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$F(2)$ type structures in the complex Finsler spaces

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Abstract. There are lot of papers and books ([1]–[3], [6]–[12] etc.) in which the almost complex, almost product and tangent structures are studied in the tangent bundles of Riemannian or Finsler spaces. Here, the $F(2)$ type structures on the tangent bundle of complex Finsler spaces are defined ([4], [5]). They have the property, that for different values of parameters are almost complex, almost product or tangent structures. It is proved that they are tensors of type (1,1) with respect to the coordinate transformations. The invariant subspaces of one structure are determined.

1. Complex Finsler spaces

Let us consider two n-dimensional Finsler spaces $F_1(x,\dot{x})$ and $F_2(y,\dot{y})$. The allowable coordinate transformations in F_1 and F_2 are given by

$$
\begin{aligned}
&x^{a'} = x^{a'}(x) && y^{i'} = y^{i'}(y)\\
&\dot{x}^{a'} = A_a^{a'}(x)\dot{x}^a && \dot{y}^{i'} = B_i^{i'}(y)\dot{y}^i\\
&A_a^{a'} = \frac{\partial x^{a'}}{\partial x^a} && B_i^{i'} = \frac{\partial y^{i'}}{\partial y^i},
\end{aligned}
\tag{1.1}
$$

where

$$\operatorname{rank}[A_a^{a'}] = n \qquad \operatorname{rank}[B_i^{i'}] = n,$$

so the inverse transformations exist.

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The adapted basis of $T(F_1)$ is $B_1 = \left\{ \frac{\delta}{\delta x^a}, \frac{\partial}{\partial \dot{x}^a} \right\}$ and the adapted basis of $T(F_2)$ is $B_2 = \left\{ \frac{\delta}{\delta y^i}, \frac{\partial}{\partial \dot{y}^i} \right\}$, where

$$\frac{\delta}{\delta x^a} = \frac{\partial}{\partial x^a} - N_a^b(x,\dot{x})\frac{\partial}{\partial \dot{x}^b}, \quad \frac{\delta}{\delta y^i} = \frac{\partial}{\partial y^i} - \bar{N}_i^j(y,\dot{y})\frac{\partial}{\partial \dot{y}^j}.$$

$N_a^b(x,\dot{x})$ and $\bar{N}_i^j(y,\dot{y})$ are coefficients of the non-linear connections, which satisfy the usual transformation law with repsect to (1.1).

The complex Finsler space $E'(x,\dot{x},y,\dot{y})$ is formed in such a way that B', the adapted basis of $T(E')$, is given by $B' = B_1 \cup iB_2$.

For the further exploration we shall use five kinds of indices

$$\begin{array}{ll} a,b,c,d,e,f,g = 1,2,\ldots,n, & i,j,k,l,m,p,q = n+1,\ldots,2n \\ A,B,C,D,E,F,G = 2n+1,\ldots,3n, & I,J,K,L,M,P,Q = 3n+1,\ldots,4n \\ \alpha,\beta,\gamma,\delta,\kappa,\nu,\mu = 1,2,\ldots,4n. & \end{array}$$

The following equalities are valid

$$\begin{gathered} a = i = A = I (\operatorname{mod} n), \qquad b = j = B = J (\operatorname{mod} n) \\ c = h = C = H (\operatorname{mod} n). \end{gathered} \tag{1.2}$$

Using these indices, B' and its dual B'^* can be written in the form

$$\text{(a) } B' = \{\partial_\alpha\} = \left\{ \frac{\delta}{\delta x^a}, i\frac{\delta}{\delta y^i}, \frac{\partial}{\partial \dot{x}^A}, i\frac{\partial}{\partial \dot{y}^I} \right\} \tag{1.3}$$

$$\text{(b) } B'^* = \left\{ d^\alpha \right\} = \left\{ dx^b, -idy^j, \delta\dot{x}^B, -i\delta\dot{y}^J \right\},$$

where

$$\delta\dot{x}^B = d\dot{x}^B + N_c^B(x,\dot{x})dx^c, \;\; \delta\dot{y}^J = d\dot{y}^J + N_i^J(y,\dot{y})dy^i.$$

2. The complete list of $F(2)$ type structures

Definition 2.1. The tensor field F of type (1,1) defined on E' is the structure of $F(k)$ type, if in the basis B its matrix can be decomposed on 4×4 blocks of form $n \times n$, such that in each row and each column are k scalar matrices and $4 - k$ zero blocks.

Notation. Every one of scalar fields a,b,c,d denotes the corresponding real or complex matrix of type $n \times n$ (for example $a = a(x,y,\dot{x},\dot{y})I$.

