On the parallel displacement of arc holding some intrinsic properties.

To Professor O. Varga on his 50th birthday.

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Introduction.

The present author has already introduced the so called parallel displacement of arc ([1])[1]) in a space with affine connection $\Gamma_j{}^i{}_k$ with the aid of the extended connection parameter [2]. However such a parallel displacement could not hold properties of the arc, in the general space except a flat space. The aim of the present paper is to find out a new connection parameter of an extensor which defines a parallel displacement of arc holding some of them, making use of the quantities $D^\alpha \delta^i{}_{|\beta j}$ introduced by H. V. CRAIG [3].

In the present paper we use certain of the ideas, notations and results given in the papers ([1]—[3]) without explanation.

§ 1. The intrinsic connection parameter of an extensor.

Let us consider an n-dimensional space L_n with an affine connection $\Gamma_j{}^i{}_k$, then at each point on a parametrized curve $x^i = x^i(t)$ of class $P(P>M)$, we can express the α-th order intrinsic derivative of a contravariant vector v^i in the following extensor form introduced by H. V. CRAIG [3]

$$(1.1)\qquad \frac{\delta^\alpha v^i}{d t^\alpha} = \sum_{\beta=0}^{\alpha} D^\alpha \delta^i{}_{|\beta j}\, v^{(\beta) j} \quad {}^{2})$$

where the symbols $D^\alpha \delta^i{}_{|\beta j}$ are obviously the coefficients of the $v^{(\beta)j}$, the quantities defined by δ^j_i, $\Gamma_j{}^i{}_k x^{(1)k}$ and these derivatives do not exceed order α, and for fixed α they are the components of the extensor of the type

[1]) Numbers in brackets refer to the references at the end of the paper.

[2]) Numbers in parentheses denote times of differentiations with respect to t.

indicated by the indices, that is, excovariant of order one and contravariant of order one. To illustrate this, we have

$$D^0\delta^i_{|0j}=\delta^i_j \quad \text{for} \quad \alpha=0,$$
$$D^1\delta^i_{|1j}=\delta^i_j,\ \ D^1\delta^i_{|0j}=L^i_j \quad \text{for} \quad \alpha=1,\ (L^i_j \text{ being } \Gamma^i_{jk}x^{(1)k}),$$
$$D^2\delta^i_{|2j}=\delta^i_j,\ \ D^2\delta^i_{|1j}=2L^i_j,\ \ D^2\delta^i_{|0j}=L^{(1)i}_j+L^k_jL^i_k \quad \text{for} \quad \alpha=2,$$
$$D^3\delta^i_{|3j}=\delta^i_j,\ \ D^3\delta^i_{|2j}=3L^i_j,\ \ D^3\delta^i_{|1j}=3(L^{(1)i}_j+L^k_jL^i_k),$$
$$D^3\delta^i_{|0j}=(L^{(1)i}_j+L^k_jL^i_k)^{(1)}+(L^{(1)k}_j+L^{(1)l}_jL^k_l)L^i_k \quad \text{for} \quad \alpha=3,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

etc.,
where the delta denotes the Kronecker delta.

On putting $\overset{\alpha}{g}{}^i_{\beta j}\equiv D^\alpha\delta^i_{|\beta j}$, since the $(M+1)n$-rowed determinant g of the quantities $\overset{\alpha}{g}{}^i_{\beta j}$ $\alpha=0,\ldots,M$ does not vanish, i. e., $g=1$, we obtain the quantities $\underset{\alpha}{g}{}^{\gamma k}_i$ determined by

$$\sum_{\alpha=0}^{M}\overset{\alpha}{g}{}^i_{\beta j}\underset{\alpha}{g}{}^{\gamma k}_i=\delta^\gamma_\beta\delta^k_j. \tag{1.2}$$

Then from the fact that $\overset{\alpha}{g}{}^i_{\beta j}\equiv 0$ for $\alpha<\beta$, we get easily the following

Theorem 1.1. *The relation* $\underset{\alpha}{g}{}^{\gamma k}_i\equiv 0$ *holds for* $\alpha>\gamma$

Corollary. *If* $\gamma<M$, *then there exists the relation*

$$\sum_{\alpha=0}^{M}\overset{\alpha}{g}{}^i_{\beta j}\underset{\alpha}{g}{}^{\gamma k}_i\equiv\sum_{\alpha=0}^{\gamma}\overset{\alpha}{g}{}^i_{\beta j}\underset{\alpha}{g}{}^{\gamma k}_i.$$

