The solution of a minimum problem

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1. In his papers [1] and [2] G. SZEGŐ solves among others the following problem: Let $f(x)$ be a nonnegative (L) integrable function defined on the interval $(0, 2\pi)$, for which

$$A(f) = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx > 0$$

holds.

Let moreover α be an arbitrary complex number. Suppose $p_n(z) = \sum_{k=0}^{n} a_k z^k$ runs through all polynomials of degree n, for which

$$p_n(\alpha) = 1 \tag{1}$$

holds. What will then be the lower limit of the integrals

$$\frac{1}{2\pi}\int^{2\pi} |p_n(z)|^2 f(x)\,dx \qquad (z = e^{ix}).$$

SZEGŐ has proved that when

$$\mu_n(\alpha, f) = \text{Min}\,\frac{1}{2\pi}\int_0^{2\pi} |p_n(z)|^2 f(x)\,dx \qquad (z = e^{ix}),$$

while $p_n(z)$ runs through all the polynomials of degree n satisfying (1), then

$$\mu_n(\alpha, f) = \frac{1}{s_n(\alpha, \alpha)},$$

and the minimizing polynomial is

$$p_n(z) = \mu_n(\alpha, f)\, s_n(\alpha, z),$$

where $s_n(\alpha, z) = \sum_{\nu=0}^{n} \overline{\varphi_\nu(\alpha)}\,\varphi_\nu(z)$. The polynomial $\varphi_\nu(z)$ $(\nu = 0, 1, \ldots, n)$

occurring in this expression is the $\nu+1$-th member of the system of polynomials, which we get by orthonormalizing the trigonometrical system

$$1, e^{ix}, \ldots, e^{inx}$$

by Schmidt's procedure with respect to the weight function $f(x)$.

As is known, the above results play an important role in the theory of stochastic processes. (See e. g. [3].)

The generalization of the theory of stochastic processes to matrix-valued variables made it necessary to extend the above result of SZEGŐ to quadratic matrices. This generalization is contained in Theorem 3. Theorems 4, 5 and 6 can also be regarded as generalizations of corresponding theorems of SZEGŐ.

2. Here we give a brief survey of the notations and well-known matrix-theoretical concepts used throughout the paper.

The zero matrix resp. the unit matrix of order r (we say also of type $r\times r$) will be denoted by $(\mathbf{0})_r$ and $\mathbf{E}_r$ respectively. The conjugate transpose of a matrix $\mathbf{A}$ of order r will be denoted by $\mathbf{A}^*$, and the inverse of the regular matrix $\mathbf{A}$ by $\mathbf{A}^{-1}$.

The matrix $\mathbf{A}$ of order r will be said to be positive definite, positive semidefinite resp. Hermitian, if for any row-vector $\mathbf{z}=(z_1, z_2, \ldots, z_r)$ the condition

$$\mathbf{z}\mathbf{A}\mathbf{z}^*>0, \ \mathbf{z}\mathbf{A}\mathbf{z}^*\geqq 0 \quad \text{resp.} \quad \mathbf{A}=\mathbf{A}^*$$

holds.

A matrix $\mathbf{A}(x)$ will be said to be bounded or continuous, if all its elements are bounded or continuous. By the integral $\int_a^b \mathbf{A}(x)\,dx$ of a matrix $\mathbf{A}(x)$ we understand the matrix formed of the integrals of its elements.

By the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_r$ of the matrix $\mathbf{A}$ of order r we understand the roots of the equation

$$\operatorname{Det}(\lambda\mathbf{E}_r-\mathbf{A})=0.$$

By the spur of the matrix $\mathbf{A}=(a_{ik})$ $(i, k=1, 2, \ldots, r)$ of order r we understand the sum of the elements standing in the main diagonal, i. e.

$$\operatorname{Sp}\mathbf{A}=\sum_{k=1}^{r} a_{kk}.$$

As is known, $\sum_{k=1}^{r} a_{kk}=\sum_{k=1}^{r}\lambda_k$. The following relations clearly hold:

a) $\operatorname{Sp}(a\mathbf{A}+b\mathbf{B})=a\operatorname{Sp}\mathbf{A}+b\operatorname{Sp}\mathbf{B}$,

where a and b are arbitrary complex numbers,

b) $\operatorname{Sp} \mathbf{AB} = \operatorname{Sp} \mathbf{BA}$,

c) $\operatorname{Sp} \mathbf{U}^{-1} \mathbf{A} \mathbf{U} = \operatorname{Sp} \mathbf{A}$

where $\mathbf{U}$ is an arbitrary regular matrix of order r.

