Discussion of the geometry of affinely connected spaces by direct method

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§ 1. Introduction

We call direct tensor calculus the method grounded on the geometrical definition and properties of tensors, which does not use coordinates in demonstrating the theorems and properties independent of the system of coordinates. The most consistent cultivators of this method, mostly Italian mathematicians, called this method absolute. The method was applied to discussion of n-dimensional euclidean and Riemannian spaces first by BURALI-FORTI and BOGGIO (see [1], [2]). They discussed by this method Riemannian spaces embedded into greater dimensional euclidean spaces, and in their treatment the embedding euclidean space was thoroughly employed.

In the years of 1920—1930 there was a debate about the question of methods in differential geometry. From that time on the direct method was scarcely employed in the investigations of differential geometrical spaces. However, the direct method has its role, and significance beside the other important methods of differential geometry. It makes superfluous the introduction of coordinates in investigations and demonstrations of properties independent of the system of coordinates. Hence, it makes superfluous investigations connected with invariance, covariance, etc. At the same time it gives a deeper insight into the geometrical essence of the theorems and makes possible the demonstration, and the underlining of essential features, which are hidden behind the formulae of the Ricci calculus.

This paper contains the setting up of the geometry of affinely connected spaces by direct method, without introducing coordinates. The spaces are not embedded into greater dimensional affine spaces. The affine space, on which our affinely connected space is mapped, plays an important part. For instance, vectors are defined by means of that affine space. In the definition of the affinely connected space we make use of the concept of parallel displacement due to LEVI-CIVITA. We may say briefly that we attain the affinely connected space by modifying the geometry of the affine space making use of the parallelism of LEVI-CIVITA. We give a geometrical definition of the tensors and use directly this definition in demonstrations.

§ 2. Tensor calculus

Consider the topological space L of the points P, and suppose that L can be topologically mapped upon the n-dimensional affine space A or upon a connected, open part of it. We denote the topological transformation of L upon A with f. If $g \subset L$ and the image of g by the mapping f is a curve g' of A, then g is called a *curve* of the space L. If $L_m \subset L$ $(m < n)$ and the image of L_m by the mapping f is an m-dimensional surface L'_m of A, then L_m is called an *m-dimensional surface* of L. The $(n-1)$-dimensional surfaces of the space L are called the *hypersurfaces* of L.

Let the point $P' \in A$ be the image of the point $P \in L$. The vectors of A setting out off the point P' are called the *vectors* of the space L in the point P. The algebraic operations *among* these vectors are defined in the same way as in the space A. Only, the vectors of L in the point P are considered as attached to the point. Hence, if P_1 and P_2 are two distinct points of L, then apriori no relation exists among the vectors of L in P_1 and in P_2 respectively.

Let $P = P(t)$ be the equation of the curve g of L, and $P_0 = P(t_0)$ a point of g. Denote the image of g and P_0 respectively in A with g' and P'_0. Since the equation of g' is $P' = f(P(t)) = P'(t)$, the parameter t can be considered as a parameter on the curve g. Furthermore denote the vector $\overrightarrow{P'_0 P'(t)}$ with $\overline{\triangle P}$ and $t - t_0$ with $\triangle t$. The vector

$$\frac{\overline{dP}}{dt} = \lim_{\triangle t \to 0} \frac{\overline{\triangle P}}{\triangle t}$$

as a vector of L in the point P_0 is called the tangent vector of the curve g in the point P_0. The tangent vectors of an m-dimensional surface L_m of L are similarly defined.

We say that α is a *first order tensor* of the space L in the point P, if α represents a linear homogeneous transformation of the vectors in P to the vectors in P, i. e. if α possesses the following properties:

a) If $\overline{a}$ is a vector of L in the point P, then $\overline{v} = \alpha\overline{a}$ is also a vector of L in the point P.

b) If $\overline{a}$ and $\overline{b}$ are two vectors in P, and r and s are two real numbers, the equation

$$\alpha(r\overline{a} + s\overline{b}) = r(\alpha\overline{a}) + s(\alpha\overline{b})$$

holds.

We say that the first order tensors α_1 and α_2 are equal, $\alpha_1 = \alpha_2$, if $\alpha_1\bar{a} = \alpha_2\bar{a}$ for every vector $\bar{a}$ in P. We define the tensors $r\alpha$ and $\alpha_1 + \alpha_2$ respectively (r is a real number, $\alpha, \alpha_1, \alpha_2$ are first order tensors) by the equations

$$(r\alpha)\bar{a} = r(\alpha\bar{a}); \quad (\alpha_1 + \alpha_2)\bar{a} = \alpha_1\bar{a} + \alpha_2\bar{a}.$$

We say that β is a *second order tensor* of the space L in the point P, if β represents a linear homogeneous transformation of the vectors in P to the first order tensors in P, i. e. if β possesses the following properties:

a) If $\bar{a}$ is a vector of L in the point P, then $\alpha = \beta\bar{a}$ is a first order tensor of L in P.

b) If $\bar{a}$ and $\bar{b}$ are two vectors in P, and r and s are two real numbers, the equation

$$\beta(r\bar{a} + s\bar{b}) = r(\beta\bar{a}) + s(\beta\bar{b})$$

holds.

We define the equality of second order tensors, the product of a real number and a second order tensor, and the sum of two second order tensors similarly as in the case of first order tensors.

In general, we say that μ is a p-th *order tensor* of the space L in the point P, if it possesses the following properties:

a) If $\bar{a}$ is a vector of L in the point P, then $\varkappa = \mu\bar{a}$ is a $(p-1)$-th order tensor of L in P.

b) If $\bar{a}$ and $\bar{b}$ are two vectors in P, and r and s are two real numbers, the equation

$$\mu(r\bar{a} + s\bar{b}) = r(\mu\bar{a}) + s(\mu\bar{b})$$

holds.

We say that the p-th order tensors μ_1 and μ_2 are equal, $\mu_1 = \mu_2$, if $\mu_1\bar{a} = \mu_2\bar{a}$ for every vector $\bar{a}$ in P. The *product* of a p-th order tensor μ and a real number r and the *sum* of the p-th order tensors μ and ν are defined by the equations

$$(r\mu)\bar{v} = r(\mu\bar{v}); \quad (\mu + \nu)\bar{v} = \mu\bar{v} + \nu\bar{v}$$

respectively.

If we apply the p-th order tensor μ to a vector $\bar{v}_1$, we get the $(p-1)$-th order tensor $\mu\bar{v}_1$; let us apply the latter tensor to a vector $\bar{v}_2$, we get the $(p-2)$-th order tensor $(\mu\bar{v}_1)\bar{v}_2 = \mu\bar{v}_1\bar{v}_2$; continuing this procedure, after the p-th step we get the vector

$$\bar{u} = \mu\bar{v}_1\bar{v}_2\cdots\bar{v}_p.$$

Hence, by a p-th order tensor to every sequence of p vectors there is made to correspond a well defined vector. Clearly, this relation is linear homogeneous in each vector argument, i. e.

$$\mu \bar{v}_1 \cdots \bar{v}_{i-1}(r\bar{v}_{i_1} + s\bar{v}_{i_2})\bar{v}_{i+1} \cdots \bar{v}_p =$$
$$= r\mu \bar{v}_1 \cdots \bar{v}_{i-1}\bar{v}_{i_1}\bar{v}_{i+1} \cdots \bar{v}_p + s\mu \bar{v}_1 \cdots \bar{v}_{i-1}\bar{v}_{i_2}\bar{v}_{i+1} \cdots \bar{v}_p \qquad (i = 1, \ldots, p).$$

It can easily be seen that the definition of the p-th order tensor given above is equivalent to the following one: μ is called a p-th order tensor of L in the point P, if for every sequence $\bar{v}_1, \ldots, \bar{v}_p$ of p vectors in P the vector $\bar{u} = \mu \bar{v}_1 \cdots \bar{v}_p$ is defined and is a linear homogeneous function of each vector argument.

It is also easy to see that $\nu = \mu \bar{v}_1 \cdots \bar{v}_k$ $(k < p)$ is a tensor of order $p - k$.

The *product* $\varrho = \mu\nu$ of a p-th order tensor μ and a q-th order tensor ν is defined as follows: it is made to correspond by ϱ to the vectors $\bar{u}_1, \ldots, \bar{u}_{p+q-1}$ the vector $\bar{v} = \mu(\nu \bar{u}_1 \cdots \bar{u}_q)\bar{u}_{q+1} \cdots \bar{u}_{q+p-1}$, i. e.

$$(\mu\nu)\bar{u}_1 \cdots \bar{u}_{p+q-1} = \mu(\nu \bar{u}_1 \cdots \bar{u}_q)\bar{u}_{q+1} \cdots \bar{u}_{q+p-1}.$$

Clearly $\varrho = \mu\nu$ is a tensor of order $p + q - 1$.

The product of the tensors $\mu_1, \ldots, \mu_m$ is similarly defined. If the order of the tensor μ_i is p_i $(i = 1, \ldots, m)$ and $p = \sum_{i=1}^{m} p_i - (m - 1)$, then the product $\prod_{i=1}^{m} \mu_i$ of these tensors is a tensor of order p for which the equation

$$\bar{v} = \left(\prod_{i=1}^{m} \mu_i\right)\bar{u}_1 \cdots \bar{u}_p =$$
$$= \mu_1(\mu_2 \cdots (\mu_m \bar{u}_1 \cdots \bar{u}_{p_m})\bar{u}_{p_m+1} \cdots)\bar{u}_{p-p_1+2} \cdots \bar{u}_p$$

holds for every sequence of p vectors $\bar{u}_1, \ldots, \bar{u}_p$.

It can be demonstrated as usual that the multiplication of tensors is an operation distributive and associative, i. e.

$$\nu(\mu_1 + \mu_2) = \nu\mu_1 + \nu\mu_2; \quad (\mu_1 + \mu_2)\nu = \mu_1\nu + \mu_2\nu,$$

where the tensor ν is of order p, and the tensors μ_1 and μ_2 are of order q,

$$\nu_1(\nu_2\nu_3) = (\nu_1\nu_2)\nu_3 = \nu_1\nu_2\nu_3,$$

where the tensor ν_i is of order p_i $(i = 1, 2, 3)$.

The p-th order tensor $k_{ij}\mu$ $(i < j \leqq p)$ is called an *transposed tensor* of the p-th order tensor μ, if for every sequence of p vectors $\bar{u}_1, \ldots, \bar{u}_p$ the equation

$$(k_{ij}\mu)\bar{u}_1 \cdots \bar{u}_i \cdots \bar{u}_j \cdots \bar{u}_p = \mu \bar{u}_1 \cdots \bar{u}_{i-1}\bar{u}_j\bar{u}_{i+1} \cdots \bar{u}_{j-1}\bar{u}_i\bar{u}_{j+1} \cdots \bar{u}_p$$

holds. Clearly, k_{ij} is a linear homogeneous operator on tensors, i. e. if μ and ν are tensors of order p, and r and s are real numbers, the equation

$k_{ij}(r\mu+s\nu)=rk_{ij}\mu+sk_{ij}\nu$ holds. The operators k_{12} and $k_{p-1\,p}$, which occur several times, we denote briefly with k^* and k respectively.

