)$ $(\mu=1, \ldots, n,\ 0<\sigma^* \leqq b^*)$ are the coordinates in Y of the point $p^*(\sigma^*)$ of L^* where σ^* is a distinguished parameter, then

$$\lim_{\sigma^* \to 0} y_\mu(\sigma^*) = 0 \quad \text{for} \quad \mu = 1, \ldots, n;$$

(ii) If $y^*_\mu(\sigma)$ $(\mu=1, \ldots, n,\ 0<\sigma \leqq b)$ are the coordinates in Y^* of the point $p(\sigma)$ of L' where σ is a distinguished parameter, then

$$\lim_{\sigma \to 0} y^*_\mu(\sigma) = 0 \quad \text{for} \quad \mu = 1, \ldots, n.$$

It follows from the definition that if L^* is associated directly with L then L is associated directly with L^*. Condition (ii) is essential for symmetry and cannot be omitted; in the last section we shall give a counterexample which will show that (ii) is in fact independent of (i).

Generally, given two geodesic arcs L and L^* of R, we say that they are *associated* if and only if there exists a finite chain of geodesic arcs $L=L_0, L_1, \ldots, L_m = L^*$ with the property that L_ν and $L_{\nu+1}$ $(\nu = 0, 1, \ldots, m-1)$ are directly associated. It is clear from the definition that being associated is an equivalence relation and that the equivalence class of an interior arc L consists of exactly those interior arcs which have the same origin as L. As a corollary we find that the equivalence class of a boundary arc contains only boundary arcs; hence it is permissible to identify boundary points which are defined by associated boundary arcs. Thus by definition, a boundary point of R is a class of associated boundary arcs. The definition is relative to R and distinct boundary points may become fused in a suitable extension of R.

If the boundary point defined by the arc L is not a singularity of R then none of the associated arcs will define a singularity. Hence we can safely speak of a boundary point being a singularity, and also of being an ordinary singularity, namely if at least one of the associated arcs defines an ordinary singularity.

Non-ordinary singularities cannot be detected by an examination of the limits (17) alone, and study of these singularities should present many interesting problems. More favourable is the situation with analytic manifolds which have the property that the $g_{\mu\nu}$ are determined in a whole neighbourhood of a non-singular boundary point at which the partial derivatives are known. This remark may be utilised to construct a unique greatest extension or universal covering manifold of an analytical Riemannian manifold. However, we shall not go further into this question but conclude with the discussion of some examples.

§ 3. Examples.

To illustrate the concepts developed in the previous section, we shall examine once more the singularities of the de Sitter and Schwartzschild manifolds. The de Sitter Universe offers no particular problems: all the boundary points of (3) can be made interior points in a suitable frame, and the complete manifold (11) is free of singularities.[8])

The Schwartzschild manifold (1) is more interesting and we shall examine its boundary points in greater detail. The geodesic equations of (1) are well known; they are (in the hyperplane $\theta = 0$, which is obviously no restriction in generality),

$$\frac{d\varphi}{ds} = \frac{h}{r^2} \tag{18}$$

$$\frac{dt}{ds} = \frac{ar}{r-2m} \tag{19}$$

$$\frac{dr}{ds} = \left(a^2 - \left(\varepsilon + \frac{h^2}{r^2}\right)\left(1 - \frac{2m}{r}\right)\right)^{\frac{1}{2}} \tag{20}$$

where a, h are constants and ε has the value $+1$, -1 or 0, depending on whether the geodesic is time-like, space-like or null.

[8]) It is not entirely free of boundary points, as there are exceptional points not covered by (11), but these can easily be transformed into interior points by a suitable rotation of the frame.

There are several types of boundary arcs, in both the exterior and interior regions. In the exterior region $r > 2m$ we have:

a) Space-like arcs

$$t = t_0, \quad \frac{d\varphi}{dr} = h(r^2 - h^2)^{-\frac{1}{2}}(r^2 - 2mr)^{-\frac{1}{2}}, \quad h < 2m, \quad 2m < r \leqq 2m + d,$$

originating at $t = t_0, \varphi = \varphi_0, r = 2m$.

b) Arcs of the form

$$\frac{d\varphi}{dr} = hr^{-2}\left(a^2 - \left(\varepsilon + \frac{h^2}{r^2}\right)\left(1 - \frac{2m}{r}\right)\right)^{\frac{1}{2}}$$

$$\frac{dt}{dr} = a\left(1 - \frac{2m}{r}\right)^{-1}\left(a^2 - \left(\varepsilon + \frac{h^2}{r^2}\right)\left(1 - \frac{2m}{r}\right)\right)^{-\frac{1}{2}},$$

$$2m < r \leqq 2m + d$$

where $a \neq 0$, $\varepsilon = \pm 1$ or 0, and d is a suitable positive number. It originates at $r = 2m, \varphi = \varphi_0, t = \pm\infty$.