Any Hermitian matrix $\mathbf{A}$ of order r can be represented in the form

$$\mathbf{A} = \mathbf{U} \Lambda \mathbf{U}^*, \tag{2}$$

where $\mathbf{U}\mathbf{U}^* = \mathbf{E}_r$, and Λ is the diagonal matrix containing the eigenvalues of $\mathbf{A}$. The representation (2) is said to be the canonical representation of $\mathbf{A}$. By the square root of the positive semidefinite Hermitian matrix $\mathbf{A}$ of order r we understand the matrix $\mathbf{A}^{\frac{1}{2}}$ which can be obtained from the canonical representation of $\mathbf{A}$, by replacing the diagonal matrix Λ containing the eigenvalues by the diagonal matrix containing the positive square roots of the eigenvalues.

3. Let L_2 denote the totality of the matrices of type $r \times r$ defined on the interval $(-\pi, \pi)$, measurable and with integrable square in the sense of Lebesgue. Let moreover $\mathfrak{M}$ denote the totality of the square matrices of order r which can be built of complex numbers. If on the set L_2 addition is to mean the usual matrix addition, while by multiplication we understand matrix multiplication, then — as it can easily be seen — these operations satisfy conditions 1) a) and b) of § 2 in [4], i. e. L_2 forms a linear space.

To any two elements $\mathbf{f}(x)$ and $\mathbf{g}(x)$ of L_2 we make correspond the matrix

$$(\mathbf{f}, \mathbf{g}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathbf{f}(x) \mathbf{g}^*(x) \, dx$$

from $\mathfrak{M}$. This correspondence satisfies conditions (8) in [4], and therefore we call the integral $(\mathbf{f}, \mathbf{g})$ the inner product of the elements $\mathbf{f}(x)$ and $\mathbf{g}(x)$.

In the metrized linear space L_2, we understand by the norm of $\mathbf{f}(x)$ the matrix $\|\mathbf{f}\| = (\mathbf{f}, \mathbf{f})^{\frac{1}{2}}$, and in case this is regular, we call the matrix $\|\mathbf{f}\|^{-1} \mathbf{f}(x)$ the normed element arising out of the element $\mathbf{f}(x)$.

4. *Definition* 1. We call polynomial matrices the expressions of the form

$$\mathbf{P}_n(z) = \sum_{k=0}^{n} \mathbf{A}_k z^k$$

where z is a complex variable, n a nonnegative integer, and $\mathbf{A}_k$ $(k = 0, 1, \ldots, n)$ an element from $\mathfrak{M}$.

If at least one element of the matrix $\mathbf{A}_n$ is different from zero, then $\mathbf{P}_n(z)$ is a polynomial matrix of exactly the degree n. If, in particular,

$z=e^{ix}$, $-\pi \leqq x \leqq \pi$ then the matrix $\mathbf{P}_n(z)=\mathbf{P}_n(e^{ix})$ is said to be a trigonometric polynomial matrix.

Definition 2. A system

$$\boldsymbol{\varphi}_0(z), \boldsymbol{\varphi}_1(z), \ldots, \boldsymbol{\varphi}_n(z)$$

of polynomial matrices of order r will be called orthonormed with respect to the functional matrix $\mathbf{f}(x)$ of order r, defined on the interval $[-\pi, \pi]$ Hermitian and (L) integrable, if

$$(3) \qquad \frac{1}{2\pi}\int_{-\pi}^{\pi} \boldsymbol{\varphi}_j(z)\mathbf{f}(x)\boldsymbol{\varphi}_k^*(z)\,dx = \begin{cases} (\mathbf{0})_r & \text{for } j \neq k \\ \mathbf{E}_r & \text{for } j=k \end{cases}$$

$$(j, k=0, 1, \ldots, n;\ z=e^{ix}).$$

The condition expressed in Definition 2 is clearly equivalent to the orthonormedness of the functional matrix system

$$\boldsymbol{\varphi}_0(z)\mathbf{f}^{\frac{1}{2}}(x), \boldsymbol{\varphi}_1(z)\mathbf{f}^{\frac{1}{2}}(x), \ldots, \boldsymbol{\varphi}_n(z)\mathbf{f}^{\frac{1}{2}}(x).$$

Consider now the system

$$\mathbf{E}_r, \mathbf{E}_r e^{ix}, \ldots, \mathbf{E}_r e^{inx}$$

of trigonometric polynomial matrices. We orthonormalize this system with respect to the positive definite Hermitian (L) integrable functional matrix $\mathbf{f}(x)$, i. e. by the generalized Schmidt method we orthonormalize the system of functional matrices

$$(4) \qquad \mathbf{f}^{\frac{1}{2}}(x), \mathbf{f}^{\frac{1}{2}}(x)e^{ix}, \ldots, \mathbf{f}^{\frac{1}{2}}(x)e^{inx}.$$

The Schmidt method can only be applied, if the system (4) is independent. By Theorem 3 of [4] it is a necessary and sufficient condition of independence that the Gramian determinant of the system (4) be positive definite. In connection with this we may state the following

