Abstract linear dependence relations[1])

By M. N. BLEICHER (New Orleans, Louisiana) and
G. B. PRESTON (Shrivenham, England)[2])

In 1935 H. WHITNEY [1], using what he called a rank function, an integral valued function on the subsets of a set, initiated an abstract treatment of the notion of linear dependence. An equivalent generalization of this concept of linear dependence over a vector space, in terms of I-functions (see below § 3), was given by RADO in 1943 [2]. In a later paper RADO extended WHITNEY's rank functions to sets with possibly infinite rank [3]. T. LAZARSON has recently shown that not every linear dependence relation defined by an I-function can be faithfully realised (in the obvious sense) as an ordinary linear dependence relation on a subset of a vector space [4] (Cf. also A. W. INGLETON [5]).

In this note we introduce another definition of an abstract linear dependence relation on a set. Our axioms are closely related to the well-known axioms found, for example, in v. D. WAERDEN's Moderne Algebra [7] and the immediate deductions from them in § 1 follow familiar lines. Linear dependence relations defined by I-functions are shown (§ 3) to be equivalent to what we call proper (linear) dependence relations. We show (Theorem 1), as does RADO [3], that any set, over which a proper dependence relation is defined, has a uniquely defined rank or dimension.[3]) As corollaries follow the corresponding theorems for abelian groups, vector spaces, fields etc. It seems to the authors that the present approach has certain advantages. Apart from the fact that the theorem on the existence of ranks is achieved at considerably less cost the present treatment preserves more of the flavour of vector space theory. We are lead (§ 4) to define what we call generalized

---

[1]) Presented to the American Mathematical Society, August 26, 1958. This work was supported by the National Science Foundation Grant to Tulane University.

[2]) The authors would like to thank Mr. N. F. WILLIAMSON for his interest in and constructive criticism of this work.

[3]) For a closely related discussion of linear dependence see the paper [8] by A. KERTÉSZ which appeared in the meantime.

vector spaces, and we hope to develop the properties of such spaces elsewhere. As a result in this direction we prove an analogue of an important lemma of v. B. BANASCHEWSKI [6, Lemma 4].

1. Notations, definitions and elementary results. We use $\square$ to denote the empty set, $A \setminus B$ to mean the set of elements in A but not in B, and x to denote either the element x or the set $\{x\}$ as the context demands.

A binary relation $\mathfrak{L}$ on the subsets of a set A will be called a *dependence relation on A* if it satisfies the conditions:

L1. If $X \subseteq Y$ then $X \mathfrak{L} Y$.

L2. If $X_t \mathfrak{L} Y$ for all t in any index set T then $\cup \{X_t : t \in T\} \mathfrak{L} Y$.

L3. If $X \mathfrak{L} Y$ and $Y \mathfrak{L} Z$ then $X \mathfrak{L} Z$.

L4. If $y \mathfrak{L} X$ and $y \overline{\mathfrak{L}} X \setminus x$ then $x \mathfrak{L} (X \setminus x) \cup y$, for any elements x, y in A.[4])

For example inclusion is a dependence relation on A; and, in fact, because of L1, the inclusion relation is the intersection of all dependence relations on A.

If $\mathfrak{L}$ is a dependence relation on A and if $B \subseteq A$, then it is clear that the restriction of $\mathfrak{L}$ to the subsets of B induces a dependence relation on B. It will cause no confusion to denote such an induced relation also by $\mathfrak{L}$.

The following definitions are now the natural ones. X is an *$\mathfrak{L}$-independent* set if for all x in X, $x \overline{\mathfrak{L}} X \setminus x$. X is *$\mathfrak{L}$-dependent* if it is not $\mathfrak{L}$-independent. If $X \mathfrak{L} Y$ and $Y \mathfrak{L} X$ then X is *$\mathfrak{L}$-equivalent* to Y. The relation of $\mathfrak{L}$-equivalence is clearly an equivalence relation on the set of subsets of A.

We now give some immediate implications of the axioms L1—L4.