Theorem 2.1. *There exist 90 $F(2)$ type structures on E'.*

Proof. There are $36 = \binom{4}{2}\binom{4}{2}$ matrices formed in such a way, that in the first two columns the chosen elements are always in different rows

$$\begin{bmatrix} a & 0 & e & 0 \\ b & 0 & f & 0 \\ 0 & c & 0 & g \\ 0 & d & 0 & h \end{bmatrix} \cdots \begin{bmatrix} 0 & c & 0 & g \\ a & 0 & e & 0 \\ b & 0 & 0 & h \\ 0 & d & f & 0 \end{bmatrix} \cdots \begin{bmatrix} 0 & c & 0 & g \\ 0 & d & 0 & h \\ a & 0 & e & 0 \\ b & 0 & f & 0 \end{bmatrix}.$$

The next 48 matrices have the property, thet the first two columns have once two elements in the same row (ac) and two elements in different rows:

$$\begin{bmatrix} a & c & 0 & 0 \\ b & 0 & e & 0 \\ 0 & d & 0 & g \\ 0 & 0 & f & h \end{bmatrix} \cdots \begin{bmatrix} b & 0 & e & 0 \\ 0 & 0 & f & g \\ a & c & 0 & 0 \\ 0 & d & 0 & h \end{bmatrix} \cdots \begin{bmatrix} 0 & 0 & e & g \\ 0 & d & 0 & h \\ b & 0 & f & 0 \\ a & c & 0 & 0 \end{bmatrix}.$$

In the last 6 matrices the first and second columns have two times, two elements in the same row:

$$\begin{bmatrix} a & c & 0 & 0 \\ b & d & 0 & 0 \\ 0 & 0 & e & g \\ 0 & 0 & f & h \end{bmatrix} \cdots \begin{bmatrix} a & c & 0 & 0 \\ 0 & 0 & e & g \\ 0 & 0 & f & h \\ b & d & 0 & 0 \end{bmatrix} \cdots \begin{bmatrix} 0 & 0 & e & g \\ 0 & 0 & f & h \\ a & c & 0 & 0 \\ b & d & 0 & 0 \end{bmatrix}.$$

Definition 2.2. The tensor field F of type (1,1) defined on E' is almost complex structure (a.c.s.) iff $F^2 = -I$, almost product structure (a.p.s.) iff $F^2 = I$, or tangent structure (t.s.) iff $F^2 = 0$.

Theorem 2.2. *There are only six $F(2)$ type structures defined on E', which for special values of parameters can be a.c.s., or a.p.s., or t.s.*

Proof. Some F_i $(i = 1, \dots, 90)$ from the above list of $F(2)$ type structures can be a.c.s., or a.p.s., or t.s. if F_i^2 has the property, that all elements, which are not on the main diagonal are equal to zero. By direct calculation can be proved that 84 of them are such, that F_i^2 has at least on one place (j,k) $j \neq k$, the product of two elements. This product is equal to zero if at least one of the factor is equal to zero, but in this case F_i is not $F(2)$ type structure. The exceptions are (for special values of parameters):

$$F_1 = \begin{bmatrix} a & 0 & b & 0 \\ 0 & c & 0 & d \\ e & 0 & -a & 0 \\ 0 & g & 0 & -c \end{bmatrix} \qquad F_2 = \begin{bmatrix} a & 0 & 0 & b \\ 0 & c & d & 0 \\ 0 & e & -c & 0 \\ f & 0 & 0 & -a \end{bmatrix}$$

$$F_3 = \begin{bmatrix} 0 & a & b & 0 \\ -ce & 0 & 0 & cd \\ cd & 0 & 0 & -ca \\ 0 & d & e & 0 \end{bmatrix} \quad F_4 = \begin{bmatrix} 0 & a & 0 & -b \\ ce & 0 & bc & 0 \\ 0 & d & 0 & e \\ -cd & 0 & ac & 0 \end{bmatrix}$$

$$F_5 = \begin{bmatrix} a & b & 0 & 0 \\ c & -a & 0 & 0 \\ 0 & 0 & e & f \\ 0 & 0 & d & -e \end{bmatrix} \qquad F_6 = \begin{bmatrix} 0 & 0 & ae & -ab \\ 0 & 0 & -ac & d \\ a^{-1}b & b & 0 & 0 \\ c & e & 0 & 0 \end{bmatrix}.$$

