# On $\mathfrak{n}$-algebraically closed groups.

To Professor O. Varga on his 50th birthday.

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## § 1. Introduction.

The concepts of algebraically closed and of weakly algebraically closed group have been introduced by W. R. SCOTT [5]. B. H. NEUMANN has shown [3] that these two concepts in fact coincide and that an algebraically closed group is necessarily simple. For an infinite cardinal $\mathfrak{n}$ we shall call $\mathfrak{n}$-*algebraically closed* those groups which in SCOTT's terminology are "weakly algebraically closed $\mathfrak{n}$-groups". By [5] every group is a subgroup of an $\mathfrak{n}$-algebraically closed group. It is the purpose of this note to show that for $\mathfrak{n}$-algebraically closed groups we have an analogue of the well-known property of so called algebraically closed abelian groups and of algebraically closed operator modules (see e. g. [6] and [2]) according to which any such group, module is a direct summand of any abelian group, operator module which contains it as a subgroup, submodule. As a corollary we get that if a group is $\mathfrak{n}$-algebraically closed for every infinite cardinal $\mathfrak{n}$, then it can only be the group consisting of one element.

## § 2. Preliminaries.

Let $G$ be a group with identity 1. If $S$ is a set of elements of $G$, then let us denote by $\{S\}$ the subgroup and by $\{\{S\}\}$ the normal subgroup of $G$ generated by $S$. If $\Delta$ is a set of indices, then by $(a_\alpha)_{\alpha\in\Delta}$ we understand the set of elements $a_\alpha$ indexed by the elements of $\Delta$. We denote by $X_\Delta$ the free group generated by the symbols $x_\alpha$ $(\alpha\in\Delta)$.

A subgroup $A$ of $G$ is called a *semi-direct factor* of $G$ if there exists a normal subgroup $N$ of $G$ such that $G=\{A, N\}$ and $A\cap N=1$.

Let $H$ be an extension[1]) of the group $G$, and let $h_\alpha(\in H, \alpha\in\Delta)$ be a

[1]) We call any group $H$ an extension of the group $G$ if $G$ is a subgroup of $H$.

system of elements, for which

$$H=\{G, h_\alpha\}_{\alpha\in\Delta}$$

holds. We establish a one-to-one correspondence between the elements $h_\alpha$ and the elements $x_\alpha$ ($\alpha\in\Delta$), and we consider the free group $X_\Delta$. The mapping

$$g\to g;\quad x_\alpha\to h_\alpha \qquad (g\in G;\ \alpha\in\Delta)$$

induces a homomorphism onto $H$ of the group $G_\Delta^*$.[2]) Consequently we have

$$(*) \qquad H\simeq G_\Delta^*/N,$$

where $N$ is the kernel of the homomorphism considered. To the representation (*) of the group $H$ we associate a power $\mathfrak{p}$ in the following way: let $\mathfrak{p}$ be the smallest infinite cardinal, which is greater than the minimal power of the generating systems of the normal subgroup $N$. Consider all representations of the form (*) of the group $H$, and the set $P$ of cardinal numbers $\mathfrak{p}$ belonging to these representations. Let $\mathfrak{n}$ be the smallest element of the set $P$. Then we say that $H$ is an $\mathfrak{n}$-extension of the group $G$.

Consider a non empty set $(f_\beta(x_\alpha))_{\beta\in\Gamma}$ of elements of $G_\Delta^*$; we call the system of formal equalities

$$(1) \qquad f_\beta(x_\alpha)=1 \qquad (\beta\in\Gamma)$$

*a system of equations over* $G$. We say that (1) is *solvable* in some extension $H$ of the group $G$ (which, of course, can coincide with $G$ itself) if $H$ has a system of elements $h_\alpha$ ($\alpha\in\Delta$) such that the kernel of the homomorphism defined by $g\to g$, $x_\alpha\to h_\alpha$ contains all the $f_\beta(x_\alpha)$, in other words such that $f_\beta(h_\alpha)=1$ (for all $\beta\in\Gamma$); such a set $(h_\alpha)_{\alpha\in\Delta}$ is called a *solution* of (1). A system of equations (1) over $G$ is said to be *compatible*, if it is solvable in some extension of $G$. A more explicit characterization of compatibility is given by the following lemma.