If $k_{ij}\mu=\mu$, the tensor μ is called *symmetric* in its i-th and j-th argument. If $k_{ij}\mu=-\mu$, the tensor μ is called *skew-symmetric* in its i-th and j-th argumentum.

Up to this point we dealt with operations on vectors and tensors defined in a single point P of the space L. Now, let us see the most important definitions in tensor analysis.

Consider the vectors $\bar{v}_{P_1}, \bar{v}_{P_2}, \dots, \bar{v}_{P_k}, \dots$ given in the points $P_1, P_2, \dots, P_k, \dots$ of the space L respectively, and let $\lim\limits_{k\to\infty} P_k=P$. If the sequence $\bar{v}_{P_k}$ is convergent as a vector sequence in the space A and $\lim\limits_{k\to\infty}\bar{v}_{P_k}=\bar{v}$, then the sequence $\bar{v}_{P_k}$ is said to be *convergent* in the space L, and the vector $\bar{v}$ of the L space in the point P is called the *limit vector* of the sequence.

If in each point P of L there is given a vector $\bar{v}(P)$, we say that a *vector field* is defined in the points of L. The vector field $\bar{v}(P)$ is said to be *continuous* in P, if for every sequence of points P_k convergent to P

$$\lim_{P_k\to P}\bar{v}(P_k)=\bar{v}(P)$$

holds.

Consider the p-th order tensors $\mu_{P_1}, \mu_{P_2}, \dots, \mu_{P_k}, \dots$ given in the points $P_1, P_2, \dots, P_k, \dots$ of the space L respectively, and let $\lim\limits_{k\to\infty} P_k=P$. The sequence μ_{P_k} is called to be *convergent,* and the p-th order tensor μ_P of L in the point P is called the *limit tensor* of the sequence, if for every convergent sequence of p vectors $\bar{u}_{1P_k}, \bar{u}_{2P_k}, \dots, \bar{u}_{pP_k}$ ($\bar{u}_{iP_k}$ is defined in the point P_k and $\lim\limits_{P_k\to P}\bar{u}_{iP_k}=\bar{u}_{iP}$, where $\bar{u}_{iP}$ is a vector in the point P, $(i=1,2,\dots,p)$) the vector sequence $\bar{v}_{P_k}=\mu_{P_k}\bar{u}_{1P_k}\cdots\bar{u}_{pP_k}$ is convergent and $\lim\limits_{P_k\to P}\bar{v}_{P_k}=$ $=\mu_P\bar{u}_{1P}\cdots\bar{u}_{pP}=\bar{v}_P$.

If in each point of L there is given a p-th order tensor $\mu(P)$, we speak of a *tensor field* defined in points of L. The tensor field $\mu(P)$ is called to be *continuous* in P, if for every sequence of points P_k convergent to P

$$\lim_{P_k\to P}\mu(P_k)=\mu(P)$$

holds.

We say that the vector field $\bar{v}(P)$ is *differentiable,* if it is differentiable as a vector field in the space A, i. e. if to the vector field $\bar{v}(f^{-1}(P'))$ a first order tensor field $\frac{d\bar{v}}{dP}$ *of the space* A can be found, such that

$$\Delta\bar{v}=v(f^{-1}(P'+\overline{\Delta P}))-\bar{v}(f^{-1}(P'))=\frac{d\bar{v}}{dP}\overline{\Delta P}+\varepsilon(\overline{\Delta P})\overline{\Delta P}, \tag{1}$$

where the tensor $\varepsilon(\overline{\Delta P})$ of the space A tends to zero if $\overline{\Delta P}\to 0$. $\dfrac{d\bar{v}}{dP}$ is called the *derivative* of $\bar{v}$.

Note that a tensor of L can be considered also as a tensor of A, but the inverse statement does not hold. A linear homogeneous vector function defined on the vectors of the space A, that is a tensor of A, can be considered as a tensor of the space L if and only if it assigns to vectors of the point P' of A vectors of the same point P'.

We say that the p-th order tensor field $\mu(P)$ is *differentiable,* if it is differentiable as a tensor field in the space A, i. e. if a $(p+1)$-th order tensor field $\dfrac{d\mu}{dP}$ *of the space* A can be found, such that

$$\Delta\mu=\mu(f^{-1}(P'+\overline{\Delta P}))-\mu(f^{-1}(P'))=\frac{d\mu}{dP}\overline{\Delta P}+\varepsilon(\overline{\Delta P})\overline{\Delta P} \tag{2}$$

where the tensor $\varepsilon(\overline{\Delta P})$ of the space A tends to zero, if $\overline{\Delta P}\to 0$. $\dfrac{d\mu}{dP}$ is called the *derivative* of μ.

§ 3. Parallel displacement; the definition of the affinely connected space

Let be given in every point P of the space L a second order, linear homogeneous operator $\gamma(P)$ associating to every pair of vectors of L in P a vector of the space A. Of course, $\gamma(P)$ can be considered, just like the tensor fields of L, as a tensor field of A. $\gamma(P)$ is assumed to be arbitrary many times differentiable, i. e. we suppose that a third order tensor $\dfrac{d\gamma}{dP}$ of the space A can be found for which the equation

$$\Delta\gamma=\gamma(f^{-1}(P'+\overline{\Delta P}))-\gamma(f^{-1}(P'))=\frac{d\gamma}{dP}\overline{\Delta P}+\varepsilon(\overline{\Delta P})\overline{\Delta P} \tag{3}$$

holds, where $\varepsilon(\overline{\Delta P})\to 0$, if $\overline{\Delta P}\to 0$, etc.

Definition. We say that the vector $\bar{v}$ of the space L in the point P is obtained from the vector $\bar{v}_0$ of the space L in the point P_0 by *γ-parallel displacement*, if the difference of the vectors $\bar{v}$ and $\bar{v}_0$ considered as vectors of A can be expressed by the formula

$$\Delta\bar{v}=\bar{v}-\bar{v}_0=-\gamma(P_0)\bar{v}_0\overline{\Delta P}+\varepsilon(\overline{\Delta P})\bar{v}_0\overline{\Delta P}, \tag{4}$$

where the vector $\overline{\Delta P}=\overrightarrow{P_0'P'}$ is considered as a vector of L in the point P_0, $\varepsilon(\overline{\Delta P})$ is a second order tensor of A, and $\varepsilon(\overline{\Delta P})\to 0$, if $\overline{\Delta P}\to 0$. Of course (4) is not a vector of the space L.

Consider a curve g in L. Let the equation of the curve g be $P=P(t)$ and $P_0=P(t_0)$ be a point of the curve. We shall define in accordance with the preceding definition the parallel displacement of a vector along the curve g.

Definition. We say that the vector field $\bar{v}(P(t))=\bar{v}(t)$ defined in the points of the curve g of L is obtained from the vector $\bar{v}_0$ of the point P_0 by parallel *displacement along the curve g*, if the equations

$$\frac{d\bar{v}}{dt}=-\gamma(P)\bar{v}\,\overline{\frac{dP}{dt}},\quad \bar{v}(P(t_0))=\bar{v}_0$$

hold, where $\frac{d\bar{v}}{dt}$ is the derivative of the vector field $\bar{v}(t)$ considered in A with respect to the parameter t.

Note that $\frac{d\bar{v}}{dt}$ cannot be considered as a vector of L, since

$$\frac{d\bar{v}}{dt}=\lim_{\Delta t\to 0}\frac{\Delta\bar{v}}{\Delta t}=\lim_{\Delta t\to 0}\frac{\bar{v}(P(t+\Delta t))-\bar{v}(P(t))}{\Delta t},$$

i. e. in forming $\frac{d\bar{v}}{dt}$ there are subtracted vectors of different points. This cannot be done in L.

It is easy to show that the definitions (4) and (5) are equivalent. However (4) can be applied only to point sequences tending to P_0, for about ε we know nothing except that it tends to zero in case of such a point sequence. It is seen from (5) that γ-parallel displacement in L depends on the path along which it is done,

Theorem. *The γ-parallel displacement along a given curve g is uniquely determined.*

Proof. Let us suppose that the equations (5) are satisfied by the vector fields $\bar{v}_1$ and $\bar{v}_2$ defined in points of g:

$$\frac{d\bar{v}_1}{dt}=-\gamma\bar{v}_1\overline{\frac{dP}{dt}},\quad \frac{d\bar{v}_2}{dt}=-\gamma\bar{v}_2\overline{\frac{dP}{dt}}$$

$$\bar{v}_1(t_0)=\bar{v}_2(t_0)=\bar{v}_0.$$

Hence by subtracting the second equation from the first we get the equation

$$\frac{d(\bar{v}_1-\bar{v}_2)}{dt}=-\gamma(\bar{v}_1-\bar{v}_2)\overline{\frac{dP}{dt}}. \tag{6}$$

Since $\bar{v}_1(t_0)-\bar{v}_2(t_0)=0$, from equation (6) and from its successive derivatives there follows that all derivatives of the vector field $\bar{v}_1(t)-\bar{v}_2(t)$ are equal to zero in the point t_0. Thus, at least in some neighbourhood of the point, t_0 the equation $\bar{v}_1(t)-\bar{v}_2(t)\equiv 0$ holds, which proves our statement.

An immediate consequence of the foregoing result is the following

Theorem. *Let P_0 and P_1 be two points of the curve g, and denote by $\bar{v}_1$ the vector arising in the point P_1 by γ-parallel displacement of the vector $\bar{v}_0$ of the point P_0 along g. Displacing the vector $\bar{v}_1$ γ-parallelly along g from P_1 to P_0 we obtain in P_0 the vector $\bar{v}_0$.*

It will be demonstrated, that γ-parallel displacement conserves the linear dependence of vectors, i. e. the following theorem is valid.

Theorem. *Suppose that among the vectors $\bar{v}_{10}, \bar{v}_{20}, \dots, \bar{v}_{m0}$ given in the point P_0 of the curve g the linear relation*

$$\sum_{i=1}^{m} a_i \bar{v}_{i0} = 0 \tag{7}$$

holds. Displace the vectors $\bar{v}_{i0}$ γ-parallelly along the curve g and denote the vector fields obtained in this manner by $\bar{v}_i(t)$ $(i=1, \dots, m)$. Then, at least in some neighbourhood of t_0 there holds the relation

$$\sum_{i=1}^{m} a_i \bar{v}_i(t) = 0.$$

Proof. From equations

$$\frac{d\bar{v}_i}{dt} = -\gamma \bar{v}_i \frac{\overline{dP}}{dt}, \quad \bar{v}_i(t_0) = \bar{v}_{i0}, \qquad (i=1, \dots, m)$$

it follows, that

$$\frac{d}{dt}\left(\sum_{i=1}^{m} a_i \bar{v}_i\right) = \sum_{i=1}^{m} a_i \frac{d\bar{v}_i}{dt} = \sum_{i=1}^{m} a_i \left(-\gamma \bar{v}_i \frac{\overline{dP}}{dt}\right) = -\gamma \left(\sum_{i=1}^{m} a_i \bar{v}_i\right) \frac{\overline{dP}}{dt}.$$

Hence

$$\left[\frac{d}{dt} \sum_{i=1}^{m} a_i \bar{v}_i\right]_{t=t_0} = 0.$$

From the foregoing relation it follows, that all the derivatives of $\sum_{i=1}^{m} a_i \bar{v}_i(t)$ are equal to zero in t_0. Hence we get immediately the theorem which was to be proved.

