A comment on E. Fried's Galois modules

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In [2], E. FRIED examines the following situation: K is a field, Γ is a finite-dimensional algebra over K and L is a finitely generated right Γ-module satisfying the following conditions

(1) If $M = \mathrm{Hom}_K(L, L)$ is the endomorphism ring of L, Then $\Gamma M = M$.

(2) If J is any right ideal of Γ, $d(JM)/d(M) = d(J)/d(\Gamma)$,

where d denotes dimension over K. The aim of this note is to show (in Theorem 2 and its corollaries) that these modules L can be characterized as special injective modules which are uniquely determined by their dimensions, and that $L = \Gamma$ satisfies (1) and (2) if and only if Γ is a Frobenius algebra (cf. [1]).

We shall assume throughout that Γ is an algebra with unit and that all modules are unitary, so that (1) is automatic. It is probably possible to avoid this assumption by judicious adjunction of a unit.

If M is any Γ-module and m is a nonnegative integer, we shall use mM to denote the direct sum of m isomorphic copies of M. In these terms we define a left Γ-module M to be *semifree* (called „regular" by NAKAYAMA [3]) if $mM \simeq n\Gamma$ for some nonzero m and n ($\simeq$ means isomorphism of left Γ-modules). If $m = 1$, M is free. We shall call a left Γ-module M a *Fried module* if M satisfies (2). The following two lemmas are then obvious:

Lemma 1. *The left Γ-module Γ is both semifree and a Fried module.*

Lemma 2. *If $mM \simeq nN$ then M is a Fried module (resp. semifree) if and only if N is a Fried module (resp. semifree).*

Theorem 1. *M is a Fried module if and only if M is semifree.*

PROOF. If M is semifree, then $mM \simeq n\Gamma$ and the fact that M is a Fried module follows immediately from Lemmas 1 and 2.

Conversely, consider first the case where Γ is a simple algebra. Then Γ has a unique simple (= irreducible) left module V and every finitely generated Γ-module is isomorphic to some nV. Thus every finitely generated module is semifree, by Lemmas 1 and 2.

Next, if Γ is semisimple, write $\Gamma = J_1 \oplus \cdots \oplus J_k$ where each J_i is a simple algebra. Then $M = \oplus J_i M$ and, by the previous case, $m_i J_i M \simeq n_i J_i$ for some m_i and n_i. But $d(\Gamma)d(J_i M) = d(M)d(J_i)$ since M is a Fried module. Thus $n_i/m_i = d(M)/d(\Gamma)$ is independent of i.

If m is the least common multiple of the m_i, write $n_i/m_i = n/m$; then $mM = \oplus mJ_iM = \oplus (m/m_i)m_iJ_iM \simeq \oplus (m/m_i)n_iJ_i = \oplus nJ_i = n\Gamma$, so that M is semifree.

Finally, if Γ has a radical R, we consider the semisimple algebra Γ/R. If M is a Fried module over Γ, then M/RM is a Fried module over Γ/R, for if J/R is any right ideal in Γ/R (J a right ideal in Γ containing R), then

$$\begin{aligned}
&d([J/R][M/RM])/d(M/RM) = d(JM/RM)/d(M/RM) = \\
&= [d(JM) - d(RM)]/[d(M) - d(RM)] = \\
&[d(JM)/d(M) - d(RM)/d(M)]/[1 - d(RM)/d(M)] = \\
&[d(J)/d(\Gamma) - d(R)/d(\Gamma)]/[1 - d(R)/d(\Gamma)] = d(J/R)/d(\Gamma/R).
\end{aligned}$$

Thus M/RM is semifree; say $m(M/RM) \simeq n(\Gamma/R)$. Let $M' = mM$ and $N = n\Gamma$. Then $N/RN \simeq M'/RM'$. This gives a Γ-homomorphism α of N to M'/LM'. Since N is free, α can be lifted to a homomorphism α' of N to M'. Then $\alpha'(N) + RM' = M'$, whence, by usual techniques, $\alpha'(N) + R^k M' = M'$, $\alpha(N) = M'$. Since M' is a Fried module,

$$d(M'/RM') = d(M') - d(RM') = d(M')\big(1 - d(R)/d(\Gamma)\big) = d(M')d(\Gamma/R)/d(\Gamma).$$

Similarly,

$$d(N)d(\Gamma/R)/d(\Gamma) = d(N/RN) = d(M'/RM') = d(M')d(\Gamma/R)/d(\Gamma),$$

which proves $d(N) = d(M')$. This, paired with $\alpha'(N) = M'$ shows that α' is an isomorphism, $mM = M' \simeq N = n\Gamma$, M is semifree.