Lemma 1. *If X is an $\mathfrak{L}$-independent subset of A then every subset of X is $\mathfrak{L}$-independent.*

PROOF. If Y is a subset of X which is not $\mathfrak{L}$-independent, then there exists y in Y such that $y \mathfrak{L} Y \setminus y$. But then $y \in X$ and by L1 and L3 we have $y \mathfrak{L} X \setminus y$ in contradiction to the $\mathfrak{L}$-independence of X:

Lemma 2. *If X is $\mathfrak{L}$-independent and $y \overline{\mathfrak{L}} X$, where y is an element of A, then $X \cup y$ is $\mathfrak{L}$-independent.*

PROOF. If $X \cup y = Y$, say, is not $\mathfrak{L}$-independent there exists $x (\neq y)$ in Y such that $x \mathfrak{L} (Y \setminus x)$. Since X is $\mathfrak{L}$-independent $x \overline{\mathfrak{L}} \{(Y \setminus x) \setminus y\}$ $(= X \setminus x)$ and hence, by L4, $y \mathfrak{L} \{((Y / x) \setminus y) \cup x\} = X$, which is contrary to hypothesis.

Lemma 3. *Any two maximal (relative to inclusion) $\mathfrak{L}$-independent subsets of A are $\mathfrak{L}$-equivalent.*

---

[4]) $\overline{\mathfrak{L}}$ denotes the negation of $\mathfrak{L}$.

Proof. Let X, Y be two maximal $\mathfrak{L}$-independent subsets of A. Let $y \in Y$. Then $y \mathfrak{L} X$, for otherwise, by Lemma 2, X would not be maximal. Hence, by L2, $Y \mathfrak{L} X$. Similarly $X \mathfrak{L} Y$.

The next lemma is a generalization of the Steinitz Exchange Theorem.

Lemma 4. *Let X, Y be $\mathfrak{L}$-equivalent, $\mathfrak{L}$-independent sets. Let $X \cap Y = K$. Then for each x in $X \setminus K$ there is a y in $Y \setminus K$ such that $(X \setminus x) \cup y$ is $\mathfrak{L}$-independent and $\mathfrak{L}$-equivalent to X.*

Proof. Let $x \in X \setminus K$. Select y in $Y \setminus K$ such that $y \bar{\mathfrak{L}} X \setminus x$. There exists such an element y, for in the contrary case, by L1—L3, $x \mathfrak{L} X \mathfrak{L} Y = (Y \setminus K) \cup K \mathfrak{L} (X \setminus x) \cup K = X \setminus x$, so that $x \mathfrak{L} X \setminus x$, which contradicts the hypothesis that X is $\mathfrak{L}$-independent. From Lemma 2 it follows that $(X \setminus x) \cup y$ is $\mathfrak{L}$-independent. Using L4 we have $x \mathfrak{L} (X \setminus x) \cup y$; whence we easily have that $(X \setminus x) \cup y$ is $\mathfrak{L}$-equivalent to X.

Denote by $|X|$ the cardinal of the set X. Then we have the following corollary to the preceeding lemma.

Corollary. *Let X, Y be $\mathfrak{L}$-equivalent, $\mathfrak{L}$-independent sets one of which is finite. Then $|X| = |Y|$.*

2. Proper dependence relations. A dependence relation $\mathfrak{L}$ on A will be said to be *proper* if the property of $\mathfrak{L}$-independence of subsets of A is a property of finite character. Throughout the rest of this note we will restrict the discussion to proper dependence relations. Since the empty set is always an independent subset we have immediately by Tukey's lemma the following result:

Lemma 5. *If $\mathfrak{L}$ is proper, then any $\mathfrak{L}$-independent subset of A is contained in a maximal $\mathfrak{L}$-independent subset of A. Hence, in particular, A contains a maximal $\mathfrak{L}$-independent subset.*

The importance of proper dependence relations rests on the following alternative characterization of them.