Theorem 1. *If* $\mathbf{f}(x)$ *is a positive definite Hermitian* (L) *integrable functional matrix, then the Gramian matrix of the system* (4) *is positive definite.*

PROOF. The Gramian matrix of the system (4) is the matrix

$$\mathbf{R}_{n+1}=\left((\mathbf{f}^{\frac{1}{2}}(x)e^{ikx}, \mathbf{f}^{\frac{1}{2}}(x)e^{ilx})\right)=\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(k-l)x}\mathbf{f}(x)\,dx\right)$$

$$(k, l=0, 1, \ldots, n)$$

of order $(n+1)r$. We show that under the conditions mentioned $\mathbf{R}_{n+1}$ is positive definite. Let indeed $\mathbf{z}=(\mathbf{z}_0, \mathbf{z}_1, \ldots, \mathbf{z}_n)$, $\mathbf{z}_j=(z_j^1, z_j^2, \ldots, z_j^r)$ be an arbitrary row vector of dimension $(n+1)r$ for which $\mathbf{z}\mathbf{z}^*\neq 0$. Then

$$\mathbf{z}\,\mathbf{R}_{n+1}\mathbf{z}^*=\frac{1}{2\pi}\sum_{k=0}^{n}\sum_{l=0}^{n}\int_{-\pi}^{\pi}\mathbf{z}_k e^{ikx}\mathbf{f}(x)e^{-ilx}\mathbf{z}_l^*\,dx=$$

$$=\frac{1}{2\pi}\sum_{k=0}^{n}\sum_{l=0}^{n}\int_{-\pi}^{\pi}\mathbf{z}_k e^{ikx}\mathbf{U}(x)\mathbf{\Lambda}(x)\mathbf{U}^*(x)e^{-ilx}\mathbf{z}_l^*\,dx=$$

$$=\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sum_{k=0}^{n}\mathbf{z}_k e^{ikx}\mathbf{U}(x)\right)\mathbf{\Lambda}(x)\left(\sum_{k=0}^{n}\mathbf{z}_k e^{ikx}\mathbf{U}(x)\right)^*dx,$$

where $\mathbf{U}(x)\mathbf{\Lambda}(x)\mathbf{U}^*(x)$ is the canonical representation of $\mathbf{f}(x)$. Denote the components of the vector $\sum_{k=0}^{n}\mathbf{z}_k e^{ikx}\mathbf{U}(x)$ by $a_k(x)$ and the elements of the diagonal matrix $\mathbf{\Lambda}(x)$ by $\lambda_k(x)$ $k=1, 2, \ldots, r$, then

$$\mathbf{z}\,\mathbf{R}_{n+1}\mathbf{z}^*=\frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{k=1}^{r}|a_k(x)|^2\lambda_k(x)\,dx. \tag{5}$$

If now $\mathbf{z}\mathbf{z}^*\neq 0$, then from the positive definiteness of $\mathbf{f}(x)$ the positivity of (5) follows.

Now we can effect the orthonormalizing of the system (4) as follows.

Let $\mathbf{f}_k(z)=\mathbf{f}_k(e^{ix})=\mathbf{f}^{\frac{1}{2}}(x)e^{ikx}$ $(k=0, 1, \ldots, n)$. Since the functional matrices $\mathbf{f}_k(z)$ are linearly independent, we have

$$\operatorname{Det}(\mathbf{f}_k, \mathbf{f}_k)>0.$$

If the square of the norm of $\mathbf{f}_0(z)$ is denoted by $\mathbf{R}(0)$, then the norm of the matrix $\boldsymbol{\varphi}_0(z)=\mathbf{R}^{-\frac{1}{2}}(0)\mathbf{f}_0(z)$ is $\mathbf{E}_r$, and so the orthonormed system has the first element

$$\boldsymbol{\Phi}_0(z)=\boldsymbol{\varphi}_0(z)\mathbf{f}^{-\frac{1}{2}}(x)=\mathbf{R}^{-\frac{1}{2}}(0). \tag{6}$$

Let now

$$\boldsymbol{\psi}_1(z)=\mathbf{f}_1(z)-\boldsymbol{\alpha}_1\boldsymbol{\varphi}_0(z),$$

where the matrix $\boldsymbol{\alpha}_1$ has been chosen so that $(\boldsymbol{\psi}_1(z), \boldsymbol{\varphi}_0(z))=(\mathbf{O})_r$. By the independence of the system (4) the matrix $\|\boldsymbol{\psi}_1(z)\|^{-1}$ exists. If

$$\boldsymbol{\varphi}_1(z)=\|\boldsymbol{\psi}_1(z)\|^{-1}\boldsymbol{\psi}_1(z),$$

then $\boldsymbol{\varphi}_1(z)$ is orthogonal to $\boldsymbol{\varphi}_0(z)$ and the norm of $\boldsymbol{\varphi}_1(z)$ is $\mathbf{E}_r$. The second element of the orthonormed system is

$$\boldsymbol{\Phi}_1(z) = \boldsymbol{\varphi}_1(z)\,\mathbf{f}^{-\frac{1}{2}}(x) \qquad (z = e^{ix}).$$