**Lemma.**[3]) *The system* (1) *of equations over* $G$ *is compatible, if and only if for the normal subgroup* $M$ *of* $G_\Delta^*$ *generated by all of the elements* $f_\beta(x_\alpha)\,(\beta\in\Gamma)$ *the relation* $M\cap G=1$ *holds.*

Proof. First let (1) be a system of equations which is solvable in some extension $H$ of $G$: let $f_\beta(h_\alpha)=1$. The normal subgroup $M$ coincides with the subgroup of $G_\Delta^*$ generated by the conjugates in $G_\Delta^*$ of the left hand sides of (1). Clearly $M$ is contained in the kernel $N$ of the homomorphism defined

[2]) We denote by $G_\Delta^*$ the free product $G*X_\Delta$. We shall find it convenient to, denote the elements of $G_\Delta^*$ by $f(x_\alpha)$.

[3]) This is an analogue of a theorem of G. Pollák [4] and of O. Villamayor [7] on rings and of a theorem of A. Kertész [2] on modules.

by $g \to g, x_\alpha \to h_\alpha$; since $G$ is fixed under this homomorphism, $N \cap G = 1$ and a fortiori $M \cap G = 1$. Conversely, $M \cap G = 1$ implies that in $\bar{G} = G^*_\Delta / M$ the cosets belonging to the elements of $G$ form a subgroup isomorphic to $G$. So $\bar{G}$ is an extension of the group $G$, in which the system (1) is clearly solvable.

We call a group $G$ $\mathfrak{n}$-*algebraically closed,* if any compatible system of equations (1) over $G$, such that the cardinality of $\Gamma$ is less than $\mathfrak{n}$, is solvable in $G$.

## § 3. $\mathfrak{n}$-algebraically closed groups.

First of all we prowe the following generalization of a theorem of S. Gacsályi (see [1], Theorem 2):

**Theorem 1.** *A subgroup A of the arbitrary group G is a semi-direct factor of G if and only if any system of equations over A, solvable in G is solvable also in A.*

**Corollary 1.**[4]) *The normal subgroup A of the arbitrary group G is a direct factor of G if and only if any system of equations over A solvable in G is solvable also in A.*

**Corollary 2.** *Let the group G be the free product of its subgroups A and B: $G = A * B$. Then A is a semi-direct factor of G.*

Proof. Let $A$ be a subgroup and $D$ a normal subgroup of $G$ such that $G = \{A, D\}$ and $A \cap D = 1$, and let

$$f_\beta(x_\alpha) = 1 \quad (f_\beta \in A^*_\Delta;\ \alpha \in \Delta;\ \beta \in \Gamma) \tag{2}$$

be an arbitrary system of equations over $A$, solvable in $G$. Then a solution $g_\alpha (\in G;\ \alpha \in \Delta)$ of the system (2) can be written in a uniquely determined way in the form

$$g_\alpha = a_\alpha d_\alpha \quad (a_\alpha \in A;\ d_\alpha \in D;\ \alpha \in \Delta) \tag{3}$$

and so the equalities

$$f_\beta(a_\alpha d_\alpha) = 1 \qquad (\beta \in \Gamma) \tag{4}$$

hold. Since $D$ is normal in $G$, for any $g \in G$ and $d \in D$ there exists a $d' \in D$

[4]) I am indebted to A. Kertész who has called my attention to the fact that this generalization of the theorem of Gacsályi has already been put forward by H. Leptin in his Zentralblatt-review of the paper of S. Balcerzyk: Remark on a paper of S. Gacsályi *Publ. Math. Debrecen* **4** (1956), 357—358 (*Zb. Math.* **70** (1957), 20—21).

for which $gd' = dg$, so that the equalities (4) can be written in the form

$$f_\beta(a_\alpha) = d_\beta \qquad (d_\beta \in D, \beta \in \Gamma). \tag{5}$$

The elements on the left-hand side of (5) belong to $A$, while those on the right-hand side belong to $D$, and so in view of $A \cap D = 1$ we must have for every $\beta \in \Gamma$ the equality

$$f_\beta(a_\alpha) = 1 \qquad (\beta \in \Gamma).$$

Thus the system of equations (2) is solvable also in $A$.