Lemma 6. *$\mathfrak{L}$ is a proper dependence relation on A if and only if the following property holds:*

(P): *for any element a in A and any subset X of A $a \mathfrak{L} X$ (if and) only if $a \mathfrak{L} F$, where F is a finite subset of X.*

Proof. Suppose (P) holds. If $\mathfrak{L}$ is not proper then there exists a set X which is $\mathfrak{L}$-dependent and such that every finite subset of X is $\mathfrak{L}$-independent Thus there exists x in X such that $x \mathfrak{L} X \setminus x$. Property (P) then implies that $x \mathfrak{L} F \subseteq X \setminus x$ for some finite set F, and hence $F \cup x$ is a finite subset of X which is $\mathfrak{L}$-dependent. This is a contradiction; and so $\mathfrak{L}$ must be proper.

Conversely suppose that $\mathfrak{L}$ is proper. Let $a\mathfrak{L}X$. Regarding $\mathfrak{L}$ as a dependence relation on X we may apply Lemma 5 and select W as a maximal $\mathfrak{L}$-independent subset of X. Then since $X\mathfrak{L}W$ we have $a\mathfrak{L}W$, and so $W\cup a$ is a $\mathfrak{L}$-dependent set. Since $\mathfrak{L}$ is proper there exists a finite subset F of $W\cup a$ which is also $\mathfrak{L}$-dependent. F cannot be a subset of W because W is $\mathfrak{L}$-independent. Hence $a\in F$. Since F is $\mathfrak{L}$-dependent there is an element y in F such that $y\mathfrak{L}G=F\setminus y$. If $y=a$, then $a\mathfrak{L}G$, a finite subset of X. If $y\neq a$ then, noting that $y\overline{\mathfrak{L}}G\setminus a$, we deduce from L4 that $a\mathfrak{L}(G\setminus a)\cup y$, again a finite subset of X. Thus we have shown that property (P) holds.

We can now extend to infinite sets the Corollary to Lemma 4.

Theorem 1. *Let $\mathfrak{L}$ be a proper dependence relation on a set A. Let X, Y be two $\mathfrak{L}$-equivalent, $\mathfrak{L}$-independent subsets of A. Then $|X|=|Y|$.*

PROOF. Since $X\mathfrak{L}Y$, by Lemma 6, for each x in X we may pick a definite finite subset F_x, say, of Y such that $x\mathfrak{L}F_x$. Then the family of sets $\mathfrak{F}=\{F_x\colon x\in X\}$ covers Y. For otherwise $Y\mathfrak{L}X\mathfrak{L}\cup\{F_x\colon x\in X\}$, a proper subset of Y, which conflicts with the hypothesis that Y is $\mathfrak{L}$-independent. If either $|X|$ or $|Y|$ is finite we know that $|X|=|Y|$ by the Corollary to Lemma 4. When $|X|$ and $|Y|$ are infinite the fact that $\mathfrak{F}$ is a cover of Y by finite sets implies that $|\mathfrak{F}|=|Y|$. Since $x\to F_x$ is a single valued mapping of X onto $\mathfrak{F}$, therefore $|X|\geqq|\mathfrak{F}|$. Thus $|X|\geqq|Y|$. Similarly we obtain $|Y|\geqq|X|$; and hence $|X|=|Y|$.

3. Dependence relations and *I*-functions. RADO [2] defines an *I-function* on a set A to be a mapping f of the set of finite sequences $(x_1,\dots,x_n)$ of elements of A (including the empty sequence which we denote by Φ) into the two element set consisting of the integers 0, 1 satisfying the following conditions.

I1. $f(x_1,\dots,x_n)=f(x_1',\dots,x_n')$ if $x_1',\dots,x_n'$ is any permutation of $x_1,\dots,x_n$.

I2. $f(x,x)=0$.

I3. $f(x_1,\dots,x_n)\geqq f(x_1,\dots,x_n,y)$.

I4. $f(x_1,\dots,x_n)=1=f(y_1,\dots,y_{n+1})$ implies there exists t, $1\leqq t\leqq n+1$, such that $f(x_1,\dots,x_n,y_t)=1$.

I5. $f(\Phi)=1$.

An *I*-function f on a set A determines a binary relation f^+ on the subsets of A defined thus: Xf^+Y if and only if for each x in X there is a finite, possibly empty, sequence α of elements of Y such that $f(\alpha)=1$ and $f(x,\alpha)=0$. This relation f^+ is, in Rado's sense, (Cf. the remarks at the end of this section) the dependence relation on A corresponding to the *I*-function f.