Definition. The space L in which we defined the parallel displacement in the foregoing manner is called an *affinely connected space of n dimensions*. γ is called the *operator of connection*.

Let $\bar{v}(P)$ be a differentiable vector field of L. Displace γ-parallelly the vector $\bar{v}(P)$ from the point P to the "neighbouring" point P_0 and denote by $\bar{u}$ the vector obtained in such manner in the point P_0. First, we shall determine the difference of the vectors $\bar{u}$ and $\bar{v}(P_0)$ of the point P_0. From equation (4) it follows, that

$$\bar{u}-\bar{v}(P)=-\gamma(P)\bar{v}(P)(-\overline{\Delta P})+\varepsilon(\overline{\Delta P})\overline{\Delta P},$$

where $\overline{\Delta P}=\overrightarrow{P_0'P'}$ (' denotes everywhere the image in the space A of the object in question of the space L). Hence

$$\bar{u}=\bar{v}(P)+\gamma(P)\bar{v}(P)\overline{\Delta P}+\varepsilon(\overline{\Delta P})\bar{v}\overline{\Delta P}=$$
$$=\bar{v}(P_0)+\left.\frac{d\bar{v}}{dP}\right|_{P_0}\overline{\Delta P}+\gamma(P_0)\bar{v}(P_0)\overline{\Delta P}+\varepsilon_1(\overline{\Delta P})\overline{\Delta P},$$

where the first order linear operator $\varepsilon_1(\overline{\Delta P})$ tends to zero, if $\overline{\Delta P}\to 0$. Thus

$$\bar{u}-\bar{v}(P_0)=\left[\left.\frac{d\bar{v}}{dP}\right|_{P_0}+\gamma(P_0)\bar{v}(P_0)\right]\overline{\Delta P}+\varepsilon_1(\overline{\Delta P})\overline{\Delta P}.$$

If the points P_0 and P are connected by a curve $P=P(t)$, $P_0=P(t_0)$, and we divide the foregoing equation by $\Delta t=t-t_0$, and the point P tends to P_0 along the curve $P=P(t)$, we get the relation

$$\lim_{\Delta t\to 0}\frac{\bar{u}-\bar{v}(P)}{\Delta t}=\left[\frac{d\bar{v}}{dP}+\gamma\bar{v}\right]_{P_0}\left.\frac{\overline{dP}}{dt}\right|_{t_0}. \tag{8}$$

Since $\frac{\overline{dP}}{dt}$ can be any vector of L in the point P_0, and the left side of equation (8) is obviously a vector of L in P_0, the first order linear homogeneous operator

$$\nabla\bar{v}=\frac{d\bar{v}}{dP}+\gamma\bar{v} \tag{9}$$

is a first order tensor of L in the point P_0. The tensor $\nabla\bar{v}$ is called the *absolute derivative* of the vector field $\bar{v}$. The vector

$$D\bar{v}=\nabla\bar{v}\overline{\Delta P} \tag{10}$$

of the space L in the point P_0 is called the *absolute differential* of $\bar{v}$. On the basis of the foregoing results we may say, that $D\bar{v}$ is the difference of the vector obtained by γ-parallel displacement of the vector $\bar{v}(P)$ to the point P_0 and of the vector $\bar{v}(P_0)$, if the second order terms are neglected. In general a linear homogeneous function of vectors is said to be of order m in $\overline{\Delta P}$, if m quantities (vectors or tensors) tend to zero in it, if $\overline{\Delta P}\to 0$.

Clearly, a vector field $\bar{v}(P)$ is γ-parallelly displaced along a curve $P=P(t)$ if and only if

$$\nabla\bar{v}\frac{\overline{dP}}{dt}=0. \tag{11}$$

§ 4. Geodesics

Definition. We say that the *direction* of the vector field $\bar{v}(t)$ defined in the points of the curve $P=P(t)$ is γ-parallelly displaced along the curve, if a function $c(t)$ twice continuously differentiable can be found which is nowhere equal to zero, and for which the vector field $c(t)\bar{v}(t)$ is γ-parallelly displaced along the given curve, i. e.

$$\frac{d[c(t)\bar{v}(t)]}{dt}=\gamma c(t)\bar{v}(t)\frac{\overline{dP}}{dt}=0. \tag{12}$$

From equation (12) there follows

$$c(t)\left[\frac{d\bar{v}}{dt}+\gamma\bar{v}(t)\frac{\overline{dP}}{dt}\right]=-c'(t)\bar{v}(t),$$

or, introducing the notation $a(t)=-\frac{c'(t)}{c(t)}$,

$$\frac{d\bar{v}}{dt}+\gamma\bar{v}(t)\frac{\overline{dP}}{dt}=a(t)\bar{v}(t). \tag{13}$$

Let the direction of $v(t)$ be γ-parallelly displaced along the curve $P=P(t)$. From equations (8) and (13) there follows, that γ-parallelly displacing the vector $\bar{v}(t)$ from the point $P=P(t)$ to the "neighbouring" point $P_0=P(t_0)$, the difference of the vector thus obtained and of the vector $\bar{v}(t_0)$ is a vector directed in the direction of $\bar{v}(t_0)$ (the second order terms neglected).

Definition. The curve g of the space L is called a *geodesic* of the space, if the direction of the tangent vector of the curve is γ-parallelly displaced along the curve.

If $P=P(t)$ is the equation of the geodesic g, then it follows from the definition that for the vector $\bar{v}(t)=\frac{\overline{dP}}{dt}$ the equation (13) holds, which now takes the form

$$\frac{\overline{d^2P}}{dt^2}+\gamma\frac{\overline{dP}}{dt}\frac{\overline{dP}}{dt}=a(t)\frac{\overline{dP}}{dt}. \tag{14}$$

This is the differential equation of geodesic lines given in general parameters.

We introduce a new parameter s on the geodesic line g, which satisfies the equation

$$\frac{ds}{dt}a(t)-\frac{d^2s}{dt^2}=0. \tag{15}$$

Clearly such a parameter s can always be found. Expressing the derivatives of $P = P(t)$ with respect to t by derivatives with respect to s in equation (14), and making use of equation (15) we get the differential equation of the geodesics in the form

$$\overline{\frac{d^2P}{ds^2}} + \gamma \overline{\frac{dP}{ds}}\, \overline{\frac{dP}{ds}} = 0. \tag{16}$$

Hence it follows that the vector $\overline{\frac{dP}{ds}}$ is γ-parallelly displaced along the geodesic. The parameter s in which the tangent vector possesses the foregoing property is called a *canonic parameter.* It is easy to show, that if s is a canonic parameter on a geodesic, then $r = as + b$ is also a canonic parameter (a and b are constant numbers).

§ 5. The tensor of torsion

Let P_0', P_1', P_2', P_3' be the vertices of a "small" parallelogram in the space A (the vertex opposite to P_0' is denoted by P_3'). This parallelogram can be derived by moving the edge vector $\overrightarrow{P_0'P_1'} = \overline{\Delta_1 P}$ to the point P_2' and the edge vector $\overrightarrow{P_0'P_2'} = \overline{\Delta_2 P}$ to the point P_1'. Denote the image of the point P_i' in L by P_i ($i = 0, 1, 2, 3$). If we displace γ-parallelly the vectors $\overline{\Delta_1 P}$ and $\overline{\Delta_2 P}$ (considered as vectors of L in P_0) to the points P_2 and P_1 respectively, we get the vectors $\overline{\Delta_1 P} - \gamma(P_0)\overline{\Delta_1 P}\,\overline{\Delta_2 P} + \varepsilon \overline{\Delta_1 P}\,\overline{\Delta_2 P}$ and $\overline{\Delta_2 P} - \gamma(P_0)\overline{\Delta_2 P}\,\overline{\Delta_1 P} + \varepsilon \overline{\Delta_2 P}\,\overline{\Delta_1 P}$. Let us consider again what does this process yield in the space A.

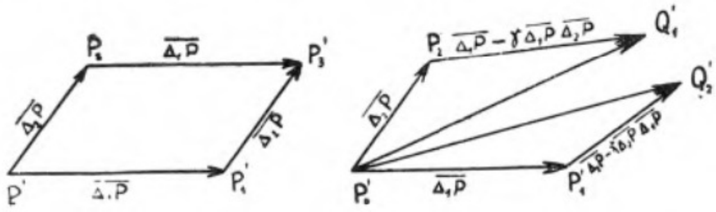

Fig. 1

The third order terms will be neglected. The vectors

$$\overrightarrow{P_0'Q_1'} = \overline{\Delta_2 P} + (\overline{\Delta_1 P} - \gamma \overline{\Delta_1 P}\,\overline{\Delta_2 P})$$

and

$$\overrightarrow{P_0'Q_2'} = \overline{\Delta_1 P} + (\overline{\Delta_2 P} - \gamma \overline{\Delta_2 P}\,\overline{\Delta_1 P})$$

will not be equal, that is the point Q_1' does not coincide with the point Q_2', since

$$\overrightarrow{Q_1'Q_2'} = \overrightarrow{P_0'Q_2'} - \overrightarrow{P_0'Q_1'} = \overline{\Delta_1 P} + \overline{\Delta_2 P} - \gamma \overline{\Delta_2 P}\,\overline{\Delta_1 P} -$$
$$- \overline{\Delta_2 P} - \overline{\Delta_1 P} + \gamma \overline{\Delta_1 P}\,\overline{\Delta_2 P} = (\gamma - k\gamma)\overline{\Delta_1 P}\,\overline{\Delta_2 P}.$$

Thus the parallelogram "opened". See Fig. 1. The vectors $\overrightarrow{P_0'Q_2'}$ and $\overrightarrow{P_0'Q_1'}$ can be considered as vectors of L in the point P_0. Their difference, which is a linear homogeneous function of the vectors $\overline{\Delta_1 P}, \overline{\Delta_2 P}$, is also a vector of L in P_0. Thus the operator $\tau = \gamma - k\gamma$ is a second order tensor of the space L, and it is called the *tensor of torsion* of L. The torsion shows in a certain manner the difference between the geometry of the space A and that of the space L. If the operator of connection γ is symmetric, $\tau = 0$, and the vector $\overrightarrow{Q_1'Q_2'}$ is at most a third order small quantity. In this case the space L is called *torsion-free*. If L is torsion-free, the departure of its geometry from the geometry of the space A is "in small" less striking, than in case of spaces with torsion different from zero.

Here we note, that if γ is symmetric, its derivative $\frac{d\gamma}{dP}$ is also symmetric in its last two arguments. So, if $\overline{\Delta P} = \Delta t \overline{a}$ is substituted in the relation (3), where $\overline{a}$ is a fixed vector and Δt a scalar number, division by Δt yields the equation

$$\frac{\gamma(f^{-1}(P' + \Delta t\overline{a})) - \gamma(f^{-1}(P'))}{\Delta t} = \frac{d\gamma}{dP}\overline{a} + \varepsilon(\Delta t).$$

Hence we get

$$\lim_{\Delta t \to 0} \frac{\gamma(f^{-1}(P' + \Delta t\overline{a})) - \gamma(f^{-1}(P'))}{\Delta t} = \frac{d\gamma}{dP}\overline{a}.$$

Since $\gamma(f^{-1}(P'))$ and $\gamma(f^{-1}(P' + \Delta t\overline{a}))$ are symmetric, $\frac{d\gamma}{dP}\overline{a}$ is also symmetric for every $\overline{a}$, which proves our statement.