Definition. With the aid of the $M+1$ quantities $\overset{\alpha}{g}{}^i_{\beta j}$ $\alpha=0,\ldots,M$, we define the so called *tensorial components of the α-th kind* for an extensor $v^{\alpha i}(x, x^{(1)},\ldots,x^{(\beta)})$ $\beta=0,\ldots,\alpha$ of functional order M and of range M with the following quantities

$$\overset{\alpha}{v}{}^i=\sum_{\beta=0}^{\alpha}\overset{\alpha}{g}{}^i_{\beta j}v^{\beta j} \qquad \alpha=0,\ldots,M. \tag{1.3}$$

This conception is generalized similarly for a general extensor of the type indicated by $T^{\gamma_1 i_1\ldots\gamma_A i_A}{}_{\delta_1 j_1\ldots\delta_B j_B}$.

We are desirous to determine the condition for a connection parameter $\Gamma^{\gamma i}_{\beta j\delta k}$ of an extensor to satisfy the following relation

$$\delta\overset{\alpha}{v}{}^i=\sum_{\beta=0}^{\alpha}\overset{\alpha}{g}{}^i_{\beta j}\,\delta v^{\beta j}, \tag{1.4}$$

$\delta\overset{\alpha}{v}{}^i$ being the covariant differential of the vector $\overset{\alpha}{v}{}^i$:

$$\delta\overset{\alpha}{v}{}^i = d\overset{\alpha}{v}{}^i + \Gamma^i_{jk}\overset{\alpha}{v}{}^j dx^k. \tag{1.5}$$

On putting

$$\delta v^{\beta j} = dv^{\beta j} + \Gamma^{\beta j}_{\gamma k\delta l} v^{\gamma k} dx^{(\delta)l}, \tag{1.6}$$

where the displacement $dx^{(\delta)l}$ means difference of the line element at any two infinitesimally near points lying on any two infinitesimally near curves, and on making use of the following relation obtained from (1.3)

$$d\overset{\alpha}{v}{}^i = \sum_{\beta=0}^{\alpha} d\overset{\alpha}{g}{}^i_{\beta j} v^{\beta j} + \sum_{\beta=0}^{\alpha} \overset{\alpha}{g}{}^i_{\beta j} dv^{\beta j}, \tag{1.7}$$

we have

$$d\overset{\alpha}{g}{}^i_{\beta j} v^{\beta j} + \overset{\alpha}{g}{}^i_{\beta j} dv^{\beta j} + \sum_{\gamma=0}^{\alpha} \Gamma^i_{jl}\overset{\alpha}{g}{}^j_{\gamma k} v^{\gamma k} dx^l = \overset{\alpha}{g}{}^i_{\beta j}(dv^{\beta j} + \Gamma^{\beta j}_{\gamma k\delta l} v^{\gamma k} dx^{(\delta)l})$$

according to (1.4), (1.5), (1.6) and (1.7). Consequently it follows that

$$\left(\frac{\partial \overset{\alpha}{g}{}^i_{\beta j}}{\partial x^{(\gamma)l}} + \Gamma^i_{kl}\overset{\alpha}{g}{}^k_{\beta j}\delta^0_\gamma - \overset{\alpha}{g}{}^i_{\delta k}\Gamma^{\delta k}_{\beta j\gamma l}\right) v^{\beta j} dx^{(\gamma)l} = 0.$$

Since $v^{\beta j}$ and $dx^{(\delta)l}$ are arbitrary, we have

$$\overset{\alpha}{g}{}^i_{\delta k}\Gamma^{\delta k}_{\beta j\gamma l} = \frac{\partial \overset{\alpha}{g}{}^i_{\beta j}}{\partial x^{(\gamma)l}} + \Gamma^i_{kl}\overset{\alpha}{g}{}^k_{\beta j}\delta^0_\gamma,$$

and from the corollary of Theorem 1.1, we can arrive at the following result

$$\Gamma^{\delta k}_{\beta j\gamma l} = \sum_{\alpha=0}^{\delta} \underset{\alpha}{g}{}^{\delta k}_i \left(\frac{\partial \overset{\alpha}{g}{}^i_{\beta j}}{\partial x^{(\gamma)l}} + \Gamma^i_{hl}\overset{\alpha}{g}{}^h_{\beta j}\delta^0_\gamma\right). \tag{1.8}$$

These quantities $\Gamma^{\delta k}_{\beta j\gamma l}$ are called the *intrinsic connection parameters of an extensor*, by which we define the excovariant differential, the excovariant derivative and the parallel displacement of an extensor. Thus we can see the following

