Two of the problems of L. Fuchs.

To Professor O. Varga on his 50th birthday.

By D. K. HARRISON (Haverford, Pa.).

L. FUCHS in his recent book (see [1]) on Abelian groups has listed 86 unsolved problems. It is the purpose of this paper to answer two of these problems:

Problem 42. Do groups G and H exist whose endomorphism rings are not isomorphic, but their endomorphism groups are isomorphic?

Problem 56. Find the structure of the tensor product of a torsion group and a mixed group.

We answer the first question in the affirmative. Theorems 65.3 and 65.4 of [1] give the structure of the tensor product of two torsion groups and of a torsion group and a torsion-free group respectively. Hence we are able to answer the second problem with the somewhat unsuspected fact that the tensor product of a torsion group G with any group H is isomorphic to $(G\otimes T)\oplus(G\otimes(H/T))$ where T is the torsion subgroup of H.

§ 1. Endomorphism group of a torsion group.

Theorem 56.1 of [1] states that two p-groups with isomorphic endomorphism rings are necessarily isomorphic. Hence our search is for two nonisomorphic p-groups G and H with $\mathrm{Hom}(G, G)\simeq \mathrm{Hom}(H, H)$.

Lemma 1. *If T and L are groups and T is a torsion group, then* $\mathrm{Hom}(T, L)$ *is a co-torsion group* (see section 2 of [2]).

PROOF. It is enough to show that $\mathrm{Ext}(Q, \mathrm{Hom}(T, L))=0$ and $\mathrm{Hom}(Q, \mathrm{Hom}(T, L))=0$ where Q is the group of rational numbers. $Q\otimes T=0$ since Q is divisible and T is torsion, and $\mathrm{Tor}(Q, T)=0$ since Q is torsion-free. By pages 116 and 28 respectively of [3]

$$\mathrm{Ext}(Q, \mathrm{Hom}(T, L))\oplus \mathrm{Hom}(Q, \mathrm{Ext}(T, L))\simeq \mathrm{Ext}(Q\otimes T, L)\oplus \mathrm{Hom}(\mathrm{Tor}(Q, T), L)$$
$$=\mathrm{Ext}(O, L)\oplus \mathrm{Hom}(O, L)=0.$$

and

$$\mathrm{Hom}(Q, \mathrm{Hom}(T, L)) \simeq \mathrm{Hom}(Q \otimes T, L) = \mathrm{Hom}(O, L) = 0.$$

Lemma 2. *If T is a torsion group with basic subgroup T_b* (see chap. 5 of [1]) *and B is a divisible group, then* $\mathrm{Hom}(T, B) \simeq \mathrm{Hom}(T_b, B) \oplus \oplus \mathrm{Hom}(T/T_b, B)$.

PROOF. If n is any integer we let Z_n denote the integers modulo n. Since $\mathrm{Ext}(T/T_b, B) = 0$, we have the exact sequence

(1) $0 \to \mathrm{Hom}(T/T_b, B) \to \mathrm{Hom}(T, B) \to \mathrm{Hom}(T_b, B) \to 0$ and since T_b is pure in T, we also have the exact sequence

$$0 \to Z_n \otimes T_b \to Z_n \otimes T \to Z_n \otimes (T/T_b) \to 0$$

and thus

$$\begin{array}{ccccccc} 0 \leftarrow \mathrm{Hom}(Z_n \otimes T_b, B) & \leftarrow & \mathrm{Hom}(Z_n \otimes T, B) & \leftarrow & \mathrm{Hom}(Z_n \otimes (T/T_b), B) \leftarrow 0 \\ \wr & & \wr & & \wr \\ \mathrm{Hom}(Z_n, \mathrm{Hom}(T_b, B)) & \leftarrow & \mathrm{Hom}(Z_n, \mathrm{Hom}(T, B)) & \leftarrow & \mathrm{Hom}(Z_n, \mathrm{Hom}(T/T_b, B)). \end{array}$$

Hence (1) is a pure sequence. But $\mathrm{Hom}(T/T_b, B)$ is a torsion-free co-torsion group, and thus by propositions 2.1 and 3.6 of [2] together with the properties of Pext, any pure extension of $\mathrm{Hom}(T/T_b, B)$ is trivial. The lemma now follows by the sequence (1).