The relation f^+ is also a dependence relation in our sense. In fact we have the following result.

Lemma 7. *If f is an I-function on A then f^+ is a proper dependence relation on A.*

PROOF. We have to show that f^+ satisfies the conditions L1—L4 of § 1 It is clear, in view of Lemma 6, that f^+ will then be proper.

L1. Let $X \subseteq Y$ and let $x \in X$. We have either $f(x)=1$ and $f(x,x)=0$ (where we note that $x \in Y$), or $f(\Phi)=1$ and $f(x,\Phi)=f(x)=0$. Thus Xf^+Y.

L2. That L2 holds for f^+ is immediate from the definition of f^+.

L3. Let Xf^+Y and Yf^+Z. If $x \in X$ then there exists a sequence α, say, of elements of Y such that $f(\alpha)=1$, $f(x,\alpha)=0$. If α is the empty sequence then clearly xf^+Z. Otherwise $\alpha=(y_1,\dots,y_n)$ and there exist sequences β_i of elements of Z such that $f(\beta_i)=1$, $f(y_i,\beta_i)=0$ for $i=1,\dots,n$. Let C denote the set of all elements of Z which occur in the sequences β_i, $i=1,\dots,n$, and, for each i, let γ_i be a sequence of elements of C which (1) contains β_i as a subsequence (2) satisfies $f(\gamma_i)=1$ and (3) is of maximal length satisfying properties (1) and (2).

From the collection of γ_i select γ of maximal length. Then $f(y_i,\gamma)=0$, $i=1,\dots,n$. For if $f(y_i,\gamma)=1$ then, firstly, $\gamma \neq \gamma_i$ for γ_i contains β_i and $f(y_i,\beta_i)=0$; and, secondly, if $\gamma=\gamma_j$, $j \neq i$, then $f(y_i,\gamma)=1=f(\gamma_i)$ and this implies, using I4 and the fact that γ is of length greater than or equal to the length of γ_i, that there is a term z in the sequence γ such that $f(\gamma_i,z)=1$. Since $z \in C$ this contradicts the defining condition (3) of γ_i.

We can now show further that $f(x,\gamma)=0$. Let η be a subsequence of γ which is maximal with respect to the property $f(\alpha,\eta)=1$. Then the length of the sequence (α,η) is not greater than the length of γ, for otherwise $f(z,\gamma)=1$ for either z in C or z one of the elements y_i; and both of these are impossible. Hence, since $f(x,\alpha)=0$ implies $f(x,\alpha,\eta)=0$, it follows, if $f(x,y)=1$, that $f(\alpha,\eta,z)=1$ where (η,z) is a subsequence (or a permutation thereof) of γ. This conflicts with the maximality of η subject to $f(\alpha,\eta)=1$. Hence we have as asserted, $f(x,\gamma)=0$, and this, together with $f(\gamma)=1$, shows that xf^+Z. This completes the proof that L3 holds for f^+.

L4. Let yf^+X and suppose that y is not in relation f^+ to $X \setminus x$. Then there exists a sequence $(x_1,\dots,x_n)$ of elements of X such that $f(x_1,\dots,x_n)=1$ and $f(y,x_1,\dots,x_n)=0$. Further it is clear that one of the x_i, x_1, say, must be x. Since $f(x_2,\dots,x_n)=1$ we must then have $f(y,x_2,\dots,x_n)=1$ for otherwise $yf^+X \setminus x$. But $f(y,x_2,\dots,x_n)=1$ and $f(x,y,x_2,\dots,x_n)=0$ together imply that $xf^+(X \setminus x) \cup y$. Thus property L4 holds for f^+.

This completes the proof of the lemma.

