On limit operations in a certain topology for endomorphism rings of abelian groups.

To Professor O. Varga on his 50th birthday.

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§ 1. Preliminaries.

Let G be an arbitrary (additively written) abelian group, and let $E(G)$ be the complete endomorphism ring of G. Consider an arbitrary subring P of $E(G)$, which may in particular coincide with $E(G)$. Following T. SZELE [1], we introduce in P a concept of convergence by the following definition:

An infinite system $\Phi = \{\varphi_\nu | \nu \in N\}$ of elements of the ring P is said to converge to the limit $\varphi \in P$, if for any $x \in G$ the relation $x(\varphi_\nu - \varphi) = 0$ holds with the exception of a finite number of ν's.[1])

If Φ converges to φ, we write $\lim\limits_{\nu \in N} \varphi_\nu = \varphi$. It is easy to see that the limit is uniquely determined.

It will perhaps be worth while to point out that the concept of convergence just introduced gives rise in a natural way to a topology. Let us indeed make the following conventions:

A subset $A (\emptyset \subseteq A \subseteq P)$ of the set P is said to have the accumulation point $\varphi \in P$, if A converges to φ. (We see that by this definition a finite set can have no accumulation point, while an infinite set can have at most one such point.)

Closed sets can now be defined in the usual way:

A set in P is closed, if it contains all its accumulation points. (I. e. a set is closed if it contains its accumulation point, in case such a point exists.)

More explicitly this definition can be stated as follows:

A subset A of the set P is called closed if $\beta \in P$ belongs to A whe-

[1]) We remark that a similar phenomenon of convergence arises also in purely algebraical investigations on endomorphism rings. See [2].

never for every $x \in G$ all but a finite number of the elements α of A satisfy $x\beta = x\alpha$.

From this definition it follows trivially that

1. $\emptyset$ and P are closed sets,

2. any finite subset of P is closed,

since the requirement expressed by the definition is irrelevant in these cases. Moreover it is easily seen that

3. the union of two closed sets is closed,

4. the intersection of any class of closed sets is closed.

Thus P becomes a topological space, in fact a T_1-space.

It is the purpose of the present note to clarify a certain aspect of the concept of convergence just exposed. We shall indeed consider double limits and investigate their interrelations.

§ 2. Convergence in two variables.

The definition of convergence to a limit just given carries over in a natural way to double limits, in the sense of the following

Definition 1. An infinite system $\Phi = \{\varphi_{\mu\nu} | \mu \in M, \nu \in N\}$ (here, as well as in the sequel, both index sets are supposed to be infinite) of elements of the ring P is said *to converge* to the limit $\varphi \in P$, if for any $x \in G$ the relation

$$x(\varphi_{\mu\nu} - \varphi) = 0$$

holds with the exception of a finite number of pairs (μ, ν).[2])

It is also possible to introduce iterated limits by the

Definition 2. An infinite system $\Phi = \{\varphi_{\mu\nu}\}$ is said to have the *iterated limit*

$$\lim_{\mu \in M} \lim_{\nu \in N} \varphi_{\mu\nu} = \alpha,$$

if the following conditions hold:

I. for all but a finite number of the μ's there exist elements $\alpha_\mu \in P$ such that

$$\lim_{\nu \in N} \varphi_{\mu\nu} = \alpha_\mu,$$

and

II. there exists an $\alpha \in P$ such that

$$\lim_{\mu \in M} \alpha_\mu = \alpha.$$

[2]) To different elements of G there can correspond of course different sets of "exceptional pairs". We require only that the set of exceptional values be finite for any $x \in G$.

The other iterated limit,

$$\lim_{\nu \in N} \lim_{\mu \in M} \varphi_{\mu\nu} = \beta,$$

can of course be defined in an analogous manner.[3])

Comparing these conditions with the one expressed by Definition 1, we immediately get the following

Theorem 1. *If the double limit*

$$\lim_{\substack{\mu \in M \\ \nu \in N}} \varphi_{\mu\nu}$$

exists, then both iterated limits exist and they are equal to each other and to the double limit.