§ 6. The curvature tensor

Let $P = P(t)$ be a curve of the space L, and $P_0 = P(t_0)$, $P_1 = P(t_1)$ two points of the curve. We suppose that P_0 and P_1 are "neighbouring" points (by that we mean, that $t_1 - t_0$ is small enough). Let $\overline{v}_0$ be a vector of the space L in the point P_0, and let us displace $\overline{v}_0$ γ-parallelly along the curve $P = P(t)$ to the point P_1. The vector obtained in such way in P_1 is denoted by $\overline{v}_1$. We determine the change $\Delta\overline{v} = \overline{v}_1 - \overline{v}_0$ of the vector $\overline{v}_0$ by this parallel displacement taking into account the *second order terms* too.

The vector field obtained by γ-parallel displacement of $\overline{v}_0$ along the curve $P = P(t)$ is denoted by $\overline{v}(t)$; $\overline{v}(t_0) = \overline{v}_0$, $\overline{v}(t_1) = \overline{v}_1$. The vector $\overline{v}(t)$ satisfies equation (5), i. e.

$$\frac{d\overline{v}}{dt} = -\gamma(P(t))\overline{v}(t)\frac{\overline{dP}}{dt},$$

hence

$$\Delta\bar{v}=\bar{v}_1-\bar{v}_0=\int_{t_0}^{t_1}\frac{d\bar{v}}{dt}dt=-\gamma(P(t))\bar{v}(t)\int_{t_0}^{t_1}\frac{\overline{dP}}{dt}dt.$$

For determining the integral taking into account also the second order terms, it is enough to determine the integrand taking into account only the first order terms. From equations (4) and (3) it follows, that within first order terms

$$\bar{v}(t)=\bar{v}_0-\gamma(P_0)\bar{v}_0\overline{\Delta P},$$

$$\gamma(t)=\gamma(P_0)+\frac{d\gamma}{dP}\bigg|_{P_0}\overline{\Delta P},$$

where $\overline{\Delta P}=\overrightarrow{P_0'P'}(t)$ ('denotes the image of the respective point in A). Substituting these expressions we get $\Delta\bar{v}$ in the form

$$\Delta\bar{v}=-\int_{t_0}^{t_1}\left[\gamma(P_0)+\frac{d\gamma}{dP}\bigg|_{P_0}\overline{\Delta P}\right][\bar{v}_0-\gamma(P_0)\,\bar{v}_0\overline{\Delta P}]\,\frac{\overline{dP}}{dt}\,dt=$$

$$=-\int_{t_0}^{t_1}\gamma(P_0)\bar{v}_0\frac{\overline{dP}}{dt}dt+\int_{t_0}^{t_1}\left[\gamma(P_0)\gamma(P_0)\bar{v}_0\overline{\Delta P}\frac{\overline{dP}}{dt}-\right.$$

$$\left.-\frac{d\gamma}{dP}\bigg|_{P_0}\overline{\Delta P}\bar{v}_0\frac{\overline{dP}}{dt}\right]dt+\int_{t_0}^{t_1}\frac{d\gamma}{dP}\bigg|_{P_0}\overline{\Delta P}\,\gamma(P_0)\bar{v}_0\overline{\Delta P}\frac{\overline{dP}}{dt}\,dt.$$

Here the last term is a third order small quantity, since its integrand is of second order. Thus this term can be neglected and we can write the preceding expression in the form

$$\Delta\bar{v}=-\gamma(P_0)\bar{v}_0\overrightarrow{P_0'P_1'}+$$
$$+\int_{t_0}^{t_1}\left[\gamma(P_0)\gamma(P_0)-k^*\frac{d\gamma}{dP}\bigg|_{P_0}\right]\bar{v}_0\overline{\Delta P}\frac{\overline{dP}}{dt}\,dt. \tag{17}$$

Suppose now, that the curve $P=P(t)$ is closed and $P_0=P(t_0)=P_1=$ $=P(t_1)$. Equation (17) now yields the change of the vector $\bar{v}_0$ in second order terms if we displace parallelly the vector $\bar{v}_0$ round along the curve $P=P(t)$ till we get back to the point P_0. In this case both $\bar{v}_0$ and $\bar{v}_1$ are vectors of the point P_0, thus their difference $\Delta\bar{v}$ is also a vector of L in the point P_0. Since from $P_0 = P_1$ it follows that the first term on the right side of equation (17) is zero, we get

$$\Delta\bar{v} = \int_{t_0}^{t_1} \left[\gamma(P_0)\gamma(P_0) - k^* \frac{d\gamma}{dP}\bigg|_{P_0}\right] \bar{v}_0 \overline{\Delta P}\, \frac{\overline{dP}}{dt}\, dt. \tag{18}$$

If $P = P(u, v)$ is the equation of a two dimensional surface of L containing the curve $P = P(t)$, the equation

$$\frac{\overline{dP}}{dt} = \frac{\overline{\partial P}}{\partial u}\dot{u} + \frac{\overline{\partial P}}{\partial v}\dot{v}$$

holds, and equation (18) takes the form

$$\Delta\bar{v} = \int_{t_0}^{t_1} \left\{ \left[\gamma(P_0)\gamma(P_0) - k^* \frac{d\gamma}{dP}\bigg|_{P_0}\right] \bar{v}_0 \overline{\Delta P}\, \frac{\overline{\partial P}}{\partial u}\dot{u} + \right.$$

$$\left. + \left[\gamma(P_0)\gamma(P_0) - k^* \frac{d\gamma}{dP}\bigg|_{P_0}\right] \bar{v}_0 \overline{\Delta P}\, \frac{\overline{\partial P}}{\partial v}\dot{v} \right\} dt =$$

$$= \oint \left\{ \left[\gamma(P_0)\gamma(P_0) - k^* \frac{d\gamma}{dP}\bigg|_{P_0}\right] \bar{v}_0 \overline{\Delta P}\, \frac{\overline{\partial P}}{\partial u}\, du + \right.$$

$$\left. + \left[\gamma(P_0)\gamma(P_0) - k^* \frac{d\gamma}{dP}\bigg|_{P_0}\right] \bar{v}_0 \overline{\Delta P}\, \frac{\overline{\partial P}}{\partial v}\, dv \right\}.$$

Using Green's formula we get

$$\Delta\bar{v} = \iint \left(\frac{\partial}{\partial u} \left\{ \left[\gamma(P_0)\gamma(P_0) - k^* \frac{d\gamma}{dP}\bigg|_{P_0}\right] \bar{v}_0 \overline{\Delta P}\, \frac{\overline{\partial P}}{\partial v} \right\} - \right.$$

$$\left. - \frac{\partial}{\partial v} \left\{ \left[\gamma(P_0)\gamma(P_0) - k^* \frac{d\gamma}{dP}\bigg|_{P_0}\right] \bar{v}_0 \overline{\Delta P}\, \frac{\overline{\partial P}}{\partial u} \right\} \right) du\, dv,$$

where the double integral must be calculated on the uv range corresponding to the surface bounded by the curve $P = P(t)$. From equations $\frac{\partial \overline{\Delta P}}{\partial u} = \frac{\overline{\partial P}}{\partial u}$, $\frac{\partial \overline{\Delta P}}{\partial v} = \frac{\overline{\partial P}}{\partial v}$ it follows, that

$$\Delta\bar{v} = \iint \left\{ \left[\gamma(P_0)\gamma(P_0) - k^* \frac{d\gamma}{dP}\bigg|_{P_0}\right] \bar{v}_0 \frac{\overline{\partial P}}{\partial u}\, \frac{\overline{\partial P}}{\partial v} - \right.$$

$$\left. - \left[\gamma(P_0)\gamma(P_0) - k^* \frac{d\gamma}{dP}\bigg|_{P_0}\right] \bar{v}_0 \frac{\overline{\partial P}}{\partial v}\, \frac{\overline{\partial P}}{\partial u} \right\} du\, dv.$$

In our second order accuracy we may replace here the integrand by its value in the point P_0, then we get

$$\Delta\bar{v}=\left\{\left[\gamma\gamma-k^*\frac{d\gamma}{dP}\right]\bar{v}_0\frac{\overline{\partial P}}{\partial u}\frac{\overline{\partial P}}{\partial v}-\left[\gamma\gamma-k^*\frac{d\gamma}{dP}\right]\bar{v}_0\frac{\overline{\partial P}}{\partial v}\frac{\overline{\partial P}}{\partial u}\right\}s,$$

where $s=\iint du\,dv$ is the area of the uv range corresponding to the surface bounded by the curve $P=P(t)$. The argument P_0 was already everywhere omitted. Finally, neglecting the terms which are of higher order in s, we may write the change of a vector displaced γ-parallelly along a closed curve in the form

$$\Delta\bar{v}=s\left[\gamma\gamma-k(\gamma\gamma)+kk^*\frac{d\gamma}{dP}-k^*\frac{d\gamma}{dP}\right]\bar{v}_0\frac{\overline{\partial P}}{\partial u}\frac{\overline{\partial P}}{\partial v}. \tag{19}$$

From this we get the exact equation

$$\lim_{s\to 0}\frac{\Delta\bar{v}}{s}=\left[\gamma\gamma-k(\gamma\gamma)+kk^*\frac{d\gamma}{dP}-k^*\frac{d\gamma}{dP}\right]\bar{v}_0\frac{\overline{\partial P}}{\partial u}\frac{\overline{\partial P}}{\partial v}.$$

Since $\bar{v}_0, \dfrac{\overline{\partial P}}{\partial u}, \dfrac{\overline{\partial P}}{\partial v}$ and $\lim\limits_{s\to 0}\dfrac{\Delta\bar{v}}{s}$ are vectors of L in P_0, the linear homogeneous operator

$$\varrho=\gamma\gamma-k(\gamma\gamma)+kk^*\frac{d\gamma}{dP}-k^*\frac{d\gamma}{dP} \tag{20}$$

is a third order tensor of the space L. ϱ is called the *curvature tensor* of the space L.

If $\gamma\equiv 0$, i. e. if L is the n dimensional affine space, then the tensor of torsion τ and the curvature tensor ϱ are identically equal to zero. The geometrical interpretation of the latter fact is obvious: if a vector is displaced parallelly along a closed curve in the affine space, its change is equal to zero. The more ϱ differs from the zero tensor the greater will be the change of the vector γ-parallelly displaced along a closed curve, i. e. the greater will be the departure of the geometry of the affinely connected space from the geometry of the affine space.