Theorem 1.2. *If and only if an extensor $v^{\beta i}(x, x^{(1)}, \dots, x^{(\gamma)})$ of range M is displaced parallelly, that is, $\delta v^{\beta i} = 0$, then its tensorial components of the α-th kind are also displaced parallelly, that is, $\delta\overset{\alpha}{v}{}^i = 0$, $\alpha = 0, \dots, M$.*

The theorem is self-evident from that the determinant g does not vanish. Especially if we write $x^{(\beta+1)i}$ for the extensor $v^{\beta i}$, then it follows that

$$\frac{\delta^\alpha x^{(1)i}}{dt^\alpha} = \overset{\alpha}{g}{}^i_{\beta j} x^{(\beta+1)j}$$

and we have the following

Corollary. *If and only if* $\delta x^{(\beta+1)j}=0$, $\beta=0,\ldots,R$, *then* $\delta\left(\frac{\delta^\alpha x^{(1)i}}{dt^\alpha}\right)=0$ $\alpha=0,\ldots,R$ *where* $R<M$.

If the curve is analytic and if the connection parameter Γ^i_{jk} is also analytic, then we can consider the quantities $\Gamma^{\delta k}_{\beta j \gamma l}$ $\delta=0,\ldots,\infty$.

§ 2. The intrinsic and parallel displacement of arc.

We consider two analytic curves C and $\tilde{C}$ passing through two infinitesimally near points P and $\tilde{P}$ in a certain domain of L_n respectively, which are written in the form of power series in $t-t_0$ in the same interval $0\leqq t-t_0\leqq\varepsilon$:

$$x^i(t)=x^i+\sum_{\alpha=1}^{\infty}\frac{1}{\alpha!}x^{(\alpha)i}(t-t_0)^\alpha, \tag{2.1}$$

$$\tilde{x}^i(t)=\tilde{x}^i+\sum_{\alpha=1}^{\infty}\frac{1}{\alpha!}\tilde{x}^{(\alpha)i}(t-t_0)^\alpha, \tag{2.2}$$

where x^i and $\tilde{x}^i$ are the coordinates of P and $\tilde{P}$ respectively and correspond to the same parameter value t_0. If the coefficients $x^{(\alpha)i}$ and $\tilde{x}^{(\alpha)i}$ satisfy the following relations

$$\tilde{x}^i=x^i+dx^i,\quad \tilde{x}^{(\alpha)i}=x^{(\alpha)i}-\Gamma^{\alpha-1\, i}_{\beta j\gamma k}x^{(\beta+1)j}dx^{(\gamma)k}$$
$$dx^{(\gamma)i}=\tilde{x}^{(\gamma)i}-x^{(\gamma)i},\quad \alpha,\gamma=1,2,\ldots,\infty,$$

then we call one of these curves the *parallelly displaced curve* of the other curve.

When the curve (2.1) is an affine path passing through P, then, as is well-known, the differential equation of the affine path is written as follows

$$\frac{\delta x^{(1)i}}{dt}=\varphi_1(t)x^{(1)i}. \tag{2.3}$$

Differentiating covariantly the above equation along the curve and substituting the above result, we have

$$\frac{\delta^2 x^{(1)i}}{dt^2}=\varphi_1^{(1)}x^{(1)i}+\varphi_1\frac{\delta x^{(1)i}}{dt}=(\varphi_1^{(1)}+\varphi_1^2)x^{(1)i}=\varphi_2 x^{(1)i}\ \text{(putting } \varphi_2\equiv\varphi_1^{(1)}+\varphi_1^2\text{)}.$$

Similarly we can see the following relation

$$\frac{\delta^\alpha x^{(1)i}}{dt^\alpha}=\varphi_\alpha(t)x^{(1)i}\qquad \alpha=1,2,\ldots \tag{2.4}$$

φ_α being a function of t. Conversely if the coefficients $x^{(\alpha)i}$ of (2.1) satisfy the relation (2. 4) at the point P, then it is self-evident that (2. 1) is the affine path passing through the point P with the direction $x^{(1)i}$. Here we can state the following

Theorem 2. 1. *If a curve C is an affine path, then its parallel curve is also an affine path.*