Continuing this process we get a system

$$\boldsymbol{\Phi}_0(z), \boldsymbol{\Phi}_1(z), \ldots, \boldsymbol{\Phi}_n(z)$$

of polynomial matrices, satisfying (3). This system is however not uniquely determined. Indeed, one sees from (3) that if $\mathbf{U}_j$ is a constant matrix with complex elements for which $\mathbf{U}_j\mathbf{U}_j^* = \mathbf{E}_r$, holds, then together with $\boldsymbol{\Phi}_j(z)$ also $\mathbf{U}_j\boldsymbol{\Phi}_j(z)$ $(j = 0, 1, \ldots, n)$ satisfies (3). It is however clear from the construction that the coefficient of z^j in $\boldsymbol{\Phi}_j(z)$ is a positive definite Hermitian matrix. Let us now choose the matrix $\mathbf{U}_j$ so that the coefficient of z^j in $\mathbf{U}_j\boldsymbol{\Phi}_j(z)$ shall remain a positive definite Hermitian matrix. Since there exist one and only one such matrix (namely the unit matrix), this condition makes the system $\boldsymbol{\Phi}_j(z)$ $(j = 0, 1, \ldots, n)$ uniquely determined.

Theorem 2. *If $\mathbf{f}(x)$ is a positive definite Hermitian (L) integrable functional matrix, then the Hermitian matrix*

$$\mathbf{S}_j(\alpha, \alpha) = \sum_{k=0}^{j} \boldsymbol{\Phi}_k(\alpha)\boldsymbol{\Phi}_k^*(\alpha) \qquad (j = 0, 1, \ldots, n)$$

is positive definite, where α stands for an arbitrary complex number.

PROOF. Clearly we have for any row vector $\mathbf{z} = (z_1, \ldots, z_r)$ the relation

$$\mathbf{z}\boldsymbol{\Phi}_k(\alpha)\boldsymbol{\Phi}_k^*(\alpha)\mathbf{z}^* = \big(\mathbf{z}\boldsymbol{\Phi}_k(\alpha)\big)\big(\mathbf{z}\boldsymbol{\Phi}_k(\alpha)\big)^* \geqq 0. \tag{7}$$

By the condition for $\mathbf{f}(x)$ the matrix

$$\mathbf{R}(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathbf{f}(x)\,dx$$

and consequently also the matrix $\mathbf{R}^{-1}(0)$ is positive definite Hermitian. Thus, if $\mathbf{z}\mathbf{z}^* \neq 0$, then

$$\mathbf{z}\boldsymbol{\Phi}_0(\alpha)\boldsymbol{\Phi}_0^*(\alpha)\mathbf{z}^* = \mathbf{z}\mathbf{R}^{-1}(0)\mathbf{z}^* > 0. \tag{8}$$

(7) and (8) together say that for any row vector $\mathbf{z}\mathbf{z}^* \neq 0$ the relation

$$\mathbf{z}\mathbf{S}_j(\alpha, \alpha)\mathbf{z}^* = \sum_{k=0}^{j} \mathbf{z}\boldsymbol{\Phi}_k(\alpha)\boldsymbol{\Phi}_k^*(\alpha)\mathbf{z}^* > 0 \qquad (j = 0, 1, \ldots, n)$$

holds.

We shall need also the following inequality: If $\mathbf{A}_k$ and $\mathbf{B}_k$ are arbitrary matrices of order r $(k = 1, \ldots, n)$, then

$$\left|\operatorname{Sp}\sum_{k=1}^{n} \mathbf{A}_k\mathbf{B}_k\right|^2 \leqq \left(\operatorname{Sp}\sum_{k=1}^{n} \mathbf{A}_k\mathbf{A}_k^*\right)\left(\operatorname{Sp}\sum_{k=1}^{n} \mathbf{B}_k\mathbf{B}_k^*\right) \tag{9}$$

and we have equality in (9) if and only if $\mathbf{A}_k = \lambda \mathbf{B}_k^*$, where λ is an arbitrary complex number.