Two properties of the curvature tensor will be demonstrated. First, the curvature tensor is antisymmetric in its last two arguments,

$$k\varrho=-\varrho.$$

This follows at once from the equation (20) and from the known properties of the operator k. Secondly, if the tensor of torsion $\tau=0$ and $\bar{a}, \bar{b}$ and $\bar{c}$ are any three vectors of L in the same point, the equation

$$\varrho\bar{a}\bar{b}\bar{c}+\varrho\bar{b}\bar{c}\bar{a}+\varrho\bar{c}\bar{a}\bar{b}=0 \tag{21}$$

holds. Equation (21) is called the *equation of Ricci.* In order to prove equation (21) we introduce the notation

$$\alpha = \gamma\gamma + kk^* \frac{d\gamma}{dP}. \tag{22}$$

From the symmetry of the operators γ and $\frac{d\gamma}{dP}$, and from the definition of the operators k and k^* it follows, that the third order operator α is symmetric in its first two arguments, i. e.

$$\alpha = k^* \alpha, \quad \text{or} \quad \alpha \bar{a} \bar{b} \bar{c} = \alpha \bar{b} \bar{a} \bar{c}.$$

By using notation (22) the curvature tensor can be written in the form

$$\varrho = \alpha - k\alpha.$$

From these remarks the equations

$$\varrho \bar{a} \bar{b} \bar{c} = \alpha \bar{a} \bar{b} \bar{c} - (k\alpha) \bar{a} \bar{b} \bar{c} = \alpha \bar{a} \bar{b} \bar{c} - \alpha \bar{a} \bar{c} \bar{b},$$

$$\varrho \bar{b} \bar{c} \bar{a} = \alpha \bar{b} \bar{c} \bar{a} - (k\alpha) \bar{b} \bar{c} \bar{a} = \alpha \bar{b} \bar{c} \bar{a} - \alpha \bar{a} \bar{b} \bar{c},$$

$$\varrho \bar{c} \bar{a} \bar{b} = \alpha \bar{c} \bar{a} \bar{b} - (k\alpha) \bar{c} \bar{a} \bar{b} = \alpha \bar{a} \bar{c} \bar{b} - \alpha \bar{b} \bar{c} \bar{a}$$

follow. Summing these equations we get (21).

Definition. If in the affinely connected space L the γ-parallel displacement does not depend on the path, we say that the space L possesses *absolute parallelism.*

For instance the affine space is a space of absolute parallelism.

Theorem. *The space L possesses absolute parallelism if and only if its curvature tensor is identically equal to zero:*

$$\varrho \equiv 0. \tag{23}$$

PROOF. First we demonstrate that the condition is necessary. Suppose that (23) is not true. In this case the space L has a point P_0 and a closed curve g passing through this point, so that γ-parallelly displacing a vector of P_0 along g, the change will be different from zero. From this it follows already that dividing the closed curve g by points P_0 and P into two parts g_1 and g_2, the γ-parallel displacement from P to P_0 along g_1 and g_2 respectively does not yield the same result.

Before demonstrating that the condition is sufficient, we demonstrate, that the equation

$$\nabla\left[(\nabla \bar{w}) \overline{\frac{\partial P}{\partial u}}\right] \overline{\frac{\partial P}{\partial v}} - \nabla\left[(\nabla \bar{w}) \overline{\frac{\partial P}{\partial v}}\right] \overline{\frac{\partial P}{\partial u}} = \varrho \bar{w} \overline{\frac{\partial P}{\partial u}} \overline{\frac{\partial P}{\partial v}} \tag{24}$$

holds, where $P=P(u,v)$ is a two dimensional surface and $\overline{w}$ is a vector field of L. For

$$(\nabla\overline{w})\frac{\overline{\partial P}}{\partial u}=\frac{\partial\overline{w}}{\partial u}+\gamma\overline{w}\frac{\overline{\partial P}}{\partial u},$$

and taking the absolute derivative of this vector field and applying it to the vector $\frac{\overline{\partial P}}{\partial v}$, we get the equation

$$\begin{aligned}\nabla\left[(\nabla\overline{w})\frac{\overline{\partial P}}{\partial u}\right]\frac{\overline{\partial P}}{\partial v}=\frac{\partial^2\overline{w}}{\partial u\,\partial v}+\frac{d\gamma}{dP}\frac{\overline{\partial P}}{\partial v}\overline{w}\frac{\overline{\partial P}}{\partial u}+\gamma\frac{\partial\overline{w}}{\partial v}\frac{\overline{\partial P}}{\partial u}+\\+\gamma\overline{w}\frac{\overline{\partial^2 P}}{\partial v\,\partial u}+\gamma\frac{\partial\overline{w}}{\partial u}\frac{\overline{\partial P}}{\partial v}+\gamma\gamma\overline{w}\frac{\overline{\partial P}}{\partial u}\frac{\overline{\partial P}}{\partial v}.\end{aligned}\tag{25}$$

Interchanging the letters u and v in equation (25) we get the equation

$$\begin{aligned}\nabla\left[(\nabla\overline{w})\frac{\overline{\partial P}}{\partial v}\right]\frac{\overline{\partial P}}{\partial u}=\frac{\partial^2\overline{w}}{\partial v\,\partial u}+\frac{d\gamma}{dP}\frac{\overline{\partial P}}{\partial u}\overline{w}\frac{\overline{\partial P}}{\partial v}+\gamma\frac{\partial\overline{w}}{\partial u}\frac{\overline{\partial P}}{\partial v}+\\+\gamma\overline{w}\frac{\overline{\partial^2 P}}{\partial u\,\partial v}+\gamma\frac{\partial\overline{w}}{\partial v}\frac{\overline{\partial P}}{\partial u}+\gamma\gamma\overline{w}\frac{\overline{\partial P}}{\partial v}\frac{\overline{\partial P}}{\partial u}.\end{aligned}\tag{26}$$

Subtracting (26) from (25) and applying the operators k and k^* we get (24).

Now we demonstrate that condition (23) is sufficient. Let P_1 and P_2 be two arbitrary points of L and $\overline{w}(P_1)$ a vector in P_1. Let the points P_1 and P_2 be connected by the curves g_1 and g_2 and displace the vector $\overline{w}(P_1)$ parallelly along g_1 and g_2 respectively to P_2. It will be demonstrated, that the vectors $\overline{w}_1(P_2)$ and $\overline{w}_2(P_2)$ thus obtained in P_2 are equal.

Let g_1 and g_2 be members of a one parameter family of curves $P=P(t,c)$, connecting P_1 and P_2. Suppose that the equation of g_1 is $P=P(t,c_1)$ and that of g_2 is $P=P(t,c_2)$; suppose furthermore, that the equations

$$P(t_1,c)=P_1 \quad\text{and}\quad P(t_2,c)=P_2 \tag{27}$$

hold for each member of the family, that is for every c. We displace γ-parallelly the vector $\overline{w}(P_1)$ along each member of the family to P_2 and denote the vector of the point $P=P(t,c)$ obtained by γ-parallel displacement of $\overline{w}(P_1)$ along the curve $P=P(t,c)$ by $\overline{w}(t,c)$. In accordance with the foregoing notations we have $\overline{w}_1(P_2)=\overline{w}(t_2,c_1)$ and $\overline{w}_2(P_2)=\overline{w}(t_2,c_2)$. We write now for this case the equation (24):

$$\nabla\left[(\nabla\overline{w})\frac{\overline{\partial P}}{\partial c}\right]\frac{\overline{\partial P}}{\partial t}-\nabla\left[(\nabla\overline{w})\frac{\overline{\partial P}}{\partial t}\right]\frac{\overline{\partial P}}{\partial c}=\varrho\overline{w}\frac{\overline{\partial P}}{\partial c}\frac{\overline{\partial P}}{\partial t}.$$

From equation (23) and, since for c fixed $\overline{w}(t,c)$ is γ-parallelly displaced along $P(t, c)$, from equation (11) it follows that the right side and the second term on the left side of the last equation is equal to zero. Thus we get the equation

$$\nabla\left[(\nabla \overline{w})\,\overline{\frac{\partial P}{\partial c}}\right]\overline{\frac{\partial P}{\partial t}}=0.$$

This means, that $(\nabla \overline{w})\,\overline{\frac{\partial P}{\partial c}}$ is γ-parallelly displaced along each member of the family of curves. From equation (27)

$$\overline{\frac{\partial P(t_1, c)}{\partial c}}=\overline{\frac{\partial P(t_2, c)}{\partial c}}=0, \tag{28}$$

follows. Thus $(\nabla \overline{w})\,\overline{\frac{\partial P}{\partial c}}$ is equal to zero in the point P_1. Since γ-parallel displacement preserves linear relations among vectors, in every point $P=P(t, c)$ the equation

$$(\nabla \overline{w})\,\overline{\frac{\partial P}{\partial c}}=0$$

holds. Specially, this equation holds also in the point P_2, i. e.

$$(\nabla \overline{w})\,\overline{\frac{\partial P}{\partial c}}\bigg|_{P_2}=\frac{\partial \overline{w}(t_2, c)}{\partial c}+\gamma\overline{w}(t_2, c)\,\overline{\frac{\partial P(t_2, c)}{\partial c}}=0.$$

From this equation, in view of equation (28) we get

$$\frac{\partial \overline{w}(t_2, c)}{\partial c}=0.$$

Thus $\overline{w}(t_2, c)$ is constant, that is

$$\overline{w}_1(P_2)=\overline{w}(t_2, c_1)=\overline{w}(t_2, c_2)=\overline{w}_2(P_2).$$

This is just what was to be proved.

§ 7. Relations among affinely connected spaces

Let be introduced in the space L two operators of connection γ and γ^*. We demonstrate the following

Theorem. *The second order operator $\varkappa=\gamma^*-\gamma$ is a tensor of the space L.*

Proof. We displace γ-parallelly the vector $\overline{v}=\overline{v}(P_0)$ of L from the point P_0 to the point P. The change of the vector is

$$\Delta\overline{v}=-\gamma\overline{v}\,\overline{\Delta P}$$

where $\overline{\Delta P}=\overrightarrow{P_0'P'}$, and the second order terms on the right hand side are neglected. The vector $\bar{v}+\Delta\bar{v}$ will be displaced γ^*-parallelly from the point P back to the point P_0. Taking no account of the second order terms

$$\bar{v}+\Delta\bar{v}=\bar{v}-\gamma\bar{v}\overline{\Delta P}$$

and in view of

$$\gamma^*(P)=\gamma^*(P_0)+\frac{d\gamma^*}{dP}\overline{\Delta P},$$

the change of the vector will be

$$\begin{aligned}\Delta(\bar{v}+\Delta\bar{v})&=-\gamma^*(P)(\bar{v}+\Delta\bar{v})(-\overline{\Delta P})=\\&=\gamma^*\bar{v}\overline{\Delta P}+\gamma^*\Delta\bar{v}\overline{\Delta P}+\frac{d\gamma^*}{dP}\overline{\Delta P}\bar{v}\overline{\Delta P}+\frac{d\gamma^*}{dP}\overline{\Delta P}\Delta\bar{v}\overline{\Delta P}=\\&=\gamma^*\bar{v}\overline{\Delta P},\end{aligned}$$

where, again all the second or higher order terms are neglected. The difference between the vector $\bar{v}+\Delta\bar{v}+\Delta(\bar{v}+\Delta\bar{v})$ obtained in such way in the point P_0 and the original vector $\bar{v}(P_0)$, i. e. the change of the vector $\bar{v}$ during the whole process is

$$[\bar{v}+\Delta\bar{v}+\Delta(\bar{v}+\Delta\bar{v})]-\bar{v}=\Delta\bar{v}+\gamma^*\bar{v}\overline{\Delta P}=(\gamma^*-\gamma)\bar{v}\overline{\Delta P}. \qquad (29)$$

This equation is not exact, since the second order terms do not occur on the right hand side. Let the points P_0 and P be connected by a curve $P=P(t)$ and let $P_0=P(t_0)$, $P=P(t_0+\Delta t)$. Let us divide equation (29) by Δt, and let the point P tend to P_0 along the curve $P=P(t)$. Since the second order terms, which do not occur in equation (29), tend to zero even when divided by Δt, we get the exact equation

$$\lim_{\Delta t\to 0}\frac{\bar{v}+\Delta\bar{v}+\Delta(\bar{v}+\Delta\bar{v})-\bar{v}}{\Delta t}=(\gamma^*-\gamma)\bar{v}\,\frac{\overline{dP}}{dt}. \qquad (30)$$

Since the limes on the left hand side of equation (30) is obviously a vector of L in P_0 and $\bar{v}$ and $\dfrac{\overline{dP}}{dt}$ can be arbitrary vectors of L in P_0, we get that the second order linear homogeneous operator $\varkappa=\gamma^*-\gamma$ is a tensor of L. This is just what was to be proved.