We now proceed in the other direction, from dependence relations to I-functions. Let $\mathfrak{L}$ be a proper dependence relation on a set A. Then $\mathfrak{L}$ determines a mapping $\mathfrak{L}^*$ of the set of finite sequences of elements of A defined thus:

$$\mathfrak{L}^*(\Phi)=1;$$

$$\mathfrak{L}^*(x_1,\ldots,x_n)=\begin{cases}1, & \text{if } x_i\overline{\mathfrak{L}}\{x_1,\ldots,\hat{x}_i,\ldots,x_n\} \text{ for } i=1,\ldots,n;\\ 0, & \text{otherwise.}\end{cases}$$

Here $\{x_1,\ldots,\hat{x}_i,\ldots,x_n\}$ denotes the set $\{x$: there exists $j\neq i$ such that $x=x_j,\ j\leqq n\}$.

Lemma 8. *If $\mathfrak{L}$ is a dependence relation on A then $\mathfrak{L}^*$ is an I-function on A.*

Proof. It is immediate that $\mathfrak{L}^*$ has properties I1, I2, I5. Property I3 follows straightforwardly from axioms L1 and L3. It remains to show that condition I4 is satisfied.

Let $\mathfrak{L}^*(x_1,\ldots,x_n)=1=\mathfrak{L}^*(y_1,\ldots,y_{n+1})$. Then $X=\{x_1,\ldots,x_n)$ and $Y=\{y_1,\ldots,y_{n+1}\}$ are $\mathfrak{L}$-independent sets. By Lemma 5 there is a maximal $\mathfrak{L}$-independent subset M, say, of $X\cup Y$ which contains X, and then, by Theorem 1, $|M|\geqq|Y|$. Hence there is an element y_t, say, in Y such that $X\cup y_t$ is $\mathfrak{L}$-independent (Lemma 1). We then have $\mathfrak{L}^*(x_1,\ldots,x_n,y_t)=1$. This completes the proof of the lemma.

By a straightforward computation we can verify that for any I-function f, $(f^+)^*=f$ and that for any dependence relation $\mathfrak{L}$, $(\mathfrak{L}^*)^+=\tilde{\mathfrak{L}}$ where $\tilde{\mathfrak{L}}$ is a proper dependence relation determined by $\mathfrak{L}$ and defined thus: $X\tilde{\mathfrak{L}}Y$ if and only if for each element x in X there is a finite subset F_x of Y such that $x\mathfrak{L}F_x$. When $\mathfrak{L}$ is proper it is clear from Lemma 6 that $\mathfrak{L}=\tilde{\mathfrak{L}}$. Thus the mapping $\mathfrak{L}\to\mathfrak{L}^*$ is a $(1,1)$-mapping of the set of proper dependence relations on a set A onto the set of I-functions on A. Rado ([2], p. 84) defines a subset X of A to be $\mathfrak{L}^*$-independent if $\mathfrak{L}^*(\alpha)=1$ for all finite sequences α of distinct elements of X. Thus, if $\mathfrak{L}$ is proper, X is $\mathfrak{L}^*$-independent if and only if it is $\mathfrak{L}$-independent.

This completes our verification of the assertion made earlier that proper dependence relations and I-functions afford equivalent generalizations of the concept of linear dependence in vector spaces.

4. Generalized vector spaces. In view of the analogy with vector spaces it is natural to call a set A, on which is defined a dependence relation $\mathfrak{L}$, a *generalized vector space,* or, simply, a *space*. A subset B of A will be called a *subspace of* A if $x\mathfrak{L}B$ implies $x\in B$. It is clear that intersection of any set of subspaces of A is itself a subspace of A. Consequently, if C is any subset of A then $C^* = \cap \{B\colon C \subseteq B,\ B$ a subspace of $A\}$ is a subspace of A. C^* will be called the subspace of A *generated by* C.

Lemma 9. *$x \in C^*$ if and only if $x \mathfrak{L} C$.*

PROOF. Let $B = \{x\colon x \mathfrak{L} C\}$. Then clearly $C \subseteq B$. If $y \mathfrak{L} B$, then, since $B \mathfrak{L} C$, we have $y \mathfrak{L} C$ and so $y \in B$. Thus B is a subspace, and so $C^* \subseteq B$. Conversely, if $x \in B$ then $x \mathfrak{L} C \subseteq C^*$ implies $x \mathfrak{L} C^*$, so that, since C^* is a subspace, $x \in C^*$. Hence $B = C^*$.