Denote indeed the elements of the matrices $\mathbf{A}_k$ and $\mathbf{B}_k$ by $a_{jl}^{(k)}$ and $b_{jl}^{(k)}$ respectively $(j, l = 1, \ldots, r;\ k = 1, \ldots, n)$. Then by Cauchy's inequality

$$\left| \mathrm{Sp} \sum_{k=1}^{n} \mathbf{A}_k \mathbf{B}_k \right|^2 = \left| \sum_{k=1}^{n} \sum_{j=1}^{r} \sum_{l=1}^{r} a_{jl}^{(k)} b_{lj}^{(k)} \right|^2 \leqq$$

$$\leqq \left(\sum_{k=1}^{n} \sum_{j=1}^{r} \sum_{l=1}^{r} |a_{jl}^{(k)}|^2 \right) \left(\sum_{k=1}^{n} \sum_{j=1}^{r} \sum_{l=1}^{r} |b_{jl}^{(k)}|^2 \right) = \left(\sum_{k=1}^{n} \mathrm{Sp}\, \mathbf{A}_k \mathbf{A}_k^* \right) \left(\sum_{k=1}^{n} \mathrm{Sp}\, \mathbf{B}_k \mathbf{B}_k^* \right)$$

and here equality holds if and only if

$$a_{jl}^{(k)} = \lambda \overline{b_{lj}^{(k)}} \qquad (j, l = 1, \ldots, r;\ k = 1, \ldots, n)$$

where λ is an arbitrary complex number.

Our theorem concerning the minimum problem can now be formulated as follows:

Theorem 3. *If* $\mathbf{f}(x)$ *is a positive definite Hermitian* (L) *integrable functional matrix, then for*

$$\mathbf{P}_n(\alpha) = \mathbf{E}_r \tag{10}$$

we have

$$\mu_n(\alpha, \mathbf{f}) = \mathrm{Min}\ \mathrm{Sp} \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathbf{P}_n(z) \mathbf{f}(x) \mathbf{P}_n^*(z) dx = \mathrm{Sp}\, \mathbf{S}_n^{-1}(\alpha, \alpha) \qquad (z = e^{ix}), \tag{11}$$

where α *is an arbitrary complex number, and* $\mathbf{P}_n(z)$ *runs through all polynomial matrices of degree* n.

REMARK. This theorem is a generalization of the minimum problem already mentioned due to G. SZEGŐ. In the case $r = 1$, the integral on the right hand side of (11) is the so called n-th Toeplitzian form belonging to the function $f(x)$. In the case $r > 1$ this integral is the n-th generalized Toeplitzian form belonging to the functional matrix $\mathbf{f}(x)$. So the poblem is nothing else but to find the minimum of the spur of the n-th generalized Toeplitzian form standing on the right hand side of (11), under the condition (10).

A similar problem has been investigated in the paper [5] by HELSON and LOWDENSLAGER. Their result holds however only for the limiting case $n \to \infty$, and the assumption they make is also different from (10).

PROOF. Since the coefficient of z^k in $\boldsymbol{\Phi}_k(z)$ is a regular matrix, for any polynomial $\mathbf{P}_n(z)$ of degree n there exist matrices $\mathbf{X}_0, \mathbf{X}_1, \ldots, \mathbf{X}_n$ with constant elements, such that

$$\mathbf{P}_n(z) = \mathbf{X}_0 \boldsymbol{\Phi}_0(z) + \mathbf{X}_1 \boldsymbol{\Phi}_1(z) + \cdots + \mathbf{X}_n \boldsymbol{\Phi}_n(z)$$

holds. Then, in view of the orthonormedness with respect to $\mathbf{f}(x)$ of the polynomial matrices $\boldsymbol{\Phi}_k(z)$ $(k=0, 1, \ldots, n)$

$$\mathrm{Sp}\,\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathbf{P}_n(z)\mathbf{f}(x)\mathbf{P}_n^*(z)dx=\sum_{k=0}^{n}\sum_{l=0}^{n}\mathrm{Sp}\,\mathbf{X}_l^*\mathbf{X}_k\frac{1}{2\pi}\int_{-\pi}^{\pi}\boldsymbol{\Phi}_k(z)\mathbf{f}(x)\boldsymbol{\Phi}_l^*(z)dx=$$
$$=\mathrm{Sp}\sum_{k=0}^{n}\mathbf{X}_k\mathbf{X}_k^* \qquad (z=e^{ix}).$$

Thus our task is to determine the minimum of $\mathrm{Sp}\sum_{k=0}^{n}\mathbf{X}_k\mathbf{X}_k^*$ under the condition

$$\sum_{k=0}^{n}\mathbf{X}_k\boldsymbol{\Phi}_k(\alpha)=\mathbf{E}_r. \tag{12}$$

Instead of condition (12) let us consider the condition

$$\mathrm{Sp}\left(\sum_{k=0}^{n}\mathbf{X}_k\boldsymbol{\Phi}_k(\alpha)\right)^{-1}=\mathrm{Sp}\left(\sum_{k=0}^{n}\mathbf{X}_k\boldsymbol{\Phi}_k(\alpha)\right) \tag{13}$$

which certainly holds in case (12) is fulfilled. With (13) we have by the inequality (9)