PROOF. A necessary and sufficient condition that the curve $\tilde{C}$ be the parallel curve of the curve C, is that $\delta(x^{(\alpha)i})=0 \quad \alpha=1, 2, \ldots, \infty$ and it is equivalent to $\delta\left(\frac{\delta^\alpha x^{(1)i}}{dt^\alpha}\right)=0 \ \alpha=0, 1, \ldots, \infty$ according to the corollary of Theorem 1. 2. From $\delta(x^{(1)i})=0$, it follows that

$$\tilde{x}^{(1)i}=x^{(1)i}+\Gamma^i_{jk}x^{(1)j}dx^k.$$

Since the curve C is an affine path, on making use of the relation (2. 4), $\delta\left(\frac{\delta x^{(1)i}}{dt}\right)=0$ takes the following form

$$\begin{aligned}\frac{\delta\tilde{x}^{(1)i}}{dt}&=\frac{\delta x^{(1)i}}{dt}+\Gamma^i_{jk}\frac{\delta x^{(1)j}}{dt}dx^k\\&=(\delta^i_j+\Gamma^i_{jk}dx^k)\frac{\delta x^{(1)j}}{dt}\\&=(\delta^i_j+\Gamma^i_{jk}dx^k)\varphi_1 x^{(1)j}\\&=\varphi_1(x^{(1)i}+\Gamma^i_{jk}x^{(1)j}dx^k)\\&=\varphi_1\tilde{x}^{(1)i},\end{aligned}$$

and from $\delta\left(\frac{\delta^2 x^{(1)i}}{dt^2}\right)=0$ and $\frac{\delta^2 x^{(1)i}}{dt^2}=\varphi_2 x^{(1)i}$, we have

$$\frac{\delta^2\tilde{x}^{(1)i}}{dt^2}=\frac{\delta^2 x^{(1)i}}{dt^2}+\Gamma^i_{jk}\frac{\delta^2 x^{(1)j}}{dt^2}dx^k=(\delta^i_j+\Gamma^i_{jk}dx^k)\frac{\delta^2 x^{(1)j}}{dt^2}=\varphi_2\tilde{x}^{(1)i}.$$

Similarly we get

$$\frac{\delta^\alpha\tilde{x}^{(1)i}}{dt^\alpha}=\varphi_\alpha\tilde{x}^{(1)i} \qquad \alpha=1, 2, 3, \ldots\infty.$$

Consequently (2. 2) must be an affine path.

From this theorem, we can easily infer the following

Corollary. *An affine path is the curve obtained by parallel displacement of an arc to itself.*

If our space L_n is a Riemannian space R_n, then we obtain the following

Theorem 2.2. *If the curves C and $\tilde{C}$ in R_n are parallel, then the derivative of the are length of* (2.1) *with respect to t is equal to that of* (2.2) *at the point P and $\tilde{P}$ respectively, that is,* $\left(\frac{ds}{dt}\right)_0=\left(\frac{d\tilde{s}}{dt}\right)_0$.

PROOF. Differentiating covariantly the following equation

$$\left(\frac{ds}{dt}\right)^2=g_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt},$$

we have

$$2\left(\frac{ds}{dt}\right)\delta\left(\frac{ds}{dt}\right)=\delta g_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt}+2g_{ij}\delta\left(\frac{dx^i}{dt}\right)\frac{dx^j}{dt}=0,$$

because $\delta g_{ij}=0$ and $\delta\left(\frac{dx^i}{dt}\right)_0=0$. Therefore it follows that

$$\left(\frac{ds}{dt}\right)_0=\left(\frac{d\tilde{s}}{dt}\right)_0.$$

Theorem 2.3. *If the curves C and $\tilde{C}$ in R_n are parallel, then the first curvature ϱ of* (2.1) *is equal to the first curvature $\tilde{\varrho}$ of* (2.2) *at the point P and $\tilde{P}$ respectively.*

PROOF. Differentiating covariantly the following equation

$$\varrho^2=g_{ij}\frac{\delta^2x^i}{ds^2}\frac{\delta^2x^j}{ds^2}$$

and on making use of $\delta(g_{ij})=0$ and $\delta\left(\frac{\delta^2x^i}{ds^2}\right)=0$, we have $\delta(\varrho)_0=0$ and $\varrho_0=\tilde{\varrho}_0$.