We say that a property holds for all large cardinals α if there exists a cardinal number β with the property holding for all α with $\alpha \geqq \beta$.

Lemma 3. *If T and L are reduced p-groups, I_α is a direct sum of α copies of the p-adic integers, and T_b is a basic subgroup of T, then for all large α,* $\mathrm{Hom}(T, L) \oplus I_\alpha \simeq \mathrm{Hom}(T_b, L) \oplus I_\alpha$.

PROOF. By the theory of co-torsion groups (sec. 2 of [2]), it is enough to show that $\mathrm{Hom}(T, L)$ and $\mathrm{Hom}(T_b, L)$ have isomorphic torsion subgroups. Since L is reduced, $\mathrm{Hom}(T/T_b, L) = 0$ so the exact sequence $0 \to T_b \to T \to \to T/T_b \to 0$ gives that $\mathrm{Hom}(T/T_b, L) = 0 \to \mathrm{Hom}(T, L) \xrightarrow{\theta} \mathrm{Hom}(T_b, L)$ is exact. We must show that if $f \in \mathrm{Hom}(T_b, L)$ and $n \cdot f = 0$ for an integer n, then $\theta(g) = f$ for some $g \in \mathrm{Hom}(T, L)$. Since T/T_b is divisible, for any $a \in T$, there exist $c \in T, b \in T_b$ with $a - nc = b$. We let $g(a) = f(b)$. To show g is well defined, suppose $a - nc' = b'$ with $b' \in T_b$. Then $n(c' - c) = b - b'$ and since T_b is pure, $b - b' = nd$ for some $d \in T_b$. Hence $f(b) - f(b') = nf(d) = 0$ or $f(b) = f(b')$. It is clear that g is a homomorphism and that $\theta(g) = f$.

For any integer n, we shall use the notation L_n to denote the set of all $a \in L$ with $na = 0$.

Lemma 4. *If T_b is a direct sum of cyclic groups and if L and L' are two groups with $L_n \simeq L'_n$ for all integers n, then* $\mathrm{Hom}(T_b, L) \simeq \mathrm{Hom}(T_b L')$.

PROOF. If $T_b \simeq \Sigma Z_n$, then

$$\mathrm{Hom}(\Sigma Z_n, L) \simeq \Pi \mathrm{Hom}(Z_n, L) \simeq \Pi L_n \simeq \Pi L'_n \simeq \Pi \mathrm{Hom}(Z_n L') \simeq \mathrm{Hom}(\Sigma Z_n, L').$$

Proposition 5. *If T and T' are reduced p-groups with isomorphic basic subgroups, if L and L' are reduced p-groups with $L_n \simeq L'_n$ for all integers n, and if B_α denotes a direct sum of α copies of $Z(p^\infty)$, then for all large α*

$$\mathrm{Hom}(T \oplus B_\alpha, L \oplus B_\alpha) \simeq \mathrm{Hom}(T' \oplus B_\alpha, L' \oplus B_\alpha).$$

PROOF.

$$\mathrm{Hom}(T \oplus B_\alpha, L \oplus B_\alpha) \simeq \mathrm{Hom}(T, L) \oplus \mathrm{Hom}(B_\alpha, L) \oplus \mathrm{Hom}(T, B_\alpha) \oplus \mathrm{Hom}(B_\alpha, B_\alpha).$$

$\mathrm{Hom}(B_\alpha, L) = 0$ since L is reduced. Hence with lemmas 2 and 3 we have for α large enough.

$$\mathrm{Hom}(T \oplus B_\alpha, L \oplus B_\alpha) \simeq \mathrm{Hom}(T_b, L) \oplus \mathrm{Hom}(T_b, B_\alpha) \oplus$$
$$\mathrm{Hom}(T/T_b, B_\alpha) \oplus \mathrm{Hom}(B_\alpha, B_\alpha).$$