By direct calculation we obtain

$$F_1^2 = \mathrm{diag}[a^2 + be, c^2 + dg, a^2 + be, c^2 + dg]$$

$$F_2^2 = \mathrm{diag}[a^2 + bf, c^2 + de, c^2 + de, a^2 + bf]$$

$$F_3^2 = c(bd - ae)I$$

$$F_4^2 = c(bd + ae)I$$

$$F_5^2 = \mathrm{diag}[a^2 + bc, a^2 + bc, e^2 + df, e^2 + df]$$

$$F_6^2 = (de - abc)I.$$

From Theorem 2.2 follows

Theorem 2.3. *The $F(2)$ type structures $F_1 - F_6$ are a.c.s. if*

$$\begin{aligned} &\text{in } F_1 \quad a^2 + be = c^2 + dg = -1, \\ &\text{in } F_2 \quad a^2 + bf = c^2 + de = -1, \\ &\text{in } F_3 \quad c(bd - ae) = -1, \\ &\text{in } F_4 \quad c(bd + ae) = -1, \\ &\text{in } F_5 \quad a^2 + bc = e^2 + df = -1, \\ &\text{in } F_6 \quad de - abc = -1. \end{aligned}$$

If in the above equations -1 is everywhere replaced by 1, the structures $F_1 - F_6$ become a.p.s.; if -1 is everywhere replaced by 0, the structures $F_1 - F_6$ become t.s.

3. The tensor character of $F(2)$ type structures

Theorem 3.4. *The $F(2)$ type structures $F_1 - F_6$ are* (1,1) *tensors with respect to the coordinate transformation* (1.1).

PROOF. As the proof is the same for all mentioned $F(2)$ type structures, we shall give it only for F_1. The precise form of F_1 is the following:

$$
\begin{aligned}
F_1 = {} & a\delta^a_b \frac{\delta}{\delta x^a} \otimes dx^b + b\delta^a_\beta \frac{\delta}{\delta x^a} \otimes \delta\dot{x}^B \\
& + c\delta^i_j \left(i\frac{\delta}{\delta y^i} \right) \otimes (-idy^j) + d\delta^i_J \left(i\frac{\delta}{\delta y^i} \right) \otimes (-i\delta\dot{y}^J) \\
& + e\delta^A_b \left(\frac{\partial}{\partial \dot{x}^A} \right) \otimes dx^b - a\delta^A_B \left(\frac{\partial}{\partial \dot{x}^A} \right) \otimes (\delta\dot{x}^B) \\
& + g\delta^I_j \left(i\frac{\partial}{\partial \dot{y}^I} \right) \otimes (-idy^j) - c\delta^I_J \left(i\frac{\partial}{\partial \dot{y}^I} \right) \otimes (-i\delta\dot{y}^J),
\end{aligned}
$$

where (1.2) is valid.

Substituting the transformation laws for the basis vectors:

$$
\begin{aligned}
\frac{\delta}{\delta x^a} &= A^{a'}_a(x)\frac{\delta}{\delta x^{a'}} & dx^B &= A^B_{B'}(x')dx^{B'} \\
i\frac{\delta}{\delta y^i} &= B^{i'}_i(y)i\frac{\delta}{\delta y^{i'}} & -idy^j &= B^j_{j'}(y')(-idy^{j'}) \\
\frac{\partial}{\partial \dot{x}^A} &= A^{A'}_A(x)\frac{\partial}{\partial \dot{x}^{A'}} & \delta\dot{x}^B &= A^B_{B'}(x')\delta\dot{x}^{B'} \\
i\frac{\partial}{\partial \dot{y}^I} &= B^{I'}_I(y)\frac{\partial}{\partial \dot{y}^{I'}} & -i\delta\dot{y}^J &= B^J_{J'}(y')(-i\delta\dot{y}^{J'})
\end{aligned}
$$

we obtain F_1 in the new basis. The obtained expression shows, that F_1 is a tensor of type (1,1).

4. Invariant subspaces of structure F_1

Proposition 4.1. *The eigenvectors for the almost complex structure F_1 are:*

(4.1) $$a_{(1,b)} = (-b(a-i)^{-1}, 0, 1, 0),\ a_{(2,b)} = (0, -d(c-i)^{-1}, 0, 1),$$

which correcpond to the eigenvalue i and

(4.2) $$a_{(3,b)} = (-b(a+i)^{-1}, 0, 1, 0),\ a_{(4,b)} = (0, -d(c+i)^{-1}, 0, 1),$$

which correspond to the eigenvalue $-i$.

Remark 1. As F_1 is determined by the matrix of type $4n \times 4n$, the eigenvectors of F_1 should have $4n$ coordinates. In (4.1), (4.2) the coordinates on the place $b,\ b+n,\ b+2n,\ b+3n$ are given, the other coordinates are equal to zero and $b = 1, 2, \ldots, n$.