Corollary. *If the curves C and $\tilde{C}$ in R_n are parallel, then the derivatives of the first curvature α-times with respect to s have the same value at the point P and $\tilde{P}$ respectively.*

Les us take the arc length s in place of the parameter t of the curve C, then (2.1) becomes as follows

$$x^i(s)=x^i+\sum_{\alpha=1}^{\infty}\frac{1}{\alpha!}x^{(\alpha)i}(s-s_0)^\alpha. \tag{2.5}$$

If the curve (2.5) is a geodesic circle whose differential equation is written in the form

$$\frac{\delta^3x^i}{ds^3}=\varrho^2\frac{dx^i}{ds}, \tag{2.6}$$

then we obtain the following relations

$$(2.7)\qquad \begin{cases} \dfrac{\delta^4 x^i}{ds^4} = (\varrho^2)^{(1)} \dfrac{dx^i}{ds} + \varrho^2 \dfrac{\delta^2 x^i}{ds^2} \\ \dfrac{\delta^5 x^i}{ds^5} \{(\varrho^2)^{(2)} + \varrho^4\} \dfrac{dx^i}{ds} + 2(\varrho^2)^{(1)} \dfrac{\delta^2 x^i}{ds^2} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \text{ etc.} \end{cases}$$

Conversely if the coefficients $x^{(\alpha)i}$ of (2.5) satisfy the relation (2.6) and (2.7) at the point P, then it is self-evident that (2.5) is the geodesic circle passing through the point P with the given $x^{(1)i}$ and $x^{(2)i}$, whose arc length is s. Hence we can see the following

Theorem 2.4. *If a curve C in R_n is a geodesic circle, then its parallel curve is also a geodesic circle.*

PROOF. Let us put the parallel curve $\tilde{C}$ of the curve C given by (2.5) as follows

$$(2.8)\qquad \tilde{x}^i(s) = \tilde{x}^i + \sum_{\alpha=1}^{\infty} \frac{1}{\alpha!} \tilde{x}^{(\alpha)i} (s-s_0)^\alpha,$$

then we have

$$\tilde{x}^{(1)i} = x^{(1)i} + \Gamma^i_{jk} x^{(1)j} dx^k = (\delta^i_j + \Gamma^i_{jk} dx^k) x^{(1)j},$$

$$\frac{\delta^2 \tilde{x}^i}{ds^2} = \frac{\delta^2 x^i}{ds^2} + \Gamma^i_{jk} \frac{\delta^2 x^j}{ds^2} dx^k = (\delta^i_j + \Gamma^i_{jk} dx^k) \frac{\delta^2 x^j}{ds^2},$$

$$\frac{\delta^3 \tilde{x}^i}{ds^3} = (\delta^i_j + \Gamma^i_{jk} dx^k) \frac{\delta^3 x^j}{ds^3},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\frac{\delta^\alpha \tilde{x}^i}{ds^\alpha} = (\delta^i_j + \Gamma^i_{jk} dx^k) \frac{\delta^\alpha x^j}{ds^\alpha}.$$

Replacing (2.6) and (2.7) in the above equations, we have

$$\frac{\delta^3 \tilde{x}^i}{ds^3} = (\delta^i_j + \Gamma^i_{jk} dx^k) \varrho^2 \frac{dx^j}{ds} = \varrho^2 \frac{d\tilde{x}^i}{ds},$$

$$\frac{\delta^4 \tilde{x}^i}{ds^4} = (\delta^i_j + \Gamma^i_{jk} dx^k) \left\{ (\varrho^2)^{(1)} \frac{dx^i}{ds} + \varrho^2 \frac{\delta^2 x^i}{ds^2} \right\}$$

$$= (\varrho^2)^{(1)} \frac{d\tilde{x}^i}{ds} + \varrho^2 \frac{\delta^2 \tilde{x}^i}{ds^2},$$

$$\frac{\delta^5 \tilde{x}^i}{ds^5} = \{(\varrho^2)^{(2)} + \varrho^4\} \frac{d\tilde{x}^i}{ds} + 2(\varrho^2)^{(1)} \frac{\delta^2 \tilde{x}^i}{ds^2},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \text{ etc.}$$

Accordingly the parallel curve $\tilde{C}$ is the geodesic circle passing through the point $\tilde{P}$ with the given $\tilde{x}^{(1)i}$ and $\tilde{x}^{(2)i}$, whose arc length is s.

In particular, if our space L_n is a flat space, then in the parallel coordinate system, the connection parameter of an extensor: $\Gamma^{\alpha i}_{\beta j \gamma k}$ expressed by (1.8) vanishes identically. Therefore it becomes $\tilde{x}^{(\alpha)i} = x^{(\alpha)i}$, that is $\delta x^{(\alpha)i} = 0$. Thus the following theorem is obtained without difficulty.

Theorem 2.5. *If the curves C and $\tilde{C}$ are parallel in a flat space, then one of these must be the translation of the other curve.*

REMARK. For any analytic curve, in order to show the existence of its parallel curve, we must prove that when the right hand member of (2.1) is convergent, the right hand member of (2.2) is also convergent. We shall do it next time.

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