In the interior region $0 < r < 2m$ we have:

c) Time-like arcs

$$t = t_0, \quad \frac{d\varphi}{dr} = h(r^2 + h^2)^{-\frac{1}{2}}(2mr - r^2)^{-\frac{1}{2}}, \quad 0 < d \leqq r < 2m.$$

d) Space-like arcs

$$t = t_0, \quad \frac{d\varphi}{dr} = h(h^2 - r^2)^{-\frac{1}{2}}(2mr - r^2)^{-\frac{1}{2}}, \quad h > 2m, \quad 0 < d \leqq r < 2m.$$

e) Null arcs

$$t = t_0, \quad \frac{d\varphi}{dr} = \pm(2mr - r^2)^{-\frac{1}{2}}, \quad 0 < d \leqq r < 2m.$$

f) Arcs of the form

$$\frac{d\varphi}{dr} = hr^{-2}\left(a^2 + \left(\varepsilon + \frac{h^2}{r^2}\right)\left(\frac{2m}{r} - 1\right)\right)^{-\frac{1}{2}},$$

$$\frac{dt}{dr} = ar(2m - r)^{-1}\left(a^2 + \left(\varepsilon + \frac{h^2}{r^2}\right)\left(\frac{2m}{r} - 1\right)\right)^{-\frac{1}{2}},$$

$$a \neq 0, \quad 0 < d \leqq r < 2m, \quad \varepsilon = \pm 1 \quad \text{or} \quad 0.$$

g) Time-like arcs

$$\varphi=\varphi_0, \quad \frac{dt}{dr}=ar(2m-r)^{-1}\left(\frac{2m}{r}-1+a^2\right)^{-\frac{1}{2}},$$

a arbitrary, $0<r\leqq d$.

h) Null arcs

$$\varphi=\varphi_0, \quad \frac{dt}{dr}=\pm r(2m-r)^{-1}, \quad 0<r\leqq d.$$

i) Arcs of the form

$$\frac{d\varphi}{dr}=hr^{-2}\left(a^2+\left(\frac{h^2}{r^2}+\varepsilon\right)\left(\frac{2m}{r}-1\right)\right)^{-\frac{1}{2}},$$

$$\frac{dt}{dr}=ar(2m-r)^{-1}\left(a^2+\left(\frac{h^2}{r^2}+\varepsilon\right)\left(\frac{2m}{r}-1\right)\right)^{-\frac{1}{2}},$$

a arbitrary, $h\neq 0$, $0<r\leqq d$.

The first three of these originate at $t=t_0$, $\varphi=\varphi_0$, $r=2m$, f) originates at $r=2m$, $\varphi=\varphi_0$, $t=\pm\infty$, g), h) and i) at $r=0$, $\varphi=\varphi_0$, $t=t_0\neq\pm\infty$.

It is easy to verify that all boundary points defined by these arcs, with the exception of the last three, are interior points in (10) and therefore do not represent singularities. In fact they all lie on the lines $u^2=v^2$; a), c), d) and e) at $u=v=0$, b) and f) at $u^2=v^2\neq 0$. Of course a), c), d) and e) become non-ordinary singularities if we identify (u, v) with $(-u, -v)$.