In the basis B', the eigenvectors determined by (4.1) and (4.2) can be written in the following way:

$$\begin{aligned}
a_{(1,b)} &= -b(a-i)^{-1}\delta^a_b \frac{\delta}{\delta x^a} + \delta^A_{b+2n}\frac{\partial}{\partial \dot{x}^A}\\
a_{(2,b)} &= -d(c-i)^{-1}\delta^j_{b+n}\left(i\frac{\delta}{\delta y^j}\right) + \delta^J_{b+3n}\left(i\frac{\partial}{\partial \dot{y}^J}\right)\\
a_{(3,b)} &= -b(a+i)^{-1}\delta^a_b \frac{\delta}{\delta x^a} + \delta^A_{b+2n}\frac{\partial}{\partial \dot{x}^A}\\
a_{(4,b)} &= -d(c+i)^{-1}\delta^j_{b+n}\left(i\frac{\delta}{\delta y^j}\right) + \delta^J_{b+3n}\left(i\frac{\partial}{\partial \dot{y}^J}\right).
\end{aligned}$$

Proposition 4.2. *The invariant subspaces of $T(E')$ for the a.c.s. F_1 are T_1 and T_2, where*

$$\begin{aligned}
&\forall X \in T_1 \quad F_1 X = iX\\
&\forall Y \in T_2 \quad F_1 Y = -iY.
\end{aligned}$$

The space $T(E')$ is equal to the direct sum $T_1 \oplus T_2$. In the basis B' we have

$$\begin{aligned}
X &= -b(a-i)^{-1}\alpha^b \frac{\delta}{\delta x^b} - d(c-i)^{-1}\beta^j\left(i\frac{\delta}{\delta y^j}\right) + \alpha^B\frac{\partial}{\partial \dot{x}^B} + \beta^J\left(i\frac{\partial}{\partial \dot{y}^J}\right)\\
Y &= -b(a+i)^{-1}\gamma^b \frac{\delta}{\delta x^b} - d(c+i)^{-1}\delta^j\left(i\frac{\delta}{\delta y^j}\right) + \gamma^B\frac{\partial}{\partial \dot{x}^B} + \delta^J\left(i\frac{\partial}{\partial \dot{y}^J}\right)
\end{aligned}$$

where

$$\begin{aligned}
\alpha^B &= \alpha^{b+2n} = \alpha^b \qquad \beta^J = \beta^{j+2n} = \beta^j\\
\gamma^B &= \gamma^{b+2n} = \gamma^b \qquad \delta^J = \delta^{j+2n} = \delta^j
\end{aligned} \tag{4.3}$$

are arbitrary real numbers.

Proposition 4.3. *The eigenvectors for the almost product structure F_1 are:*

$$a_{(1,b)} = (-b(a-1)^{-1}, 0, 1, 0),\ a_{(2,b)} = (0, -d(c-1)^{-1}, 0, 1), \tag{4.4}$$

which correspond to the eigenvalue 1 and

$$a_{(3,b)} = (-b(a+1)^{-1}, 0, 1, 0),\ a_{(2,b)} = (0, -d(c+1)^{-1}, 0, 1), \tag{4.5}$$

which correspond to the eigenvalue -1*.*

The Remark 1 is valid for (4.4) *and* (4.5)*.*

Proposition 4.4. *The invariant subspaces of* $T(E')$ *for the a.p.s.* F_1 *are* $\bar{T}_1$ *and* $\bar{T}_2$*, where*

$$\forall X \in \bar{T}_1 \quad F_1 X = X$$
$$\forall Y \in \bar{T}_2 \quad F_1 Y = -Y.$$

The space $T(E')$ *is equal to the direct sum* $\bar{T}_1 \oplus \bar{T}_2$*. From* (4.4) *and* (4.5) *follows, that in the basis* B' *we have:*

$$X = -b(a-i)^{-1}\alpha^a \frac{\delta}{\delta x^a} - d(c-i)^{-1}\beta^j \left(i\frac{\delta}{\delta y^j}\right) + \alpha^A \frac{\partial}{\partial \dot{x}^A} + \beta^J \left(i\frac{\partial}{\partial \dot{y}^J}\right)$$
$$Y = -b(a+i)^{-1}\gamma^a \frac{\delta}{\delta x^a} - d(c+1)^{-1}\delta^j \left(i\frac{\delta}{\delta y^j}\right) + \gamma^A \frac{\partial}{\partial \dot{x}^A} + \delta^J \left(i\frac{\partial}{\partial \dot{y}^J}\right).$$

In the above formulae (4.3) is valid.

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