Corollary. *If $X \mathfrak{L} Y$, then $X^* \subseteq Y^*$. In particular, if $X \subseteq Y$, then $X^* \subseteq Y^*$.*

This section is devoted to proving the analogue of the theorem of BANASCHEWSKI mentioned earlier. Some preliminary lemmas will facilitate the proof.

Lemma 10. *If $\mathfrak{L}$ is proper then the union of any (inclusion) chain of subspaces of A is itself a subspace of A.*

PROOF. Let $\mathfrak{C}$ be a chain of subspaces of A. Let $B = \cup \{C\colon C \in \mathfrak{C}\}$. Let $x \mathfrak{L} B$. Then since $\mathfrak{L}$ is proper there is a finite subset F, say, of B such that $x \mathfrak{L} F$. Since $\mathfrak{C}$ is a chain and F is finite there exists C in $\mathfrak{C}$ such that $F \subseteq C$. It then follows that $x \in C$ and so $x \in B$.

Let $\mathfrak{S}(A)$ denote the set of subspaces of A. A mapping $\pi\colon \mathfrak{S}(A) \to \mathfrak{S}(A)$ will be said to be *quasi-orthogonal*[5]) if for X, Y in $\mathfrak{S}(A)$ it satisfies the conditions:

O1. $(X \cup X\pi)^* = A$;

O2. $X \cap X\pi = \square^*$;

O3. $X \subseteq Y$ implies $X\pi \supseteq Y\pi$.

If B is a subspace of A then C is a subspace of B if and only if it is a subspace of A contained in B. Thus $\mathfrak{S}(B) = \{V \cap B\colon V \in \mathfrak{S}(A)\}$.

Consider the set $\mathfrak{B}$ of all ordered pairs (B, β), where B is a subspace of A and β is a quasi-orthogonal mapping of $\mathfrak{S}(B)$. It is easy to verify that a partial order is defined on $\mathfrak{B}$ by the relation $\leqq$ defined thus: $(B, \beta) \leqq (C, \gamma)$ if $B \subseteq C$ and $(B \cap V)\beta \subseteq (C \cap V)\gamma$ for all V in $\mathfrak{S}(A)$.

Lemma 11. *If $\mathfrak{L}$ is proper then $\mathfrak{B}$ is an inductive set. In particular, let $\mathfrak{C}$ be a totally ordered subset of $\mathfrak{B}$ and let $C = \cup \{P\colon (P, \varrho) \in \mathfrak{C}\}$. Then there exists a quasi-orthogonal mapping γ of $\mathfrak{S}(C)$ such that (C, γ) is an upper bound for $\mathfrak{C}$.*

PROOF. By Lemma 10, we know that C is in fact, as assumed in the statement of the lemma, a subspace of A. Define γ thus:

---

[5]) Our thanks to Dr. W. N. EVERITT for suggesting this term.

$$X\gamma = \cup\{(X\cap P)\varrho\colon (P,\varrho)\in\mathfrak{C}\},\ (X\in\mathfrak{S}(C)).$$

Then γ is a quasi-orthogonal mapping of $\mathfrak{S}(C)$. For, firstly, for any (P,ϱ) in $\mathfrak{C}$, $X\cap P\subseteq X$ and $(X\cap P)\varrho\subseteq X\gamma$. Hence $[(X\cap P)\cup(X\cap P)\varrho]^*\subseteq(X\cup X\gamma)^*$, i. e. $P\subseteq(X\cup X\gamma)^*$. Thus $\cup\{P\colon (P,\varrho)\in\mathfrak{C}\}\subseteq(X\cup X\gamma)^*$, i. e. $C\subseteq(X\cup X\gamma)^*$. Since, necessarily, $(X\cup X\gamma)^*\subseteq C$, this proves property O1.