The arcs g), h), i) originate at $r=0$, i. e. on the hyperbola $v^2-u^2=4/e$; it is reasonable to expect that they represent true singularities. Consider first a time-like arc of the type g); for convenience we take $a=1$, $\varphi=0$, and the origin at $t=0$. If s is a distinguished parameter, we have from (19) and (20)

$$dt/ds=r/(2m-r), \qquad dr/ds=(2m/r)^{\frac{1}{2}}.$$

They give on integration

$$\text{(21)} \quad L: r=2m\sigma^2, \quad t=2m\log\frac{1+\sigma}{1-\sigma}-4m\sigma-\frac{4}{3}m\sigma^3, \quad 0<\sigma\leqq d,$$

where

$$\text{(22)} \qquad \sigma=(3s/4m)^{\frac{1}{3}}.$$

Let us first study a two-dimensional reduced model

$$\text{(23)} \qquad \pm ds^2=\left(\frac{2m}{r}-1\right)dt^2-\frac{r}{2m-r}dr^2, \qquad 0<r<2m.$$

Clearly (21) is also a boundary arc of (23). The equation of a geodesic which radiates from the point $p(\sigma)$ of L, perpendicularly to L, is

$$\frac{dt}{ds}=\frac{1}{\sigma}\frac{r}{2m-r}, \qquad \frac{dr}{ds}=\left(\frac{1}{\sigma^2}+1-\frac{2m}{r}\right)^{\frac{1}{2}} \tag{24}$$

with

$$r=2m\sigma^2, \qquad t=2m\log\frac{1+\sigma}{1-\sigma}-4m\sigma-\frac{4}{3}m\sigma^3 \quad \text{at} \quad s=0. \tag{25}$$

Normal coordinates are obtained by setting

$$y_1=s, \quad y_2=\frac{4}{3}m\sigma^3, \tag{26}$$

by (22). Equations (24) give

$$\frac{\partial t}{\partial y_1}=\frac{1}{\sigma}\frac{r}{2m-r}, \qquad \frac{\partial r}{\partial y_1}=\left(\frac{1}{\sigma^2}+1-\frac{2m}{r}\right)^{\frac{1}{2}} \tag{27}$$

where

$$\sigma=(3y_2/4m)^{\frac{1}{3}}, \tag{28}$$

and (25), (26) give

$$\frac{\partial r}{\partial y_2}=\frac{1}{\sigma}, \quad \frac{\partial t}{\partial y_2}=\frac{\sigma^2}{1-\sigma^2} \quad \text{at} \quad y_1=0. \tag{29}$$

These formulae allow us to calculate $\partial^{n+1} r/\partial y_1^n \partial y_2, \partial^{n+1} t/\partial y_1^n \partial y_2$ at $y_1=0$ (hence on L) for every n.

Now

$$g_{\mu\nu}=\left(\frac{2m}{r}-1\right)\frac{\partial t}{\partial y_\mu}\frac{\partial t}{\partial y_\nu}-\frac{r}{2m-r}\frac{\partial r}{\partial y_\mu}\frac{\partial r}{\partial y_\nu}$$

in the normal system, hence $g_{11}=1$ by (27). A simple calculation gives furthermore

$$g_{22}=-1, \quad \partial^2 g_{22}/\partial y_1^2=\frac{8}{9}y_2^{-2} \quad \text{on} \quad L; \tag{30}$$

the last expression tends to infinity when $y_2\to 0$, so that L defines an ordinary singularity. But the partial derivatives with respect to y_1 on L are the same in the complete as in the reduced model, so that L defines an ordinary singularity of (1), as expected.

Next we consider a null arc

$$L_0: t=-r+2m\log\frac{2m}{2m-r}, \qquad 0<r\leqq d \tag{31}$$

of the type h). Straightforward calculation gives in the reduced model the following normal coordinates relatively to L_0:

$$r = \frac{1}{2} y_1 + y_2 - m y_1/y_2, \tag{32}$$
$$t = -y_2 + \frac{1}{2} y_1 - m y_1/y_2 + 2m \log \frac{1 + y_1/2y_2}{1 - y_2/2m}.$$

The metric tensor in this system is

$$g_{11} = 0, \quad g_{12} = 1, \tag{33}$$
$$g_{22} = 8m\, y_2^2 (2m - y_2)^{-2} \left(\frac{1}{r} - \frac{1}{y_2}\right) - 4m\, y_1 y_2^{-1} (2m - y_2)^{-1}$$

and

$$\partial^2 g_{22} / \partial y_1^2 = 4m y_2^{-3} \quad \text{on} \quad L$$

which again indicates an ordinary singularity.