Definition. If between two n dimensional affinely connected spaces there exists a one to one correspondence in which the images of geodesics are geodesics, then the two spaces are said to be *projectively equivalent.*

Suppose that in the space L there two operators of connection, γ and γ^*, are introduced. We determine on what condition the affinely connected space L_γ defined by γ is projectively equivalent with the affinely connected space L_{γ^*} defined by γ^*.

First we demonstrate the following simple lemma. If $\bar{v}$ is an arbitrary vector of L in the point P, there can be found a canonic parameter t on the geodesic of L_γ passing through P in direction of $\bar{v}$, such that $\frac{\overline{dP}}{dt}=\bar{v}$ in P. Indeed, if s is a canonic parameter along the geodesic passing through P in direction of $\bar{v}$, the equations

$$\bar{v}=a\frac{\overline{dP}}{ds}\bigg|_P; \quad \frac{\overline{d^2P}}{ds^2}+\gamma\frac{\overline{dP}}{ds}\frac{\overline{dP}}{ds}=0$$

hold. The parameter wanted is

$$t=\frac{s}{a}-\frac{c}{a},$$

where c is an arbitrary constant. Obviously t is a canonic parameter. If the substitution $s=at+c$ is performed in the function $P=P(s)$, we get

$$\frac{\overline{dP}}{dt}=\frac{\overline{dP}}{ds}\frac{ds}{dt}=\frac{\overline{dP}}{ds}a=\bar{v},$$

which was to be proved.

If $P=P(t)$ is an arbitrary geodesic in the space L_γ and t is a canonic parameter on it, the equation

$$\frac{\overline{d^2P}}{dt^2}=-\gamma\frac{\overline{dP}}{dt}\frac{\overline{dP}}{dt} \tag{31}$$

holds. The necessary and sufficient condition of $P=P(t)$ being a geodesic also in the space L_{γ^*} is

$$\frac{\overline{d^2P}}{dt^2}=-\gamma^*\frac{\overline{dP}}{dt}\frac{\overline{dP}}{dt}+h(t)\frac{\overline{dP}}{dt}. \tag{32}$$

Subtracting equation (32) from equation (31) we get

$$(\gamma^*-\gamma)\frac{\overline{dP}}{dt}\frac{\overline{dP}}{dt}=h(t)\frac{\overline{dP}}{dt}.$$

From the lemma it follows that an arbitrary vector $\bar{v}$ of L can be considered as a tangent vector $\frac{\overline{dP}}{dt}$ of a geodesic line of L_γ given in canonic parameter. Hence the affinely connected spaces L_γ and L_{γ^*} are projectively equivalent if and only if the tensor $\varkappa=\gamma^*-\gamma$ has the property that applied to an arbitrary vector $\bar{v}$ twice it yields the vector $\bar{v}$ multiplied by a scalar number.

Theorem. *A second order tensor $\varkappa$, applied to an arbitrary vector $\bar{v}$ twice yields this vector multiplied by a scalar number, i. e. the equation*

$$\varkappa \bar{v} \bar{v} = c \bar{v} \tag{33}$$

holds for every $\bar{v}$, if and only if the equation

$$(\varkappa + k\varkappa) \bar{u} \bar{v} = c_1 \bar{u} + c_2 \bar{v} \tag{34}$$

holds for every pair of vectors $\bar{u}$ and $\bar{v}$ (c_1 and c_2 are scalar numbers).

PROOF. Suppose that the equation (33) holds for any vector $\bar{v}$. Then there hold the following equations:

$$\begin{aligned}\varkappa \bar{u} \bar{v} &= \varkappa(\bar{u} - \bar{v} + \bar{v})(\bar{v} - \bar{u} + \bar{u}) = \varkappa(\bar{u} - \bar{v})(\bar{v} - \bar{u}) + \\ &+ \varkappa(\bar{u} - \bar{v})\bar{u} + \varkappa \bar{v}(\bar{v} - \bar{u}) + \varkappa \bar{v} \bar{u} = \\ &= -\varkappa(\bar{u} - \bar{v})(\bar{u} - \bar{v}) + \varkappa \bar{u} \bar{u} - \varkappa \bar{v} \bar{u} + \varkappa \bar{v} \bar{v} - \varkappa \bar{v} \bar{u} + \varkappa \bar{v} \bar{u} = \\ &= c_3(\bar{u} - \bar{v}) + c_4 \bar{u} - \varkappa \bar{v} \bar{u} + c_5 \bar{v},\end{aligned}$$

from which we get

$$(\varkappa + k\varkappa) \bar{u} \bar{v} = (c_3 + c_4) \bar{u} + (c_5 - c_3) \bar{v},$$

i. e. equation (34) holds. Inversely, suppose that equation (34) holds and let $\varkappa + k\varkappa$ be applied to the vector $\bar{v}$ twice:

$$\begin{aligned}(\varkappa + k\varkappa) \bar{v} \bar{v} &= c_1 \bar{v} + c_2 \bar{v} = 2c \bar{v} \\ \varkappa \bar{v} \bar{v} + k\varkappa \bar{v} \bar{v} &= 2c \bar{v} \\ 2\varkappa \bar{v} \bar{v} &= 2c \bar{v},\end{aligned}$$

from which equation (33) follows.

Since conditions (33) and (34) are equivalent we get as an immediate consequence of this fact the following

Theorem. *The affinely connected spaces L_γ and L_{γ^*} are projectively equivalent, if and only if the condition* (34) *holds for the tensor $\varkappa = \gamma^* - \gamma$ and for two arbitrary vectors $\bar{u}$ and $\bar{v}$.*

Let in the space L be introduced two operators of connection γ and γ^*. These operators define the affinely connected spaces L_γ and L_{γ^*} respectively. We determine on what condition the spaces L_γ and L_{γ^*} possess the following property: if $\bar{v}(t)$ is an arbitrary vector field defined in points of an arbitrary curve $P = P(t)$ whose direction is γ-parallelly displaced along this curve, then the direction of $\bar{v}(t)$ is also γ^*-parallelly displaced along this curve and conversely.

Thus we suppose that if equation

$$\frac{d\bar{v}}{dt}+\gamma\bar{v}\,\overline{\frac{dP}{dt}}=a_1(t)\bar{v}(t)$$

holds, then the equation

$$\frac{d\bar{v}}{dt}+\gamma^*\bar{v}\,\overline{\frac{dP}{dt}}=a_2(t)\bar{v}(t)$$

too necessarily holds. If we subtract the former equation from the latter we get

$$(\gamma^*-\gamma)\bar{v}\,\overline{\frac{dP}{dt}}=b(t)\bar{v}(t) \tag{35}$$

where $b(t)=a_2(t)-a_1(t)$. We may write equation (35) in the form

$$k(\gamma^*-\gamma)\overline{\frac{dP}{dt}}\,\bar{v}=b(t)\bar{v}(t). \tag{36}$$

Since equation (36) should hold for arbitrary vectors $\bar{v}$ and $\overline{\frac{dP}{dt}}$, the tensor $k(\gamma^*-\gamma)=k\varkappa$ has to possess the following property: the tensor $k\varkappa$ applied to an arbitrary vector $\bar{a}$ yields a first order tensor which makes correspond to every vector itself multiplied by a scalar number. The necessity of this condition was proved. But it is also sufficient; this can easily be shown by accomplishing inversely the preceding procedure. Thus, there holds the following

Theorem. *By a one-to-one correspondence of two n-dimensional affinely connected spaces L_γ and L_{γ^*} vector fields whose direction is γ-parallelly displaced turn into vector fields with direction γ^*-parallelly displaced and inversely, if and only if the condition*

$$k(\gamma^*-\gamma)\bar{u}\,\bar{v}=k\varkappa\bar{u}\,\bar{v}=a\bar{v}$$

holds for arbitrary vectors $\bar{u}$ and $\bar{v}$ (a is a scalar number).

If the spaces L_γ and L_{γ^*} are torsion-free, that is γ and γ^* are symmetric, then $\varkappa$ is also symmetric. If the preceding relation exists between these spaces, i. e. if the images of vector fields γ-parallelly displaced are vector fields γ^*-parallelly displaced and conversely, then we get for arbitrary vectors $\bar{u}$ and $\bar{v}$ the equation

$$a\bar{v}=k\varkappa\bar{u}\,\bar{v}=\varkappa\bar{u}\,\bar{v}=\varkappa\bar{v}\,\bar{u}=k\varkappa\bar{v}\,\bar{u}=b\bar{u}.$$

This can be true for any vectors $\bar{u}$ and $\bar{v}$ if and only if $a=b=0$, i. e. if

$$\varkappa\bar{u}\,\bar{v}=0.$$

Hence we get $\varkappa=\gamma^*-\gamma\equiv 0$, or $\gamma^*=\gamma$. Thus the two affinely connected spaces are equal.

§ 8. Geometry of surfaces in affinely connected spaces

Let F be an m-dimensional surface in the affinely connected space L with equation $Q = Q(u_1, \dots, u_m)$. The principal aim of this Section is the definition of the parallel displacement on F "induced" by the γ-parallel displacement defined in L.

Some definitions and theorems are wanted.

The m dimensional vector space created by the tangent vectors of the smooth curves of F passing through the point Q is called the *tangent plane* of F in the point Q. Any vector of the tangent plane can be expressed by a linear combination of the vectors $\frac{\overline{\partial Q}}{\partial u_i}$ $(i = 1, 2, \dots, m)$.

Consider those vectors of L in the point Q which are linearly independent of the tangent vectors of F.

The $(n-m)$-dimensional vector space created by these vectors is called the *normal plane* of the surface F in the point Q.