Now let $x\in X\cap X\gamma$. Then $x\in X\cap P$ and $x\in(X\cap P')\varrho'$ for some (P,ϱ), (P',ϱ') in $\mathfrak{C}$. Since $\mathfrak{C}$ is totally ordered either (a) $(P,\varrho)\leqq(P',\varrho')$ or (b) $(P',\varrho')\leqq(P,\varrho)$. In case (a) $P\subseteq P'$ and so $x\in X\cap P'$; Thus $x\in(X\cap P')\cap\cap(X\cap P')\varrho'=\square^*$. In case (b) $(X\cap P')\varrho'\subseteq(X\cap P)\varrho$ and so $x\in(X\cap P)\cap\cap(X\cap P)\varrho=\square^*$. Thus $X\cap X\gamma\subseteq\square^*$. Since $X\cap X\gamma$ is a subspace we also have $\square^*\subseteq X\cap X\gamma$. Thus $X\cap X\gamma=\square^*$; and this proves that γ has property O2.

It remains to show that O3 holds. But this is clear. For if $X\subseteq Y$ then $X\cap P\subseteq Y\cap P$ and so $(X\cap P)\varrho\supseteq(Y\cap P)\varrho$ for all (P,ϱ) in $\mathfrak{C}$. Thus from the definition of γ we have $X\gamma\supseteq Y\gamma$; and this completes the proof that γ is quasi-orthogonal.

Hence we have shown that $(C,\gamma)\in\mathfrak{B}$. It is clear that (P,ϱ) in $\mathfrak{C}$ implies that $(P,\varrho)\leqq(C,\gamma)$. Consequently the set $\mathfrak{C}$ has (C,γ) as an upper bound in $\mathfrak{B}$. This completes the proof of the lemma.

Lemma 12. *Let $(M,\mu)\in\mathfrak{B}$, $a\in A\setminus M$, and $N=(M\cup a)^*$. Define λ thus:*

$$X\lambda=\begin{cases}(X\mu\cup a)^*, & \text{if } X\in\mathfrak{S}(M),\\ (X\cap M)\mu, & \text{if } X\in\mathfrak{S}(N)\setminus\mathfrak{S}(M).\end{cases}$$

Then λ is a quasi-orthogonal mapping of $\mathfrak{S}(N)$ and $(M,\mu)\leqq(N,\lambda)$.

Proof. If $X\in\mathfrak{S}(M)$, then $(X\cup X\lambda)^*=(X\cup(X\mu\cup a)^*)^*\supseteq((X\cup X\mu)^*\cup a)^*==(M\cup a)^*=N$. Hence $(X\cup X\lambda)^*=N$. If $X\in\mathfrak{S}(N)\setminus\mathfrak{S}(M)$ then there exists y in $X\setminus M$ and, since M is a subspace, $y\overline{\mathfrak{L}}M$. But $y\mathfrak{L}M\cup a$, since $y\in(M\cup a)^*$. Hence $a\mathfrak{L}M\cup y$; whence it follows that $(M\cup y)^*=N$. It now follows that $(X\cup X\lambda)^*=(X\cup(X\cap M)\mu)^*\supseteq((X\cap M)\cup(X\cap M)\mu\cup y)^*\supseteq(M\cup y)^*$. Hence again $(X\cup X\lambda)^*=N$. Thus we have shown that λ has property O1.

To prove that λ has property O2, note firstly that if $X\in\mathfrak{S}(M)$ then $(X\mu\cup a)^*\cap M=X\mu$. For otherwise there exists y in M such that $y\mathfrak{L}X\mu\cup a$ and $y\overline{\mathfrak{L}}X\mu$. This would imply that $a\mathfrak{L}X\mu\cup y\subseteq M$ which is contrary to hypothesis. Hence when $X\in\mathfrak{S}(M)$ we have $X\cap X\lambda=X\cap(X\mu\cup a)^*=X\cap(X\mu\cup a)^*\cap\cap M=X\cap X\mu=\square^*$. When $X\in\mathfrak{S}(N)\setminus\mathfrak{S}(M)$ then $X\cap X\lambda=X\cap(X\cap M)\mu==(X\cap M)\cap(X\cap M)\mu=\square^*$. Thus λ has property O2.