It is also easy to get a simple necessary and sufficient condition for the two iterated limits to exist and to be equal to each other. We indeed have the following

Theorem 2. *The two iterated limits*

$$\lim_{\mu \in M} \lim_{\nu \in N} \varphi_{\mu\nu} \quad \text{and} \quad \lim_{\nu \in N} \lim_{\mu \in M} \varphi_{\mu\nu}$$

both exist and are equal to each other if and only if there exists an $\alpha \in P$ and for any $x \in G$ there exist finite subsets M_x of M and N_x of N so that for

$$\mu \in M - M_x,\ \nu \in N - N_x$$

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[3]) It will perhaps be worth while to point out the intuitive meaning of this definition. Let us call sets of the form

$$\{\varphi_{\mu\nu} \mid \mu : \text{fixed},\ \nu \in N\}$$

rows, and sets

$$\{\varphi_{\mu\nu} \mid \nu : \text{fixed},\ \mu \in M\}$$

columns. Then, as one easily sees, our definition says that

$$\lim_{\mu \in M} \lim_{\nu \in N} \varphi_{\mu\nu} = \alpha,$$

if for each $x \in G$

$$x(\varphi_{\mu\nu} - \alpha) = 0$$

holds, if only we except a finite number of rows, and in each of the remaining rows a finite number of places. (Of course, the "tableau" determined by the required omissions is dependent in general on the element $x \in G$.) Similarly

$$\lim_{\nu \in N} \lim_{\mu \in M} \varphi_{\mu\nu} = \beta,$$

if for each $x \in G$

$$x(\varphi_{\mu\nu} - \beta) = 0$$

holds, if only we except a finite number of columns, and in each of the remaining columns a finite number of places.

the relation

$$x(\varphi_{\mu\nu}-\alpha)=0$$

holds with only so many exceptions that each single row as well as each column contains only a finite number of them.

PROOF. If both iterated limits exist and are equal to each other, then for any $x \in G$ the exceptions to $x(\varphi_{\mu\nu}-\alpha)=0$ will surely be eliminated, if we eliminate finitely many rows, and in each remaining row a finite number of places; the exceptions can however also be eliminated by cancelling finitely many columns and in each remaining column a finite number of places. Now if we take the first steps of both these procedures, i. e. if we eliminate both the (appropriately chosen) finitely many rows and the finitely many columns, then the remaining exceptions will clearly be capable of being eliminated by the second step of any of the two procedures, i. e. any (remaining) row as well as any (remaining) column will contain at most a finite number of the remaining exceptions. This proves the condition to be necessary.

On the other hand, if the condition is fulfilled, then for any $x \in G$ all exceptions to $x(\varphi_{\mu\nu}-\alpha)=0$ can clearly be eliminated by cancelling finitely many rows and in each remaining row finitely many places, as well as by an analogue for columns of this procedure.

§ 3. Uniform convergence.

Let us now introduce a concept of uniform convergence, which will allow us to establish a further (sufficient) condition for the existence and equality to each other of the two iterated limits.

DEFINITION 3. We say that the system $\Phi=\{\varphi_{\mu\nu}\}$ converges in the variable $\nu(\in N)$ uniformly to the limit τ_μ, if

a) for $\mu \in M$, with but a finite number of exceptions, there exist elements $\tau_\mu \in P$ for which

$$\lim_{\nu \in N} \varphi_{\mu\nu}=\tau_\mu$$

holds;

b) for any $x \in G$ there exists a finite subset N_x of N, such that for any $\mu \in M$ for which $\lim\limits_{\nu \in N} \varphi_{\mu\nu}=\tau_\mu$, the relation

$$x(\varphi_{\mu\nu}-\tau_\mu)=0$$

holds, if only $\nu \in N-N_x$.[4])

[4]) The uniformity of this sort of convergence consists of course in the fact that the subset N_x depends only on x but not on μ.

c) the "exceptional" subsets N_x do not exhaust the set N, in the sense that

$$N - \{\bigcup_{x \in G} N_x\}$$

is an infinite set.

Now we are able to formulate and to prove the following

Theorem 3. *Let the system $\Phi = \{\varphi_{\mu\nu}\}$ converge uniformly in each variable with respect to the other, i. e. let*

a) $$\lim_{\mu \in M} \varphi_{\mu\nu} = \gamma_\nu$$

uniformly on N, and

b) $$\lim_{\nu \in N} \varphi_{\mu\nu} = \tau_\mu$$

uniformly on M. Then both iterated limits exist and they are equal.

PROOF. By a) $\lim_{\mu \in M} \varphi_{\mu\nu} = \gamma_\nu$ for $\nu \in N - N_1$, where N_1 is a (fixed) finite subset of N. — Moreover, for any $x \in G$ there exists a finite subset M_x of M, such that $x(\varphi_{\mu\nu} - \gamma_\nu) = 0$ for $\mu \in M - M_x$, $\nu \in N - N_1$.

Similarly, by b) $\lim_{\nu \in N} \varphi_{\mu\nu} = \tau_\mu$ for $\mu \in M - M_1$, where M_1 is a (fixed) finite subset of M. — Moreover, for any $x \in G$ there exists a finite subset N_x of N, such that $x(\varphi_{\mu\nu} - \tau_\mu) = 0$ for $\mu \in M - M_1$, $\nu \in N - N_x$.