Conversely, let us suppose that any system of equations over $A$ which is solvable in $G$, is also solvable in $A$. Let $g_\alpha(\alpha \in \Delta)$ be a system of elements of $G$, for which

$$G = \{A, (g_\alpha)_{\alpha \in \Delta}\}. \tag{6}$$

Now consider all valid relations of the form

$$f_\beta(g_\alpha) = 1 \qquad (\beta \in \Gamma) \tag{7}$$

connecting the elements of $A$ with the elements $g_\alpha$. The system

$$f_\beta(x_\alpha) = 1 \qquad (\beta \in \Gamma) \tag{7'}$$

corresponding to (7) is a system of equations over $A$ which admits the solution $g_\alpha(\in G; \alpha \in \Delta)$, and so by our hypothesis (7′) has also a solution $a_\alpha$ $(\alpha \in \Delta)$ in $A$. Let us now consider the subgroup $B = \{(g_\alpha a_\alpha^{-1})_{\alpha \in \Delta}\}$ of $G$. Then on the one hand $\{A, B\} = G$ since $\{A, B\}$ contains all elements $g_\alpha(\alpha \in \Delta)$ and (6) holds. Thus a fortiori $\{A, \{\{B\}\}\} = G$. On the other hand we show that $A \cap \{\{B\}\} = 1$. Since $\{\{B\}\}$ is the subgroup generated by the conjugates in $G$ of the elements $g_\alpha a_\alpha^{-1}$ $(\alpha \in \Delta)$, any element $a$ belonging to both $A$ and $\{\{B\}\}$ can be written in the form

$$h_1^{-1}(g_{\alpha_1} a_{\alpha_1}^{-1})^{\varepsilon_1} h_1 \dots h_k^{-1}(g_{\alpha_k} a_{\alpha_k}^{-1}) h_k = a \quad (h_i \in G), \tag{8}$$

where $\varepsilon_1, \dots, \varepsilon_k = \pm 1$. Furthermore, in view of $G = \{A, B\}$ all elements $h_i$ arise as products of finitely many factors which are elements of $A$ and some elements $g_{\alpha_j}$. Thus (8) is a relation between elements of $A$ and the $g_\alpha$'s; as such, it is essentially one of the relations in (7); hence it remains valid if we replace the $g_\alpha$'s by the corresponding $a_\alpha$'s. This substitution shows that one necessarily has $a = 1$. Thus we have shown that $A$ is a semi-direct factor of $G$, completing the proof of Theorem 1.

Corollary 1 is a special case of the theorem. In order to prove Corollary 2, by Theorem 1 it will be sufficient to show that any system of equations

$$f_\beta(x_\alpha) = 1 \qquad (\alpha \in \Delta, \beta \in \Gamma) \tag{9}$$

over $A$ which is solvable in $G$ is solvable in $A$. Consider for a solution $g_\alpha(\alpha \in \Delta)$ of the system of equations (9) the representation

$$g_\alpha = a_{\alpha_1} b_{\alpha_1} a_{\alpha_2} b_{\alpha_2} \dots a_{\alpha_k} b_{\alpha_k} \qquad (\alpha \in \Delta) \tag{10}$$

arising form of the free decomposition $G = A * B$. If we substitute these elements $g_\alpha$ into (9), we get relations between elements of $A$ and of $B$, the left-hand sides of which reduce to the empty word. From this it is clear that the system of elements $g'_\alpha = a_{\alpha_1} a_{\alpha_2} \dots a_{\alpha_k} (\in A; \alpha \in \Delta)$ which arises from (10) by the substitution $b_{\alpha_i} = 1$ $(i = 1, \dots, k)$ is a solution in $A$ of the system of equations (9).