Clearly, any vector $\bar{v}$ of L in Q can be expressed as the sum of a vector of the tangent plane and a vector of the normal plane. It can be easily shown that this decomposition is uniquely determined. Indeed, suppose that

$$\bar{v} = \bar{e}_1 + \bar{n}_1$$
$$v = \bar{e}_2 + \bar{n}_2$$

where e. g. $\bar{e}_1 \neq \bar{e}_2$ are vectors of the tangent plane, $\bar{n}_1$ and $\bar{n}_2$ are vectors of the normal plane. From the assumptions we get $\bar{n}_1 \neq \bar{n}_2$. Subtracting the preceding two equations from each other we get

$$0 = e_1 - \bar{e}_2 + \bar{n}_1 - \bar{n}_2.$$

But this equation is in contradiction with the fact that $\bar{e}_1 - \bar{e}_2$ is a nonzero vector of the tangent plane, $\bar{n}_1 - \bar{n}_2$ is a nonzero vector of the normal plane, and thus they are linearly independent.

Definition. We denote by $\omega(Q)$ the operator defined in the point Q of the surface F of L, which makes to correspond to a vector $\bar{v}$ of L in Q the vector component of $\bar{v}$ in the tangent plane of F.

The operator $\omega(Q)$ is, obviously, linear homogeneous. Since the vector component of $\bar{v}$ in the tangent plane of F is a vector of L, the operator $\omega(Q)$ is a first order tensor of L.

Suppose that $\omega(Q)$ is arbitrarily many times differentiable. Of course, in the equation

$$\omega(Q) - \omega(Q_0) = \frac{d\omega}{dQ} \overline{\Delta Q} + \varepsilon \overline{\Delta Q} \tag{37}$$

defining $\dfrac{d\omega}{dQ}$ the vector $\overline{\Delta Q}=\overrightarrow{Q_0'Q'}$ cannot be arbitrary; Q_0 and Q have to be points of the surface F. On basis of equation (2) $\dfrac{d\omega}{dQ}$ can be considered as a second order tensor of the space A, but it is not defined for every pair of vectors in Q_0. However it is defined for tangent vectors of F and for vectors $\overrightarrow{Q_0'Q'}$, where Q is also a point of F. That is for us sufficient.

Theorem. *The second order tensor* $\dfrac{d\omega}{dQ}$ *of the space A defined on tangent vectors of F is symmetric.*

Proof. Let $\bar{a}$ and $\bar{b}$ be arbitrary tangent vectors of F in the point Q. The equation

$$\frac{d\omega}{dQ}\bar{a}\,\bar{b}=\frac{d\omega}{dQ}\bar{b}\,\bar{a}$$

will be demonstrated.

If $Q_1=Q_1(t)$ and $Q_2=Q_2(s)$ are curves of F passing through the point Q and the tangent vectors of these curves in Q are $\bar{a}$ and $\bar{b}$ respectively, we may write

$$Q=Q_1(t_0)=Q_2(s_0);\quad \left.\frac{\overline{dQ_1}}{dt}\right|_{t_0}=\bar{a},\quad \left.\frac{\overline{dQ_2}}{ds}\right|_{s_0}=\bar{b}.$$

Denote the vectors $\overrightarrow{Q'Q_1'}(t_0+\Delta t)$ and $\overrightarrow{Q'Q_2'}(s_0+\Delta s)$ by $\overline{\Delta Q_1}$ and $\overline{\Delta Q_2}$ respectively (Q', $Q_1'(t_0+\Delta t)$, $Q_2'(s_0+\Delta s)$ are the images in A of the points Q, $Q_1(t_0+\Delta t)$, $Q_2(s_0+\Delta s)$ respectively). On the basis of the definition of a tangent vector and of equation (37) we may write

$$\frac{d\omega}{dQ}\bar{a}\,\bar{b}=\frac{d\omega}{dQ}\frac{\overline{dQ_1}}{dt}\frac{\overline{dQ_2}}{ds}=\lim_{\Delta t,\,\Delta s\to 0}\frac{d\omega}{dQ}\frac{\overline{\Delta Q_1}}{\Delta t}\frac{\overline{\Delta Q_2}}{\Delta s}=$$

$$=\lim_{\Delta t,\,\Delta s\to 0}\frac{1}{\Delta t\,\Delta s}\left[\omega(Q_1(t_0+\Delta t))\overline{\Delta Q_2}-\omega(Q)\overline{\Delta Q_2}\right],$$

and

$$\frac{d\omega}{dQ}\bar{b}\,\bar{a}=\frac{d\omega}{dQ}\frac{\overline{dQ_2}}{ds}\frac{\overline{dQ_1}}{dt}=\lim_{\Delta t,\,\Delta s\to 0}\frac{d\omega}{dQ}\frac{\overline{\Delta Q_2}}{\Delta s}\frac{\overline{\Delta Q_1}}{\Delta t}=$$

$$=\lim_{\Delta t,\,\Delta s\to 0}\frac{1}{\Delta t\,\Delta s}\frac{d(\omega\overline{\Delta Q_2})}{dQ}\overline{\Delta Q_1}=$$

$$=\lim_{\Delta t,\,\Delta s\to 0}\frac{1}{\Delta t\,\Delta s}\left[\omega\,\overline{\Delta Q_2}\big|_{Q_1(t_0+\Delta t)}-\omega\,\overline{\Delta Q_2}\big|_{Q}\right]=$$

$$=\lim_{\Delta t,\,\Delta s\to 0}\frac{1}{\Delta t\,\Delta s}\left[\omega(Q_1(t_0+\Delta t))\overline{\Delta Q_2}-\omega(Q)\overline{\Delta Q_2}\right].$$

From the equality of the right hand sides of these equations follows the equality of the left hand sides, which was to be proved.

Definition. The field of surface vectors of F defined in points of a curve $Q=Q(t)$ of F is said to be *γ-parallelly displaced along the curve $Q=Q(t)$ with respect to the surface F,* if

$$\omega\left(\frac{d\bar{v}}{dt}+\gamma\bar{v}\,\frac{\overline{dQ}}{dt}\right)=0. \tag{38}$$

Thus $\bar{v}$ is a surface vector $\omega\bar{v}=v$. Hence we get

$$\omega\,\frac{d\bar{v}}{dt}=\frac{d}{dt}(\omega\bar{v})-\frac{d\omega}{dt}\,\bar{v}=\frac{d\bar{v}}{dt}-\frac{d\omega}{dQ}\,\frac{\overline{dQ}}{dt}\,\bar{v}.$$

Substituting this expression into equation (38) we get

$$\frac{d\bar{v}}{dt}-\frac{d\omega}{dQ}\,\frac{\overline{dQ}}{dt}\,\bar{v}+\omega\gamma\bar{v}\,\frac{\overline{dQ}}{dt}=0.$$

Since $\dfrac{d\omega}{dQ}$ is symmetric the latter equation can be written in the form

$$\frac{d\bar{v}}{dt}=-\left[\omega\gamma-\frac{d\omega}{dQ}\right]\bar{v}\,\frac{\overline{dQ}}{dt}. \tag{39}$$

Clearly, equation (39) is equivalent to equation (38). Therefore it is reasonable to make the following

Definition. The second order linear homogeneous operator $\underline{\gamma}=\omega\gamma-\dfrac{d\omega}{dQ}$, which makes correspond to any pair of surface vectors a vector of the space A, is called the *operator of connection of the parallel displacement induced on F by the $\underline{\gamma}$-parallel displacement introduced in L.*

Using this notation the formula defining the $\underline{\gamma}$-parallel displacement induced on F can be written in the form

$$\frac{d\bar{v}}{dt}=-\underline{\gamma}\bar{v}\,\frac{\overline{dQ}}{dt},$$

which is analogous to equation (5).

Theorem. *If the γ-parallel displacement defined in the space L is torsion-free, then the $\underline{\gamma}$-parallel displacement induced by that on F is also torsion-free.*

PROOF. The tensor of torsion $\underline{\tau}$ of the $\underline{\gamma}$-parallel displacement induced on F is

$$\begin{aligned}\underline{\tau}=\underline{\gamma}-k\underline{\gamma}&=\omega\gamma-\frac{d\omega}{dQ}-k\left(\omega\gamma-\frac{d\omega}{dQ}\right)=\\&=\omega\gamma-k(\omega\gamma)=\omega\gamma-\omega k\gamma=\omega(\gamma-k\gamma)=\omega\tau.\end{aligned}$$

Hence if $\tau=0$, then $\underline{\tau}=0$.

§ 9. Introduction of coordinate systems

Let in the space A be introduced the system of general coordinates $x^1, x^2, \dots, x^n$. This, clearly, can be considered as a coordinate system of the space L. Let the equation of the x^i parameter line passing through an arbitrary point $P(x_0^1, \dots, x_0^n)$ of L be

$$P = P(x^i) = P(x_0^1, \dots, x^i, \dots, x_0^n) \qquad (i=1, \dots, n).$$

We introduce the notation

$$\bar{e}_i = \overline{\frac{dP}{dx^i}} = \overline{\frac{\partial P}{\partial x^i}} \qquad (i=1, \dots, n).$$

The n vectors $\bar{e}_i$ of L in the point P form the *basis* of L in P in the system of coordinates x^i. Once given the system of coordinates the basis is uniquely determined in every point. We know from affine geometry that the vectors $\bar{e}_i$ are linearly independent. Hence an arbitrary vector $\bar{v}$ of L in P can be expressed uniquely as a linear combination

$$\bar{v} = v^1\bar{e}_1 + \cdots + v^n\bar{e}_n = v^i\bar{e}_i$$

of the basis vectors of that point. (The summation convention is used.) The vector $\bar{v}$ determines uniquely in the given system of coordinates the numbers $v^1, \dots, v^n$ and conversely. Therefore v^i is called the i-th *coordinate of the vector* $\bar{v}$.

Let α be a first order tensor of L in P, and denote the vector made to correspond by α to $\bar{e}_k$ with $\bar{a}_k$, i. e.

$$\bar{a}_k = \alpha\bar{e}_k \qquad (k=1, \dots, n). \tag{40}$$

Denote the coordinates of the vector $\bar{a}_k$ with a_k^i $(i=1. \dots, n)$ i. e.

$$\bar{a}_k = a_k^i\bar{e} \qquad (k=1, \dots, n). \tag{41}$$

If α makes correspond to the vector $\bar{v}(v^1, \dots, v^n)$ the vector $\bar{u}(u^1, \dots, u^n)$, using the equation (40) and (41) we get

$$u^i\bar{e}_i = \bar{u} = \alpha\bar{v} = \alpha v^k\bar{e}_k = v^k\alpha\bar{e}_k = v^k\bar{a}_k = v^k a_k^i\bar{e}_i.$$

This equation holds if and only if

$$u^i = a_k^i v^k \qquad (i=1, \dots, n). \tag{42}$$

With the aid of equations (42) from the coordinates of a given vector $\bar{v}$ the coordinates of the vector $\bar{u}$ can be directly determined, $\bar{u}$ being the vector made to correspond by α to $\bar{v}$. The tensor α determines uniquely the system of the quantities a_k^i $(i, k=1, \dots, n)$ with the aid of formulae (40), (41). Conversely, once given a system of quantities a_k^i the tensor which makes correspond vectors to each other by formula (42) is uniquely determined. Therefore the quantities a_k^i are called the *coordinates of the tensor* α.