It can be verified easily that in the system (32), the arc (21) is represented by

$$y_1/m = -(\sqrt{2} - 1)(y_2/m)^2 + \frac{4}{5} 2^{-3/4} (y_2/m)^{5/2} + 0(y_2^3) \tag{34}$$

where $0(y_2^3)$ refers to $y_2 \to 0$. This suggests that (21) and (31) are associated arcs. For this to be true, it would be necessary to verify that in a normal frame relatively to (21) the arc corresponding to (31) also originates at $y_1 = 0$, $y_2 = 0$. That (34) alone is not sufficient to establish equivalence is shown by the fact that there is a boundary arc of the form

$$y_1 = -y_2 + \frac{1}{2m} y_2^2 + 0(y_2^3), \qquad 0 < y_2 \leqq d \tag{35}$$

in the frame (33), corresponding to the interior arc

$$t = m(1 - 2\log 2), \qquad m < r \leqq m + d \tag{36}$$

in (23), and this is clearly not associated with L_0 since the latter determines a singularity.

It seems likely that the manifold (10) is complete and all boundary arcs originating at the same point $\varphi_0, \theta_0, u_0, v_0$ with $v_0^2 - u_0^2 = 4/e$, are associated. We shall not attempt to prove this here, but conclude with the discussion of another example,

$$\pm ds^2 = t^2 (dx_1^2 + dx_2^2 + dx_3^2) - dt^2, \qquad t > 0. \tag{37}$$

This line element was suggested by the author some time ago [5] as a possible cosmological model, with a singularity at the time origin $t = 0$. An interesting feature of this line element is that the metric tensor itself has no singularities, and the only suspicious place is at $t=0$ where the determinant vanishes.

The geodesic equations of (37) have for general solution

$$(38)\qquad x_k=\frac{1}{2}\lambda_k\log\frac{s+\alpha-\varepsilon\beta}{\beta+\varepsilon(s+\alpha)}+\xi_k,\qquad k=1,2,3,$$
$$t=[(s+\alpha-\varepsilon\beta)(\beta+\varepsilon(s+\alpha))]^{\frac{1}{2}}$$

where $\lambda_1^2+\lambda_2^2+\lambda_3^2=1$ and ξ_1, ξ_2, ξ_3 are arbitrary constants. Again $\varepsilon=+1$ if the geodesic is time-like, $\varepsilon=-1$ if it is space-like and $\varepsilon=0$ if it is a null-line.

There are two types of boundary arcs: time-like arcs

$$(39)\qquad t=s,\quad x_k=\xi_k,\quad k=1,2,3,\quad 0<s\leqq b,$$

obtained by setting $\beta=0$ in (38), and arcs with $\beta\neq 0$, originating at $t=0$, $r=\infty$. To examine the first type, we assume that it originates at $x_1=x_2=$ $=x_3=0$, i. e.

$$(40)\qquad L: t=\sigma,\quad 0<\sigma\leqq b,\quad x_k=0,\quad k=1,2,3.$$

The non-null geodesics emanating from L perpendicularly to L are space-like and have the form

$$(41)\qquad t=(\sigma^2-s^2)^{\frac{1}{2}},\qquad x_k=\frac{1}{2}\lambda_k\log\frac{s+\sigma}{\sigma-s}.$$

Hence $(x_1^2+x_2^2+x_3^2)^{\frac{1}{2}}=r=\frac{1}{2}\log\frac{s+\sigma}{\sigma-s}$, $s=\sigma\tanh r$. Normal coordinates are obtained by setting

$$y_k=\lambda_k s,\qquad k=1,2,3,\qquad y_4=\sigma,$$

i. e.

$$s=\bar{r},\qquad \lambda_k=y_k/\bar{r}$$

where $\bar{r}^2=y_1^2+y_2^2+y_3^2$. The transformation equations are

$$(42)\qquad x_k=(y_k/\bar{r})\ \operatorname{arctanh}\ (\bar{r}/y_4),\qquad k=1,2,3$$
$$t=(y_4^2-\bar{r}^2)^{\frac{1}{2}},$$

and the metric tensor in the new system is

$$g_{mn}=(y_4^2/\bar{r}^2-1)\operatorname{arctanh}^2(\bar{r}/y_4)(\delta_{mn}-y_m y_n/\bar{r}^2)$$
$$(43)\qquad +y_m y_n/\bar{r}^2,\qquad 1\leqq m<n\leqq 3,$$
$$g_{44}=-1.$$

Hence L determines an ordinary singularity.