$$\left|\mathrm{Sp}\left(\sum_{k=0}^{n}\mathbf{X}_k\boldsymbol{\Phi}_k(\alpha)\right)^{-1}\right|^2=\left|\mathrm{Sp}\left(\sum_{k=0}^{n}\mathbf{X}_k\boldsymbol{\Phi}_k(\alpha)\right)\right|^2\leqq\left(\sum_{k=0}^{n}\mathrm{Sp}\,\mathbf{X}_k\mathbf{X}_k^*\right)\mathrm{Sp}\,\mathbf{S}_n(\alpha,\alpha)$$

and here equality holds if and only if

$$\mathbf{X}_k=\lambda\,\boldsymbol{\Phi}_k^*(\alpha),$$

where the constant λ must be chosen so that

$$\mathrm{Sp}\left(\sum_{k=0}^{n}\lambda\,\boldsymbol{\Phi}_k^*(\alpha)\boldsymbol{\Phi}_k(\alpha)\right)^{-1}=\mathrm{Sp}\sum_{k=0}^{n}\lambda\,\boldsymbol{\Phi}_k^*(\alpha)\,\boldsymbol{\Phi}_k(\alpha)$$

shall be valid, i. e. that

$$\frac{1}{\lambda}\,\mathrm{Sp}\,\mathbf{S}_n^{-1}(\alpha,\alpha)=\lambda\,\mathrm{Sp}\,\mathbf{S}_n(\alpha,\alpha)$$

shall hold. Choosing from here the positive value of λ we get

$$\lambda=+\sqrt{\frac{\mathrm{Sp}\,\mathbf{S}_n^{-1}(\alpha,\alpha)}{\mathrm{Sp}\,\mathbf{S}_n(\alpha,\alpha)}}.$$

So, under condition (13), the expression

$$\mathrm{Sp}\,\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathbf{P}_n(z)\,\mathbf{f}(x)\,\mathbf{P}_n^*(z)dx \qquad (z=e^{ix}) \tag{14}$$

has the minimum $\mathrm{Sp}\,\mathbf{S}_n^{-1}(\alpha,\alpha)$, and the minimizing polynomial matrix is

$$\mathbf{P}_n(z) = +\sqrt{\frac{\mathrm{Sp}\,\mathbf{S}_n^{-1}(\alpha,\alpha)}{\mathrm{Sp}\,\mathbf{S}_n(\alpha,\alpha)}}\sum_{k=0}^{n}\boldsymbol{\Phi}_k^*(\alpha)\boldsymbol{\Phi}_k(z). \tag{15}$$

This polynomial matrix satisfies condition (13), but

$$\mathbf{P}_n(\alpha) = +\sqrt{\frac{\mathrm{Sp}\,\mathbf{S}_n^{-1}(\alpha,\alpha)}{\mathrm{Sp}\cdot\mathbf{S}_n(\alpha,\alpha)}}\,\mathbf{S}_n(\alpha,\alpha)$$

is in general no unit matrix. From this it follows only that for a polynomial matrix $\mathbf{P}_n(z)$ satisfying also (12), i. e. for which $\mathbf{P}_n(\alpha)=\mathbf{E}_r$, the inequality

$$\mu_n(\alpha,\mathbf{f}) \geqq \mathrm{Sp}\,\mathbf{S}_n^{-1}(\alpha,\alpha)$$

holds.

For the polynomial matrix $\mathbf{P}_n(z)=\mathbf{S}_n^{-1}(\alpha,\alpha)\,\mathbf{S}_n(\alpha,z)$ we have however

$$\mathrm{Sp}\,\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathbf{P}_n(z)\mathbf{f}(x)\mathbf{P}_n(z)\,dx = \mathrm{Sp}\,\mathbf{S}_n^{-1}(\alpha,\alpha) \qquad (z=e^{ix})$$

and $\mathbf{P}_n(\alpha)=\mathbf{E}_r$. Thus we have given a polynomial matrix for which the integral takes on its minimal value, and which satisfies condition (10). This completes the proof of our theorem.

We still remark that if instead of (13) we had started with the condition

$$\mathrm{Sp}\,\mathbf{P}_n(\alpha)=r$$

then the minimum obtained would have been

$$\frac{r^2}{\mathrm{Sp}\,\mathbf{S}_n(\alpha,\alpha)}.$$

It is an interesting fact that the relation existing between $\mathrm{Sp}\,\mathbf{S}_n^{-1}(\alpha,\alpha)$ and $\dfrac{r^2}{\mathrm{Sp}\,\mathbf{S}_n(\alpha,\alpha)}$ expresses the inequality between the arithmetical and geometrical mean. As a matter of fact, if the eigenvalues of $\mathbf{S}_n(\alpha,\alpha)$ are denoted by $\lambda_1,\lambda_2,\ldots,\lambda_r$ then

$$\frac{\lambda_1+\lambda_2+\cdots+\lambda_r}{r} \geqq \sqrt[r]{\lambda_1\cdot\lambda_2\cdots\lambda_r}$$

and

$$\frac{1}{\lambda_1}+\frac{1}{\lambda_2}+\cdots+\frac{1}{\lambda_r} \geqq \frac{r}{\sqrt[r]{\lambda_1\cdot\lambda_2\cdots\lambda_r}}.$$