**Theorem 2.** *A group $G$ is $\mathfrak{n}$-algebraically closed if and only if it is a semi-direct factor of any of its $\mathfrak{m}$-extensions with $\mathfrak{m} \leqq \mathfrak{n}$.*

PROOF. Let us first suppose that $G$ is a semi-direct factor of any of its $\mathfrak{m}$-extensions with $\mathfrak{m} \leqq \mathfrak{n}$. Let

$$f_\beta(x_\alpha) = 1 \qquad (\alpha \in \Delta, \beta \in \Gamma) \tag{11}$$

be a compatible system of equations over $G$, such that the cardinality of $\Gamma$ is smaller than $\mathfrak{n}$. We denote by $N$ the normal subgroup generated in $G^*_\Delta$ by the system of elements $(f_\beta(x_\alpha))_{\beta \in \Gamma}$. In view of the compatibility of the system (11) we have in $G^*_\Delta$ the relation $N \cap G = 1$, and so $G^*_\Delta / N = H$ is an extension of the group $G$, in fact an $\mathfrak{m}$-extension for some $\mathfrak{m} \leqq \mathfrak{n}$. So by hypothesis $G$ is a semi-direct factor of $H$. Now, since those elements of the group $H$, which by the natural homomorphism of $G^*_\Delta$ onto $H$ correspond to the elements $x_\alpha$ $(\alpha \in \Delta)$ give a solution of the system (11), by Theorem 1 we obtain that the system of equations (11) is solvable also in $G$.

Conversely, let $G$ be an $\mathfrak{n}$-algebraically closed group, and let $H$ be an arbitrary $\mathfrak{m}$-extension of $G$ with $\mathfrak{m} \leqq \mathfrak{n}$. Then there exists a system of elements $h_\alpha$ $(\in H, \alpha \in \Delta)$ such that $H = \{G, h_\alpha\}_{\alpha \in \Delta}$, and that the system of equations corresponding to the totality of the relations

$$f_\beta(h_\alpha) = 1 \qquad (\beta \in \Gamma) \tag{12}$$

existing between the elements of $G$ and the elements $h_\alpha(\alpha \in \Delta)$, namely the system

$$f_\beta(x_\alpha) = 1 \qquad (\beta \in \Gamma) \tag{13}$$

is equivalent to a system of equations

$$\varphi_\omega(x_\alpha) = 1 \qquad (\omega \in \Omega), \tag{14}$$

whose power is smaller than $\mathfrak{n}$. Since, on the basis of (12) and by its equivalence to the system of equations (13), the system (14) is compatible, it is, by our hypothesis, solvable in $G$. Let $g_\alpha$ $(\alpha \in \Delta)$ be a solution. Then this is a solution also of the system (13), and in exactly the same manner as in the proof of Theorem 1, we can show that for the subgroup $K = \{(h_\alpha g_\alpha^{-1})_{\alpha \in \Delta}\}$ the relations $\{G, K\} = H$ and $G \cap \{\{K\}\} = 1$ hold.

**Theorem 3.** *If the group $G$ is $\mathfrak{n}$-algebraically closed for every infinite cardinal $\mathfrak{n}$, then $G$ consits only of the identity.*

**Corollary.** *If the group $G$ is a free factor of any group in which it is contained as a subgroup, then it consits only of the identity.*

The Corollary is an immediate consequence of Theorem 3, of Theorem 2, and of Corollary 2 to Theorem 1.

PROOF OF THEOREM 3. Let $G$ be a group, for which the condition in the theorem holds. Let $H_0$ be a group containing $G$ but having greater cardinality than $G$, and let $H$ be the $\aleph_0$-algebraically closed extension of $H_0$, which exists by SCOTT [5]. By Theorem 2 $G$ is a semi-direct factor of $H$, and so there exists a normal subgroup $K$ of $H$, for which $H = \{G, K\}$ and $G \cap K = 1$. Then $H/K \simeq G$, and since by NEUMANN [3] $H$ is simple, one has either $K = \{1\}$ or $K = H$. In the first case $G \simeq H$, but this cannot be valid since the cardinality of $H$ is greater than that of $G$. Thus necessarily $K = H$, $G = \{1\}$.

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