A theorem on abstract linear dependence relations

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In the preceding paper [1] M. N. BLEICHER and G. B. PRESTON study a certain definition of an abstract linear dependence relation on a set. In the present paper, the same definition (Definition 1) is considered and some other results are obtained. The Corollary to the Theorem in section 3 below settles affirmatively a conjecture of G. B. PRESTON'S regarding a strong form of Steinitz Exchange Theorem.

§ 1. Terminology

In this paper $\emptyset$ denotes the empty set, $A \setminus B$ the set of elements in A, but not in B. Here x, instead of $\{x\}$, will be used to denote the set whose only element is x if no confusion is likely.

Definition 1. A relation $<$ between subsets of a set A ($X<Y$ will be written and X will be said to be $<$ dependent on Y) is called a linear dependence relation on A, or briefly a *dependence relation on A if* it satisfies the following four laws:

1.1. If $X \subset Y$ then $X<Y$ where $X, Y \subset A$.

1.2. If $X_t<Y$ for all t in any index set T then $(\cup\{X_t | t \in T\})<Y$, where $X_t, Y \subset A$.

1.3. If $X<Y$ and $Y<Z$ then $X<Z$.

1.4. If $y<X$ and $y \not< X \setminus x$ then $x<(X \setminus x) \cup y$ for any elements x, y in A.[1)]

In this paper $<$ is used exclusively in the sense of Definition 1. For the purpose of well ordering a set the symbol $\prec$ is used.

Set theoretic inclusion is a dependence relation on A. Other examples are given by dependence in vector spaces. Hence all results proved apply to these important examples.

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1) Compare 1.1, 1.3, and 1.4 with the well-known axioms of abstract linear dependence in [2] pp. 65—66 and [3] § 36.

The following are definitions where it is considered that everything takes place in a fixed set A on which $<$ is a dependence relation. X is a $<$ independent set if for all $x \in X$, $x \nless X \setminus x$. If X is not independent then it is a $<$ dependent set. If $X<Y$ and $Y<X$ then X is $<$ equivalent to Y. We write $I(X)$ for the class of all independent subsets of X and $M(X)$ for the class of all maximal (relative to inclusion) independent subsets of X. $S(X)$ will denote the class of all subsets of X. If $A<B\subset A$, then we say $B \in G(A)$. We say $<$ is of finite character in A if for all $B\subset A$, B is $<$ independent if and only if every finite subset of B is $<$ independent. By well known results on properties of finite character any $<$ independent subset is contained in a maximal such.

§ 2. Properties of linear dependence relations

The following result is well known [2].

Lemma 2.1. *If* $X \in I(A)$ *and* $X \cup y \notin I(A)$ *then* $y < X$.

From Lemma 2.1. we obtain the following.

Lemma 2.2. $M(A) \subset G(A)$.

PROOF. Let $M \in M(A)$, and suppose $a \in A$, $a \nless M$. Then by Lemma 2.1. $M \cup a \in I(A)$. Hence M was not maximal which contradicts $M \in M(A)$.

In the remainder of this paper $<$ will be considered to be of finite character.

§ 3. A function theorem

Theorem. *Let* $<$ *be a dependence relation of finite character on a set A. Let* $B \in G(A)$. *Then there exists a function* Π_B: $S(A) \to I(B)$ *such that*

(1) $C \cup \Pi_B(C) \in G(A)$ *for each* $C \in S(A)$.

(2) *If* $x \in C$, $x \nless \emptyset$, *then* $x \nless \Pi_B(C)$, *and if* $y \in \Pi_B(C)$ *then* $y \nless (C \cup \Pi_B(C)) \setminus y$.

(3) *If* $C_1, C_2 \in S(A)$, $C_1 < C_2$ *then* $\Pi_B(C_2) \subset \Pi_B(C_1)$.

(4) $C \cap \Pi_B(C) = \emptyset$, *for each* $C \in S(A)$.

(5) *If* $C_1, C_2 \in S(A)$ *are* $<$ *equivalent, then* $\Pi_B(C_1) = \Pi_B(C_2)$.

(6) *If* $D \in G(A)$ *then* $\Pi_B(D) = \emptyset$.

(7) $\Pi_B(\emptyset) \in M(A)$.

PROOF. Well order B with respect to $\prec$. Let $C \in S(A)$. Form $\Pi_B(C)$ as follows. Inspect each element of B in succession with respect to $\prec$ and if $b \in B$ and all elements of B less than b have already been checked, do as follows with b: If $b \nless C \cup \{$those elements of B both less than b and alread y placed in $\Pi_B(C)\}$, then place b in $\Pi_B(C)$. Otherwise leave b out of $\Pi_B(C)$.

That Π_B is into $I(B)$ will follow when (2) is proved below.

Proof of (1). By construction of $C \cup \Pi_B(C)$, if $x \in A$ then either $\{x\} \subset C \cup \Pi_B(C)$ or $x < K \subset C \cup \Pi_B(C)$. Hence by 1. 1. of Definition 1, $x < C \cup \Pi_B(C)$. (1) then follows from 1. 1. and 1. 2. of Definition 1.

Proof of (2). *A*. If $x \in C$, $x \nless \emptyset$, then $x \nless \Pi_B(C)$. For suppose $x < \Pi_B(C)$. Then $x < S \subset \Pi_B(C)$ where S is finite and of minimal cardinal (for $<$ is of finite character). Hence by 1. 4. of Definition 1 there is a $y \in S$ such that $y < (S \setminus y) \cup x$. But this contradicts the construction of $\Pi_B(C)$ in view of Lemma 2. 1. Hence $x \nless \Pi_B(C)$ if $x \in C$.