Consider $X\subseteq Y$. If $Y\in\mathfrak{S}(M)$ then $X\lambda=(X\mu\cup a)^*\supseteq(Y\mu\cup a)^*=Y\lambda$. If $X\in\mathfrak{S}(M)$, $Y\in\mathfrak{S}(N)\setminus\mathfrak{S}(M)$ then $X\lambda=(X\mu\cup a)^*=((X\cap M)\mu\cup a)^*\supseteq\supseteq(X\cap M)\mu\supseteq(Y\cap M)\mu=Y\lambda$. If $X,Y\in\mathfrak{S}(N)\setminus\mathfrak{S}(M)$ then $X=(X\cap M)\mu\supseteq\supseteq(Y\cap M)\mu=Y\lambda$. Hence in all cases $X\subseteq Y$ implies $X\lambda\supseteq Y\lambda$. This completes the proof that λ is quasi-orthogonal.

It remains to show that $(M, \mu) \leqq (N, \lambda)$. But this is clear, for $M \subseteq N$ and if $X \in \mathfrak{S}(M)$ then $X\mu \subseteq (X\mu \cup a)^* = X\lambda$ and if $X \in \mathfrak{S}(N) \setminus \mathfrak{S}(M)$ then $(X \cap M)\mu = X\lambda$.

We can now easily prove our theorem.

Theorem 2. *Let $\mathfrak{L}$ be a proper dependence relation on A. Then there exists a quasi-orthogonal mapping $\pi\colon \mathfrak{S}(A) \to \mathfrak{S}(A)$.*

Proof. Well-order in any manner the elements of A so that $A = \{x_\alpha\colon \alpha < \eta\}$. Let $X_\alpha = \{x_\beta\colon \beta < \alpha\}^*$, so that, in particular, $X_1 = \square^*$. There exists a quasi-orthogonal mapping $\pi_1\colon \mathfrak{S}(\square^*) \to \mathfrak{S}(\square^*)$ namely the mapping which maps $\square^*$ onto $\square^*$. We proceed by means of a transfinite construction.

Suppose that quasi-orthogonal mappings $\pi_\beta\colon \mathfrak{S}(X_\beta) \to \mathfrak{S}(X_\beta)$ have already been constructed for $\beta < \alpha$ and such that if $\beta < \gamma < \alpha$ then $(X_\beta, \pi_\beta) \leqq (X_\gamma, \pi_\gamma)$. We may then apply Lemmas 11 and 12 to construct $\pi_\alpha\colon \mathfrak{S}(X_\alpha) \to \mathfrak{S}(X_\alpha)$ as follows.

When α is a limit ordinal note firstly that, by Lemma 10, $X_\alpha = \bigcup \{X_\beta\colon \beta < \alpha\}$, and hence we may apply Lemma 11 to construct a quasi-orthogonal mapping $\pi_\alpha\colon \mathfrak{S}(X_\alpha) \to \mathfrak{S}(X_\alpha)$ such that $(X_\alpha, \pi_\alpha) \geqq (X_\beta, \pi_\beta)$ for $\alpha > \beta$. When α is not a limit ordinal then $X_\alpha = (X_{\alpha-1} \cup x_{\alpha-1})^*$ and we may apply Lemma 12 to obtain a pair (X_α, π_α) with the required properties.

The construction terminates at a quasi-orthogonal mapping $\pi (= \pi_\eta)\colon \mathfrak{S}(A) \to \mathfrak{S}(A)$.

Bibliography

[1] H. Whitney, On the abstract properties of linear dependence, *Amer. J. Math.* **57** (1935), 509—533.

[2] R. Rado, A theorem on independence relations, *Quart. J. Math. Oxford Ser.* **13** (1943), 83–89.

[3] R. Rado, Axiomatic treatment of rank in infinite sets, *Canad. J. Math.* **1** (1949), 337—343.

[4] T. Lazarson, The representation problem for independence functions, *J. London Math. Soc.* **33** (1958), 21—25.

[5] A. W. Ingleton, A note on independence functions and rank, *J. London Math. Soc.* **34** (1959), 49—56.

[6] v. B. Banaschewski, Totalgeordnete Moduln, *Arch. Math.* **7** (1957), 430—440.

[7] B. L. v. d. Waerden, Moderne Algebra. I, *Berlin*, 1937.

[8] A. Kertész, On independent sets of elements in algebra, *Acta Sci. Math. Szeged* **21** (1960), 260—269.

(Received July 22, 1959.)