Since each of $\mathrm{Hom}(T/T_b, B_\alpha)$, and $\mathrm{Hom}(B_\alpha, B_\alpha)$ are torsion-free co-torsion groups and since by section 2 of [2] a torsion-free co-torsion group k is completely determined by the dimension of $k/p \cdot k$, we see that by choosing α larger also than the cardinality of T/T_b, we get all of $\mathrm{Hom}(T/T_b, B_\alpha) \oplus \oplus \mathrm{Hom}(B_\alpha, B_\alpha)$ completely determined by α. By lemma 4, $\mathrm{Hom}(T_b, L) \simeq \simeq \mathrm{Hom}(T'_b, L')$ and clearly $\mathrm{Hom}(T_b, B_\alpha) \simeq \mathrm{Hom}(T'_b, B_\alpha)$. The proposition follows.

By the existence part of the theory of countable p-groups (see chap. 6 of [1]) there exist nonisomorphic p-groups T and T' with isomorphic basic subgroups and with $T_n \simeq T'_n$ for all integers n. We let $G = T \oplus B_\alpha$ and $H = T' \oplus B_\alpha$ where α is large enough and have that $\mathrm{Hom}(G, G) \simeq \mathrm{Hom}(H, H)$ with G not isomorphic to H.

§ 2. Tensor product of a mixed group with a torsion group.

Theorem 6. *If $0 \to H \to G \to L \to 0$ is a pure exact sequence and T is a torsion group, then $G \otimes T \simeq (H \otimes T) \oplus (L \otimes T)$.*

PROOF. We have the exact sequence (see [2])

$$0 \to H \otimes T \to G \otimes T \to L \otimes T \to 0. \tag{2}$$

By proposition 4.2 of [2], $\mathrm{Hom}(T, H \otimes T)$ is a direct summand of a direct product of finite cyclic groups. Hence $\mathrm{Pext}(L, \mathrm{Hom}(T, H\otimes T)) = 0$. This means we have the exact sequence

$$\begin{array}{c} 0 \to \mathrm{Hom}(L, \mathrm{Hom}(T, H\otimes T)) \to \mathrm{Hom}(G, \mathrm{Hom}(T, H\otimes T)) \to \mathrm{Hom}(H, \mathrm{Hom}(T, H\otimes T)) \to 0 \\ \wr \qquad\qquad\qquad\qquad \wr \qquad\qquad\qquad\qquad \wr \\ 0 \to \mathrm{Hom}(L\otimes T, H\otimes T) \to \mathrm{Hom}(G\otimes T, H\otimes T) \to \mathrm{Hom}(H\otimes T, H\otimes T) \to 0. \end{array}$$

Hence the identity map of $H\otimes T$ to itself comes from some $f \in \mathrm{Hom}(G\otimes T, H\otimes T)$. But from this it is standard procedure to show that the kernel of f is the image of $H\otimes T$ in the sequence (2) and that the kernel of f is a direct summand of $G\otimes T$.

Corollary. *If G is any group with torsion subgroup G_t, and if T is any torsion group, then* $G\otimes T \simeq ((G/G_t)\otimes T) \oplus (G_t\otimes T)$.

ADDED IN PROOF: 1. Let $\alpha(T, L)$ be the cardinal number determining the torsion free co-torsion part of $\mathrm{Hom}(T, L)$. We (with section 2 of [2]) have shown that $\mathrm{Hom}(T, L)$ is determined by T_b the L_n, and $\alpha(T, L)$. Hence it is very desirable to see a calculation of $\alpha(T, L)$.

2. In the meantime E. SĄSIADA (On two problems concerning endomorphism groups, *Annales Univ. Sci. Budapest* **2** (1959), 65—66) solved Problem 42 for torsion-free groups and L. FUCHS (Notes on abelian groups. I, ibid. 5—23) described the tensor product of a mixed and a torsion group, and also the torsion part of the tensor product of any two groups (cf. also L. FUCHS, Notes on abelian groups. II, Acta Math. Acad. Sci. Hungar. **11** (1960), 117—125).

Bibliography.

[1] L. FUCHS, Abelian groups, *Budapest*, 1958.
[2] D. K. HARRISON, Infinite Abelian groups and homological methods, *Ann. of Math.* (1959), 366—391.
[3] H. CARTAN and S. EILENBERG, Homological algebra, *Princeton*, 1956.

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