B. If $y \in \Pi_B(C)$, then $y \nless (C \cup \Pi_B(C)) \setminus y$. The proof of B is similar to that of A and is omitted. Note: In view of B the fact that Π_B is into $I(B)$ follows at once.

Proof of (3). Suppose (3) false. Then let $C_3 = \Pi_B(C_2) \setminus \Pi_B(C_1) \neq \emptyset$. Let x be the first element of C_3 with respect to $\prec$. Then $x < C_1 \cup \{y \mid y \in B, y \prec x, \text{ and } y \in \Pi_B(C_1)\}$. Otherwise x would be in $\Pi_B(C_1)$. But $C_1 \cup \{y \mid y \in B, y \prec x, \text{ and } y \in \Pi_B(C_1)\} < C_2 \cup \{y \mid y \in B, y \prec x, \text{ and } y \in \Pi_B(C_2)\}$ because $C_1 < C_2$ while $\{y \mid y \in B, y \prec x, \text{ and } y \in \Pi_B(C_1)\} < \{y \mid y \in B \text{ and } y \prec x\} < C_2 \cup \{y \mid y \in B, y \prec x, \text{ and } y \in \Pi_B(C_2)\}$ by 1. 1. of Definition 1 and the way $\Pi_B(C_2)$ is constructed. Then $x < ((C_2) \cup \Pi_B(C_2)) \setminus x$ because of 1. 3. of Definition 1. Hence by (2), $x \notin \Pi_B(C_2)$. But $x \in C_3 \subset \Pi_B(C_2)$. This is a contradiction. Hence x does not exist and consequently C_3 has no first element. Since C_3 is well ordered it follows that $C_3 = \emptyset$. Thus we have that $\Pi_B(C_2) \setminus \Pi_B(C_1) = \emptyset$. This implies $\Pi_B(C_2) \subset \Pi_B(C_1)$.

Proof of (4). This is immediate from the construction of $\Pi_B(C)$.

Proof of (5). By (3) if C_1 and C_2 are $<$ equivalent then $\Pi_B(C_1) \subset \subset \Pi_B(C_2) \subset \Pi_B(C_1)$. Hence $\Pi_B(C_1) = \Pi_B(C_2)$.

Proof of (6). If $D \in G(A)$ then $\Pi_B(D) = \emptyset$ by construction.

Proof of (7). This is immediate from (1) and (2).

Corollary. *Let A be a set and let $<$ be a linear dependence relation of finite character on A. Then there exists a function Π: $I(A) \to I(A)$ such that if $S, S_1, S_2 \in I(A)$ then*

(α) *$S \cup \Pi(S)$ is a maximal $<$ independent subset of A.*

(β) *If $S_1 < S_2$ then $\Pi(S_2) \subset \Pi(S_1)$.*

(γ) *$S \cap \Pi(S) = \emptyset$.*

Proof. It is clear that Π_B in the Theorem induces a function Π: $I(A) \to I(B) \subset I(A)$ defined by $\Pi = \Pi_B | I(A)$. Then (β) of this corollary follows from (3) of the Theorem and (γ) of this corollary follows from (4) of the Theorem.

To show (α) observe that if $S \in I(A)$ then by definition of Π and (1) of the Theorem, $S \cup \Pi(S) \in G(A)$. Hence, $S \cup \Pi(S)$ will be maximal independent if it is independent. Suppose $S \cup \Pi(S)$ is not independent. Then there is an $x \in S \cup \Pi(S)$ such that $x < (S \cup \Pi(S)) \setminus x$. Then by virtue of (3) of the Theorem $x \in S$. By the finite character of $<$, $x < \{x_1, \ldots, x_n\} = K$, K is of minimal cardinal (i. e. there is no set $L \subset (S \cup \Pi(S)) \setminus x$ such that $x < L$ and Card L is less than Card K), $K \subset (S \cup \Pi(S)) \setminus x$, and $K \cap \Pi(S) \neq \emptyset$ since S is $<$ independent. Hence there is a $y \in K \cap \Pi(S)$ such that $x \nless K \setminus y$. Hence $y < (K \setminus y) \cup x$ by 1.4. of Definition 1. Therefore $y < (S \cup \Pi(S)) \setminus y$ by 1.3 of Definition 1. This contradicts (3) of the Theorem. Hence $S \cup \Pi(S)$ is independent and the Corollary is true.

Remark on (5) *of the Theorem*: The question arises, „Is the converse of (5) true, that is, if $\Pi_B(C_1) = \Pi_B(C_2)$, are C_1 and C_2 necessarily $<$ equivalent?" That the answer is negative is shown by example. Let $A = \{a, b, c, d\}$. Let any subset with at least two elements be in $G(A)$. But let no subset of only one element be in $G(A)$. It is easily seen that Definition 1 is satisfied and $<$ is of finite character. Then if one lets B of the Theorem be $\{a, b\}$, well ordered so that $a \prec b$, it is seen that $\Pi_B(\{c\}) = \{a\} = \Pi_B(\{d\})$. But $\{c\}$ and $\{d\}$ are not $<$ equivalent because of the construction of $G(A)$.

A. Kertész and G. B. Preston (independently of each other) have pointed-out to me that the Corollary above implies the Theorem 2 of Bleicher and Preston [1].

Bibliography

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