Among the boundary arcs which originate at $r=\infty$, $t=0$, we shall only consider then null arc

$$(44) \qquad L_0 : x_1 = \frac{1}{2} \log \sigma, \qquad x_2 = x_3 = 0, \qquad t = \sigma^{\frac{1}{2}}, \qquad 0 < \sigma \leqq 1.$$

For the normal coordinate directions we choose the space-like vectors $(0, \sigma^{-\frac{1}{2}}, 0, 0)$, $(0, 0, \sigma^{-\frac{1}{2}}, 0)$ and the null vector $(-1, 0, 0, \sigma^{\frac{1}{2}})$. They are derived by Fermi propagation along L_0. The general form of space-like geodesics in the normal hyperplane spanned by these vectors is

$$x_1 = \frac{1}{2} \frac{\alpha}{\beta} \log \left[\left(1 + \frac{s}{\beta + \alpha}\right) \Big/ \left(1 - \frac{s}{\beta - \alpha}\right) \right] + \frac{1}{2} \log \sigma,$$

$$(45) \qquad x_m = \frac{1}{2} \lambda_m \log \left[\left(1 + \frac{s}{\beta + \alpha}\right) \Big/ \left(1 - \frac{s}{\beta - \alpha}\right) \right], \qquad m = 2, 3,$$

$$t = (\sigma - 2\alpha s - s^2)^{\frac{1}{2}}$$

where $\beta^2 - \alpha^2 = \sigma$ and $\lambda_2^2 + \lambda_3^2 = \sigma/(\alpha^2 + \sigma)$. Normal coordinates are obtained by setting

$$\sigma = y_4, \quad \alpha = \frac{1}{2} y_1 y_4 / \bar{r}, \quad \lambda_m = y_m \left(\frac{1}{4} y_4 y_1^2 + \bar{r}^2 \right)^{-\frac{1}{2}}, \qquad m = 2, 3,$$

$$s = \bar{r}, \qquad \bar{r}^2 = y_2^2 + y_3^2$$

and substituting in (45). The transformation equations are

$$x_1 = \frac{1}{4} (y_1/\eta) \log \left[\left(1 + \eta - \frac{1}{2} y_1\right) \Big/ \left(1 - \eta - \frac{1}{2} y_1\right) \right] + \frac{1}{2} \log y_4$$

$$(46) \quad x_m = \frac{1}{2} \left(y_m / y_4^{\frac{1}{2}} \eta \right) \log \left[\left(1 + \eta - \frac{1}{2} y_1\right) \Big/ \left(1 - \eta - \frac{1}{2} y_1\right) \right], \qquad m = 2, 3,$$

$$t = (y_4 - y_1 y_4 - y_2^2 - y_3^2)^{\frac{1}{2}}$$

with

$$(47) \qquad \eta = \left(\frac{1}{4} y_1^2 + (y_2^2 + y_3^2)/y_4 \right)^{\frac{1}{2}}.$$

A lengthy computation gives, for $m = 2, 3$,

$$(48) \qquad g_{mn} = t^2 \frac{x_2^2 + x_3^2}{y_2^2 + y_3^2} \delta_{mn} + \frac{1}{y_4 \eta^2} (y_m y_n - t^2 x_m x_n)$$

where r_i, x_2, x_3, t are given by (46), (47). Again it is seen that L_0 determines an ordinary singularity.

From (44) it can be verified easily that in the system (42), L_0 is represented by

$$y_1 = \frac{1}{2}(\sigma - 1), \qquad y_2 = y_3 = 0, \qquad y_4 = \frac{1}{2}(\sigma + 1),$$
$$0 < \sigma \leqq 1.$$

Hence L and L_0 are not associated directly, and probably not associated at all.

Bibliography.

[1] A. S. Eddington, The Mathematical Theory of Relativity, *Cambridge*, 1937.
[2] G. E. Lemaître, *Ann. Soc. Scient. Bruxelles* Ser A, **53** (1933), 51 - 85.
[3] E. Schrödinger, Expanding Universes, *Cambridge*, 1956.
[4] J. L. Synge, *Proc. Roy. Irish Soc.* **53** (1950), 83—114.
[5] G. Szekeres, *Phys. Rev.* **97** (1955), 212—223.

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