For an arbitrary $x \in G$ let now be

$$\mu \in M - (M_1 \cup M_x) = M_x^*,$$

and

$$\nu \in N - (N_1 \cup N_x) = N_x^*.$$

For any pair (μ, ν) satisfying these conditions,

$$x(\varphi_{\mu\nu} - \gamma_\nu) = 0 \quad \text{and} \quad x(\varphi_{\mu\nu} - \tau_\mu) = 0,$$

i. e.

$$x\varphi_{\mu\nu} = x\gamma_\nu = x\tau_\mu.$$

Now let $\mu \in M_x^*$ and $\nu_1, \nu_2 \in N_x^*$. Clearly

$$x\varphi_{\mu\nu_1} = x\gamma_{\nu_1} = x\tau_\mu,$$

$$x\varphi_{\mu\nu_2} = x\gamma_{\nu_2} = x\tau_\mu.$$

We see that $x\gamma_\nu$ remains constant while ν runs through N_x^*, and this constant value is equal to $x\tau_\mu$ for an arbitrary $\mu \in M_x^*$. If we denote this common value of the $x\gamma_\nu$'s and the $x\tau_\mu$'s by $x\alpha$, then we can say that for $\mu \in M_x^*$ and $\nu \in N_x^*$ one has $x\varphi_{\mu\nu} = x\alpha$ or equivalently $x(\varphi_{\mu\nu} - \alpha) = 0$.

Clearly $x\alpha$ is defined for every $x \in G$, so we have a mapping α of G into itself. Let us show that this mapping belongs to P; then, by Theorem

2, the existence and equality to each other of the two iterated limits will follow.

We see that by the finiteness of M_1 and of N_1, and by condition c) in Definition 3,

$$M_0 = M - \{M_1 \cup (\bigcup_{x \in G} M_x)\}$$

and

$$N_0 = N - \{N_1 \cup (\bigcup_{x \in G} N_x)\}$$

are both non-void (and even infinite) sets. Let $\mu_0 \in M_0$ and $\nu_0 \in N_0$. Then clearly $\mu_0 \in M_x^*$ and $\nu_0 \in N_x^*$ for every $x \in G$, and so $x\varphi_{\mu_0\nu_0} = x\alpha\,(x \in G)$, i. e. $\alpha = \varphi_{\mu_0\nu_0} \in P$.[5]) This completes the proof of our theorem.

Bibliography.

[1] T. Szele, On a topology in endomorphism rings of abelian groups, *Publ. Math. Debrecen* **5** (1957), 1—4.

[2] A. Kertész, On radical-free rings of endomorphisms, *Acta Univ. Debrecen* **5** (1958), 159—161.

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[5]) If we wish only to show that the mapping α is an endomorphism of G, then we can avoid making use of condition c) in the definition of uniform convergence. Let indeed for $x, y \in G$ be

$$\mu \in M - [M_1 \cup M_x \cup M_y \cup M_{x+y}],$$
$$\nu \in N - [N_1 \cup N_x \cup N_y \cup N_{x+y}].$$

Then

$$(x+y)\,\alpha = (x+y)\,\varphi_{\mu\nu} = x\,\varphi_{\mu\nu} + y\,\varphi_{\mu\nu} = x\alpha + y\alpha.$$

If the ring P considered is the centralizer of some subset Θ of the complete endomorphism ring $E(G)$ of G (the investigations of Szele's paper [1] are restricted throughout to such rings), then we can drop altogether condition c) from the definition of uniform convergence, since $\alpha \in P$ can be proved without it. In fact, for $x \in G$ and $\vartheta \in \Theta$ there exist indices μ and ν satisfying

$$\mu \in M - (M_1 \cup M_x \cup M_{x\vartheta}).$$
$$\nu \in N - (N_1 \cup N_x \cup N_{x\vartheta}),$$

and for any such pair (μ, ν) we have both $x\alpha = x\varphi_{\mu\nu}$ and $(x\vartheta)\,\alpha = (x\,\vartheta)\,\varphi_{\mu\nu}$. Since $\varphi_{\mu\nu} \in P$, i. e. $\varphi_{\mu\nu}$ commutes with any $\vartheta \in \Theta$, we get $(x\vartheta)\alpha = (x\vartheta)\,\varphi_{\mu\nu} = (x\varphi_{\mu\nu})\,\vartheta = (x\alpha)\,\vartheta$, i. e. α too commutes with every $\vartheta \in \Theta$, and this shows that $\alpha \in P$.