Corollary. *If M and N are Fried modules, then M is isomorphic to N if and only if $d(M) = d(N)$.*

Proof. Write M, N and Γ as direct sums of indecomposable modules: $M = \oplus m_i I_i$, $N = \oplus n_i I_i$, $\Gamma = \oplus c_i I_i$. Since $mM \simeq m'\Gamma$ and $nN \simeq n'\Gamma$ by Theorem 1, we have $mm_i = m'c_i$ and $nn_i = n'c_i$ by the Krull—Schmidt—Remak theorem. Thus $m_i = (m'/m)c_i$ and $n_i = (n'/n)c_i$. But $d(M) = d(N)$ implies $m'/m = n'/n$, so $m_i = n_i$ and $M \simeq N$.

Now to return to Fried's context, if L is a right Γ-module and $M = \mathrm{Hom}_K(L, L)$, let L' be the dual $\mathrm{Hom}_K(L, K)$ of L. The transposes of elements of M act on L' in the usual way, making L' a left M-module and hence also a left Γ-module. Moreover, the left (M—, hence Γ—) module M is isomorphic to a direct sum of copies of L', so M is a Fried module if and only if L' is. Translating Theorem 1 into conditions on dual modules, we have

Theorem 2. *The following are equivalent*

(3) *A right Γ-module L satisfies* (1) *and* (2).

(4) *The dual $L' = \mathrm{Hom}_K(L, K)$ of L is a semifree left Γ-module, i. e., $mL' \cong n\Gamma$ as left Γ-modules.*

(5) *There exist nonzero integers m and n such that $mL \cong n\,\mathrm{Hom}_K(\Gamma, K)$ as right Γ-modules.*

(6) *If $L = \oplus\, l_i I_i$ and $\mathrm{Hom}_K(\Gamma, K) = \oplus\, c_i' I_i$ with nonisomorphic, indecomposable I_i, then $l_i/l_j = c_i'/c_j'$ for all i, j.*

Corollary 1. *If two right Γ-modules L_1* and *L_2 satisfy the conditions of Theorem 2, then $L_1 \cong L_2$ if and only if $d(L_1) = d(L_2)$.*

This Corollary is proved exactly as is the Corollary to Theorem 1; or it can be reduced to that Corollary by taking duals.

Corollary 2. *The right Γ-module Γ satisfies the conditions of Theorem 2 if and only if Γ is a Frobenius algebra; i. e., the right module Γ (which gives the right regular representation of Γ) is isomorphic to $\mathrm{Hom}_K(\Gamma, K)$ (which gives the left regular representation).*

PROOF. The „if" part is clear from Theorem 2, (5). The „only if" follows from Corollary 1.

Thus the major theorem of [2] (Theorem 7) can be translated into the assertion that if Γ is a Frobenius algebra, if a right Γ-module L is the dual of a semifree left module, and if $d(L) = d(\Gamma)$, then $L \cong \Gamma$. The proof of the normal basis theorem then consists in showing that the group algebra is a Frobenius algebra [2, Theorem 9] (cf. also [1]), that the field L is the dual of a semifree module over the group algebra of the Galois group [1, Theorem 10] and that $d(L) = d(\Gamma)$.

Bibliography

[1] S. EILENBERG and T. NAKAYAMA, On the dimension of modules and algebras. II., *Nagoya Math. J.* **9** (1955), 1—16.

[2] E. FRIED, On Galois modules of vector spaces. *Publ. Math. Debrecen* **6** (1959), 101—110.

[3] T. NAKAYAMA, Galois theory for general rings with minimum condition, *J. Math. Soc. Japan* **1** (1949), 203—216.

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