Multiplying these two inequalities we get

$$\mathrm{Sp}\,\mathbf{S}_n(\alpha,\alpha)\,\mathrm{Sp}\,\mathbf{S}_n^{-1}(\alpha,\alpha) \geqq r^2$$

and here equality holds if and only if $\lambda_1=\lambda_2=\cdots=\lambda_r$.

Theorem 4.

$$\mu_n(\alpha, \mathbf{f}) \geqq \mu_{n+1}(\alpha, \mathbf{f}).$$

PROOF. Let $\mathbf{f}(x)$ have the canonical representation

$$\mathbf{f}(x) = \mathbf{U}(x)\,\mathbf{\Lambda}(x)\,\mathbf{U}^*(x),$$

where $\mathbf{U}(x) = (u_{jk}(x))$ $(j, k = 1, \ldots, r)$; $\mathbf{U}(x)\mathbf{U}^*(x) = \mathbf{E}_r$ and $\mathbf{\Lambda}(x)$ is the diagonal matrix containing the eigenvalues $\lambda_1(x), \ldots, \lambda_r(x)$ of $\mathbf{f}(x)$. Let moreover denote the elements of $\mathbf{P}_n(z)$ by $p_{jk}^{(n)}(z)$, then a simple computation yields

$$\mu_{n+1}(\alpha, \mathbf{f}) = \operatorname{Min} \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{l=1}^{r} \sum_{j=1}^{r} \left| \sum_{k=1}^{r} p_{jk}^{(n+1)}(z) u_{kl}(x) \right|^2 \lambda_l(x)\,dx \qquad (z = e^{ix}).$$

We have however

$$p_{jk}^{(n+1)}(z) = p_{jk}^{(n)}(z) + \varrho_{jk} z^{n+1}$$

and so

$$\left| \sum_{k=1}^{r} p_{jk}^{(n+1)}(z) u_{kl}(x) \right| \leqq \left| \sum_{k=1}^{r} p_{jk}^{(n)}(z) u_{kl}(x) \right| + \sum_{k=1}^{r} |\varrho_{jk} u_{kl}(x)|.$$

Thus for any choice of ϱ_{jk} the inequality

$$\mu_{n+1}(\alpha, \mathbf{f}) \leqq \operatorname{Min} \frac{1}{2\pi} \int_{-\pi}^{\pi} \Big\{ \sum_{l=1}^{r} \sum_{j=1}^{r} \Big[\Big| \sum_{k=1}^{r} p_{jk}^{(n)}(z) u_{kl}(x) \Big|^2 + \Big(\sum_{k=1}^{r} |\varrho_{jk} u_{kl}(x)| \Big)^2 +$$

$$+ 2 \Big| \sum_{k=1}^{r} p_{jk}^{(n)}(z) u_{kl}(x) \Big| \sum_{k=1}^{r} |\varrho_{jk} u_{kl}(x)| \Big] \lambda_l(x) \Big\}\, dx \qquad (z = e^{ix})$$

holds, and from this we get for $\varrho_{jk} = 0$ the theorem to be proved.

It is clear that if for an arbitrary row vector $\mathbf{z} = (z_1, \ldots, z_r)$

$$\mathbf{z}\mathbf{f}(x)\mathbf{z}^* \leqq \mathbf{z}\mathbf{g}(x)\mathbf{z}^*,$$

holds, where $\mathbf{f}(x)$ and $\mathbf{g}(x)$ are matrices of order r, then also the relation

$$(\mathbf{z}\mathbf{P}_n(z))\mathbf{f}(x)(\mathbf{z}\mathbf{P}_n(z))^* \leqq (\mathbf{z}\mathbf{P}_n(z))\,\mathbf{g}(x)(\mathbf{z}\mathbf{P}_n(z))^* \tag{16}$$

is valid for any polynomial matrix $\mathbf{P}_n(z)$. From this we infer the following

Theorem 5. *If $\mathbf{f}(x)$ and $\mathbf{g}(x)$ are positive definite (L) integrable Hermitian functional matrices and if for any row vector $\mathbf{z} = (z_1, \ldots, z_r)$ the relation*

$$\mathbf{z}\mathbf{f}(x)\mathbf{z}^* \leqq \mathbf{z}\mathbf{g}(x)\mathbf{z}^*$$

holds, then

$$\mu_n(\alpha, \mathbf{f}) \leqq \mu_n(\alpha, \mathbf{g}) \qquad (n = 0, 1, \ldots).$$