Let μ be a p-th order tensor of L in P, and denote the vector made to correspond to the vectors $\bar{e}_{k_1}, \bar{e}_{k_2}, \ldots, \bar{e}_{k_p}$ by $\overline{m}_{k_1 \ldots k_p}$, i. e.

$$\overline{m}_{k_1 \ldots k_p} = \mu \bar{e}_{k_1} \ldots \bar{e}_{k_p} \qquad (k_r = 1, \ldots, n\,;\; r = 1, \ldots, p). \quad (43)$$

Denote the coordinates of the vector $\overline{m}_{k_1 \ldots k_p}$ by $m^i_{k_1 \ldots k_p}$ $(i = 1, \ldots, n)$, i. e.

$$\overline{m}_{k_1 \ldots k_p} = m^i_{k_1 \ldots k_p} \bar{e}_i \qquad (k_r = 1, \ldots, n\,;\; r = 1, \ldots, p). \quad (44)$$

If μ makes correspond to the vectors $\bar{v}_1, \ldots, \bar{v}_p$ the vector $\bar{u}$, using equations (43) and (44) we get

$$\begin{aligned} u^i \bar{e}_i = \bar{u} = \mu\, v_1 \ldots \bar{v}_p = \mu(v_1^{k_1} \bar{e}_{k_1}) \ldots (v_p^{k_p} \bar{e}_{k_p}) = \\ = v_1^{k_1} \ldots v_p^{k_p} \mu \bar{e}_{k_1} \ldots \bar{e}_{k_p} = v_1^{k_1} \ldots v_p^{k_p} \overline{m}_{k_1 \ldots k_p} = \\ = v_1^{k_1} \ldots v_p^{k_p} m^i_{k_1 \ldots k_p} \bar{e}_i, \end{aligned}$$

where v_l^i is the i-th coordinate of the vector $\bar{v}_l$ $(l = 1, \ldots, p)$ and u^i is the i-th coordinate of $\bar{u}$. This equation holds, if and only if

$$u^i = m^i_{k_1 \ldots k_p} v_1^{k_1} \cdots v_p^{k_p}. \quad (45)$$

The signification of formula (45) is analogous to that of formula (42). Thus the quantities $m^i_{k_1 \ldots k_p}$ are called the *coordinates of the tensor* μ.

It can easily be shown that the operations defined among tensors (vectors) can be accomplished as follows, if the tensors (vectors) are given by their coordinates.

The coordinates of the sum of two p-th order tensors (vector) are the sums of the respective coordinates of the tensors (vectors).

We multiply a tensor (vector) with a real number by multiplying each coordinate with the given number.

Let the coordinates of the p-th order tensor μ be denoted by $m^i_{k_1 \ldots k_p}$ and those of the q-th order tensor ν by $n^i_{k_1 \ldots k_p}$. Using these notations the coordinates $r^i_{l_1 \ldots l_q k_2 \ldots k_p}$ of the tensor $\varrho = \mu \nu$ are yielded by the equations

$$r^i_{l_1 \ldots l_q k_2 \ldots k_p} = m^i_{k k_2 \ldots k_p} n^k_{l_1 \ldots l_q} \qquad (i, k_2, \ldots, k_p, l_1, \ldots, l_q = 1, \ldots, n).$$

If the coordinates of the p-th order tensor μ are $m^i_{k_1 \ldots k_p}$, the coordinates of the alternated tensor $k_{rs}\mu$ of μ will be

$$n^i_{k_1 \ldots k_r \ldots k_s \ldots k_p} = m^i_{k_1 \ldots k_s \ldots k_r \ldots k_p}.$$

Obviously, the preceding results of this Section hold also for those tensors of the space A which are not tensors of the space L.

Let now another system of coordinates $x^{i'}$ be introduced into the space. Suppose that the functions

$$x^{i'} = x^{i'}(x^k) \tag{46}$$

are twice continuously differentiable and the variables x^k can be expressed from equations (46) as uniquely determined functions of the variables $x^{i'}$. The relations

$$\bar{e}_k = \frac{\partial \overline{P}}{\partial x^k} = \frac{\partial \overline{P}}{\partial x^{i'}} \frac{\partial x^{i'}}{\partial x^k} = \bar{e}_{i'} \frac{\partial x^{i'}}{\partial x^k} \qquad (k = 1, \ldots, n) \tag{47}$$

hold among the basis vectors $\bar{e}_k = \dfrac{\partial \overline{P}}{\partial x^k}$ of the old, and the basis vectors $\bar{e}_{i'} = \dfrac{\partial \overline{P}}{\partial x^{i'}}$ of the new coordinate system.

Let the vector $\bar{v}$ of the space L be expressed in the old coordinate system by equation

$$\bar{v} = v^k \bar{e}_k \tag{48}$$

and in the new coordinate system by equation

$$\bar{v} = v^{i'} \bar{e}_{i'}. \tag{49}$$

From equation (48) we get using equation (47)

$$\bar{v} = v^k \bar{e}_k = v^k \bar{e}_{i'} \frac{\partial x^{i'}}{\partial x^k} = \frac{\partial x^{i'}}{\partial x^k} v^k \bar{e}_{i'}.$$

Comparing the latter equation with (49) we obtain the transformation formula of the coordinates of the vector $\bar{v}$, namely

$$v^{i'} = \frac{\partial x^{i'}}{\partial x^k} v^k \tag{50}$$

or inversely

$$v^k = \frac{\partial x^k}{\partial x^{i'}} v^{i'}. \tag{51}$$

With the aid of the transformation formulae of vector coordinates the transformation formulae of tensor coordinates can easily be obtained. If the coordinates of the p-th order tensor μ are in the old system $m^i_{l_1 \ldots l_p}$ and in the new system $m^{r'}_{s'_1 \ldots s'_p}$, the formulae

$$m^{r'}_{s'_1 \ldots s'_p} = \frac{\partial x^{r'}}{\partial x^i} \frac{\partial x^{l_1}}{\partial x^{s'_1}} \cdots \frac{\partial x^{l_p}}{\partial x^{s'_p}} m^i_{l_1 \ldots l_p}$$

$$(r' = 1, \ldots, n;\ s'_k = 1, \ldots, n;\ k = 1, \ldots, p)$$

and conversely

$$m^{i}_{l_1\dots l_p}=\frac{\partial x^{i}}{\partial x^{r'}}\frac{\partial x^{s'_1}}{\partial x^{l_1}}\cdots\frac{\partial x^{s'_p}}{\partial x^{l_p}}m^{r'}_{s'_1\dots s'_p}$$

$$(i=1,\dots,n;\ l_k=1,\dots,n;\ k=1,\ \dots,p)$$

hold.

From the results of this Section it is seen, that in constructing the affinely connected space we used only tensors contravariant of the first order and covariant of higher order. For this reason the multiplication of tensors was defined more specially, than usual, namely so that, by the usual phraseology, the product of tensors contravariant of the first and covariant of higher order is again a tensor contravariant of the first and covariant of higher order.

Suppose that $x^{i'}$ is an affine coordinate system in the space A and denote the coordinates of the points P'_0 and P' with $x^{i'}_0$ and $x^{i'}$ respectively. Then, the coordinates of the vector $\overline{\Delta P}=\overrightarrow{P'_0P'}$ in this coordinate system are $\Delta x^{i'}=x^{i'}-x^{i'}_0$. Using formula (51) we obtain the coordinates p^k of the vector $\overline{\Delta P}$ in a general coordinate system x^k in the form

$$p^k=\frac{\partial x^k}{\partial x^{i'}}\Delta x^{i'}.$$

Thus, neglecting the second order terms, the equation

$$p^k=\Delta x^k \tag{52}$$

holds. Hence, if the curve $P=P(t)$ of the space L is given by the functions $x^i=x^i(t)$ $(i=1,\dots,n)$, the coordinates of the tangent vector $\frac{\overline{dP}}{dt}$ of the curve are $\left(\frac{dx^1}{dt},\dots,\frac{dx^n}{dt}\right)$.

Taking into account formula (52) and writing equations (1) and (2) respectively in coordinates, we obtain that the coordinates of the tensors $\frac{d\bar{v}}{dP}$ and $\frac{d\mu}{dP}$ of the space A are

$$v^i_l=\frac{\partial v^i}{\partial x^l}\quad\text{and}\quad m^i_{kl_1\dots l_p}=\frac{\partial m^i_{l_1\dots l_p}}{\partial x^k}$$

respectively, where v^i are the coordinates of the vector $\bar{v}$ and $m^i_{l_1\dots l_p}$ are the coordinates of the p-th order tensor μ.

Now, the transformation formula of the coordinates of the operator of connection γ can be deduced. Denote the coordinates of the second order linear homogeneous operator γ with Γ^i_{jk} in the system x^i and with $\Gamma^{r'}_{s't'}$ in the system $x^{i'}$ respectively. In coordinates equation (5) takes the form

$$\frac{dv^i}{dt} = -\Gamma^i_{jk} v^j \frac{dx^k}{dt} \qquad (i=1, \ldots, n) \tag{53}$$

or

$$\frac{dv^{r'}}{dt} = -\Gamma^{r'}_{s't'} v^{s'} \frac{dx^{t'}}{dt} \qquad (r'=1, \ldots, n). \tag{54}$$

Differentiating the equation

$$v^i = \frac{\partial x^i}{\partial x^{s'}} v^{s'}$$

with respect to t we get

$$\frac{dv^i}{dt} = \frac{\partial^2 x^i}{\partial x^{t'} \partial x^{s'}} v^s \frac{dx^{t'}}{dt} + \frac{\partial x^i}{\partial x^{s'}} \frac{dv^{s'}}{dt}.$$

Substituting this expression into equation (53) and expressing the coordinates of vectors taken in the system x^i by coordinates taken in the system $x^{i'}$ equation (53) takes the form

$$\frac{\partial^2 x^i}{\partial x^{s'} \partial x^{t'}} v^{s'} \frac{dx^{t'}}{dt} + \frac{\partial x^i}{\partial x^{s'}} \frac{dv^{s'}}{dt} = -\Gamma^i_{jk} \frac{\partial x^j}{\partial x^{s'}} v^{s'} \frac{\partial x^k}{\partial x^{t'}} \frac{\partial x^{t'}}{\partial t}.$$

If the latter equation is multiplied by $\frac{\partial x^{r'}}{\partial x^i}$ and summed with respect to i we have

$$\frac{dv^{r'}}{dt} = -\left[\frac{\partial x^{r'}}{\partial x^i} \frac{\partial x^j}{\partial x^{s'}} \frac{\partial x^k}{\partial x^{t'}} \Gamma^i_{jk} + \frac{\partial x^{r'}}{\partial x^i} \frac{\partial^2 x^i}{\partial x^{s'} \partial x^{t'}}\right] v^{s'} \frac{dx^{t'}}{dt}.$$

Since $\bar{v}$ and $\overline{\frac{dP}{dt}}$ are arbitrary vectors of L in P, comparing the latter equation with (54) we obtain finally that

$$\Gamma^{r'}_{s't'} = \frac{\partial x^{r'}}{\partial x^i} \frac{\partial x^j}{\partial x^{s'}} \frac{\partial x^k}{\partial x^{t'}} \Gamma^i_{jk} + \frac{\partial x^{r'}}{\partial x^i} \frac{\partial^2 x^i}{\partial x^{s'} \partial x^{t'}}.$$

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(Received May 21, 1959.)