On the valuations of complemented modular lattices of finite length

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By a *valuation* of a lattice L is meant a single-valued function v of L into itself which has the following property: for each pair x, y of elements of L

$$v(x)+v(y)=v(x\cap y)+v(x\cup y).$$

Let L be a lattice of finite length with the least element o and let $d(x)$ denote the length of the interval sublattice $[o, x]$. Then d is a single-valued function called the *height function* of L. If L is also modular, then by a well-known result of DEDEKIND ([1], p. 68) this function is a valuation of L.

In the theory of continuous geometries there is known ([2], p. 119) the following

Theorem 1. *Let L be a complemented modular simple lattice of finite length and d the height function of L. If v is any valuation of L, then there exist real constants $\varkappa$, λ such that*

$$v(x)=\varkappa d(x)+\lambda \tag{1}$$

for every element x of L.

In this note we give a proof of elementary character for this theorem.

By our assumptions, every element x of L may be represented as

$$x=p_1\cup\cdots\cup p_r \qquad (r \text{ finite})$$

where each p_j $(j=1,\ldots,r)$ is an atom of L ([1], p. 105). Without loss of generality we may assume the set $\{p_1\ldots,p_r\}$ to be independent ([1], p. 104); then

$$d(x)=r \tag{2}$$

and

$$v(x)=v(p_1\cup\cdots\cup p_{r-1})+v(p_r)-v((p_1\cup\cdots\cup p_{r-1})\cap p_r)=$$
$$=v(p_1\cup\cdots\cup p_{r-1})+v(p_r)-v(o).$$

The subset $\{p_1,\ldots,p_{r-1}\}$ is a fortiori independent. Therefore we get, by induction,

$$v(x)=v(p_1)+\cdots+v(p_r)+(r-1)\,v(o). \tag{3}$$

Since L is simple, to each pair p_j, p_k $(j, k = 1, \ldots, r; j \neq k)$ there exists a p_l (with $1 \leqq l \leqq r$) such that

$$p_j \cup p_k = p_k \cup p_l = p_l \cup p_j$$

([1], pp. 120—121). It follows

$$\begin{aligned} v(p_j) &= v(p_j \cup p_l) + v(p_j \cap p_l) - v(p_l) = \\ &= v(p_j \cup p_l) + v(o) - v(p_l) = \\ &= v(p_k \cup p_l) + v(p_k \cap p_l) - v(p_l) = v(p_k) \end{aligned}$$

for each pair j, k; consequently

$$v(p_1) = \cdots = v(p_r) = \mu.$$

Thus (3) may be written as follows:

$$\begin{aligned} v(x) &= r\mu + (r-1)\, v(o) = \\ &= r(\mu + v(o)) - v(o). \end{aligned}$$

Taking $\mu + v(o) = \varkappa$ and $-v(o) = \lambda$, by (2) we get (1).

It is easy to see that if L is not simple, then the assertion of Theorem 1 does not remain valid. Consider, for example, the distributive lattice $D = \{o, a, b, i\}$ with the defining relations $o < a < i$, $o < b < i$, $a \cap b = o$, $a \cup b = i$ and define the function v by $v(o) = 2$, $v(a) = 6$, $v(b) = 7$, $v(i) = 11$. Then v is a valuation of D which obviously cannot be represented in the form (1).

In what follows we deal with complemented modular lattices of finite length which are not necessarily simple.

It is known ([1], pp. 120—121) that any complemented modular lattice L of finite length may be represented as a direct union

$$L = L_1 \times \cdots \times L_n \qquad (n \text{ finite}) \tag{4}$$

of some simple lattices $L_1, \ldots, L_n$. Obviously, each L_j $(j = 1, \ldots, n)$ is again a complemented modular lattice of finite length. Using Theorem 1, we prove the following

Theorem 2. *Let L be a complemented modular lattice of finite length having the direct decomposition* (4) *and let d_j $(j = 1, \ldots, n)$ denote the height function of L_j. If v is any valuation of L, then there exist real constants $\lambda_1, \ldots, \lambda_n, \mu$ such that*

$$v(x) = \sum_{j=1}^{n} \lambda_j d_j(x_j) + \mu \tag{5}$$

for every element x of L, where x_j denotes the j-th component of x in the direct decomposition (4).

PROOF. Let o_j $(j=1,\dots,n)$ denote the least element of L_j. It is easy to see that the mapping v_j defined by

$$v_j(x_j)=v(o_1,\dots,o_{j-1},\ x_j,\ o_{j+1},\dots,o_n)\qquad (x_j\in L_j)$$

is a valuation of L_j. Therefore, by a straightforward calculation we get

$$\begin{aligned} v(x)=v(x_1,\dots,x_n)&=v((x_1,\dots,x_{n-1},o_n)\cup(o_1,\dots,o_{n-1},x_n))=\\ &=v(x_1,\dots,x_{n-1},o_n)+v(o_1,\dots,o_{n-1},x_n)-v(o)=\\ &=v(x_1,\dots,x_{n-1},o_n)+v_n(x_n)-v(o).\end{aligned}$$

Hence, by induction for n,

$$v(x)=\sum_{j=1}^{n} v_j(x_j)-(n-1)\,v(o). \tag{6}$$

But, by Theorem 1 there exist real constants λ_j, μ_j $(j=1,\dots,n)$ such that

$$v_j(x_j)=\lambda_j d_j(x_j)+\mu_j\,.$$

Thus (6) may be written in the form

$$v(x)=\sum_{j=1}^{n}\lambda_j d_j(x_j)+\sum_{j=1}^{n}\mu_j-(n-1)\,v(o).$$

Taking

$$\sum_{j=1}^{n}\mu_j-(n-1)\,v(o)=\mu,$$

we get (5). Hence our theorem is proved.

Bibliography

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