PROOF. From (16) it follows in particular that for any $\mathbf{P}_n(z)$ polynomial matrix

$$\mathrm{Sp}\,\mathbf{P}_n(z)\mathbf{f}(x)\mathbf{P}_n^*(z) \leqq \mathrm{Sp}\,\mathbf{P}_n(z)\mathbf{g}(x)\mathbf{P}_n^*(z)$$

and thus

$$\mathrm{Sp}\,\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathbf{P}_n(z)\mathbf{f}(x)\mathbf{P}_n^*(z)\,dx \leqq \mathrm{Sp}\,\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathbf{P}_n(z)\mathbf{g}(x)\mathbf{P}_n^*(z)\,dx. \tag{17}$$
$$(z=e^{ix})$$

Now, if we denote by $\mathbf{Q}_n(z)$ the polynomial matrix minimizing the right hand side of (17), then

$$\mu_n(\alpha, \mathbf{g}) = \mathrm{Sp}\,\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathbf{Q}_n(z)\mathbf{g}(x)\mathbf{Q}_n^*(z)\,dx \geqq$$
$$\geqq \mathrm{Sp}\,\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathbf{Q}_n(z)\mathbf{f}(x)\mathbf{Q}_n^*(z)\,dx \geqq \mu_n(\alpha, \mathbf{f}) \qquad (z=e^{ix}).$$

From this theorem there follows the

Theorem 6. *If the positive definite Hermitian (L) integrable functional matrix* $\mathbf{f}(x)$ *is bounded, i. e. if there exist real numbers* $m \geqq 0$ *and* $M \geqq 0$, $m \leqq M$ *so that for any row vector* $\mathbf{z} = (z_1, \ldots, z_r)$ *the inequalities*

$$m\mathbf{z}\mathbf{z}^* \leqq \mathbf{z}\mathbf{f}(x)\mathbf{z}^* \leqq M\mathbf{z}\mathbf{z}^*$$

hold, then

$$\begin{cases} rm\,\dfrac{1-|\alpha|^2}{1-|\alpha|^{2n+2}} \leqq \mu_n(\alpha, \mathbf{f}) \leqq rM\,\dfrac{1-|\alpha|^2}{1-|\alpha|^{2n+2}} & \text{for} \quad |\alpha| \neq 1 \\ \textit{and} & \\ \dfrac{rm}{n+1} \leqq \mu_n(\alpha, \mathbf{f}) \leqq \dfrac{rM}{n+1} & \text{for} \quad |\alpha| = 1. \end{cases} \tag{18}$$

PROOF. By Theorem 5

$$\mu_n(\alpha, m\mathbf{E}_r) \leqq \mu_n(\alpha, \mathbf{f}) \leqq \mu_n(\alpha, M\mathbf{E}_r)$$

and if $c \geqq 0$ is constant, then $\mu_n(\alpha, c\mathbf{E}_r) = c\mu_n(\alpha, \mathbf{E}_r)$. So it suffices to determine $\mu_n(\alpha, \mathbf{E}_r)$. If, however, $f(x) = E_r$ holds, then

$$\boldsymbol{\Phi}_k(z) = z^k\mathbf{E}_r \qquad (k = 0, 1, \ldots)$$

and so

$$\mathbf{S}_n(\alpha, \alpha) = \sum_{k=0}^{n}|\alpha|^{2k}\mathbf{E}_r = \begin{cases} \dfrac{|\alpha|^{2n+2}-1}{|\alpha|^2-1}\,\mathbf{E}_r & \text{for} \quad |\alpha| \neq 1 \\ (n+1)\mathbf{E}_r & \text{for} \quad |\alpha| = 1 \end{cases}$$

and consequently

$$\mu_n(\alpha, \mathbf{E}_r) = \begin{cases} r\dfrac{|\alpha|^2-1}{|\alpha|^{2n+2}-1} & \text{for } |\alpha| \neq 1 \\ \dfrac{r}{n+1} & \text{for } |\alpha| = 1. \end{cases}$$

From this, (18) already follows.

Bibliography

[1] G. Szegő, Beiträge zur Theorie der Toeplitzschen Formen, I, *Math. Z.* **6** (1920), 167—202.

[2] G. Szegő, Beiträge zur Theorie der Toeplitzschen Formen, II, *Math. Z.* **9** (1921), 167—190.

[3] U. Grenander and G. Szegő, Toeplitz forms and their applications, *Berkeley — Los Angeles,* 1958.

[4] B. Gyires, Über die Spuren der verallgemeinerten Toeplitzschen Matrizes, *Publ. Math. Debrecen,* **8** (1961), 93—116.

[5] H. Helson and D. Lowdenslager, Prediction theory and Fourier series in several variables, *Acta Math.* **99** (1958), 165—202.

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