Connected ordered topological groupoids with idempotent endpoints[1])

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The groupoids described in this title will be called g-threads. More specifically, a g-thread is a system $G(\circ, <)$ with the following properties:

(1) G is a groupoid with respect to $\circ$, i. e., $\circ$ is a single valued binary operation defined in G for every pair of elements $a, b \in G$.

(2) $<$ is a total (i. e., linear or simple) order relation in G.

(3) The mapping $p(x, y) = x \circ y$ of $G \times G$ into G is continuous in the order topology.

(4) G is connected in the order topology.

(5) G has a zero, 0, and a unit, u [3]. Zero is the least element, and u is the greatest element of G with respect to $<$.

CLIFFORD has considered g-threads obeying the associative law. He suggested the above terminology. We will denote $a \circ b$ by ab.

Our main result states: if G is a g-thread whose elements with unique powers form a dense subset and if G satisfies the cancellation law then there exists a function F from G onto the unit interval [0, 1] of real numbers under the usual topology and the usual multiplication, which is an isomorphism as well as order preserving homemorphism. In particular any continuous multiplication on the interval [0, 1] with the usual topology is the usual multiplication if it satisfies the above conditions.

Definition 1. A groupoid with a zero is said to satisfy the *cancellation law* if and only if $xc = yc$ or $cx = cy$ implies $x = y$ for all $x, y, c \in G$ with $c \neq 0$.

Lemma 1. *Let G be a g-thread satisfying the cancellation law. Then $x < y$ implies $xc < yc$ and $cx < cy$ for all $x, y, c \in G$ with $c \neq 0$.*

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[1]) This is essentially a condensation of a part of a doctoral dissertation written at the University of Tennessee under the direction of Professor O. G. HARROLD, Jr.

PROOF. Let $x, y \in G$ such that $x<y$. Let $A=\{d\colon xc<yc$ for all $c \in [d, u]\}$. Then, $A \neq \Box$ since $u \in A$. Let $k=\inf.\{d\colon d \in A\}$. We suppose $k>0$. We first show that $xk<yk$. If $xk=yk$, then $x=y$. Suppose $xk>yk$. Then there exists a neighborhood $V(k)$ of k and a $d \in V(k) \cap A$ such that $xd>yd$. This contradicts the fact that $d \in A$. Hence, $xk<yk$. Thus, one may find an open interval $U(k)$ about k such that $xc<yc$ for all c in $U(k)$. Since there is a $d \in U(k) \cap A$, $xc<yc$ for all $c \in [d, u]$. But, there exists a $t \in U(k)$ such that $t<k$. Hence $xc<yc$ for all $c \in [t, u]$ and $t \in A$. This contradicts the definition of k. Hence, $k=0$. Thus, $xc<yc$ for all $c \neq 0$. A similar argument yields $cx<cy$ for $c \neq 0$.

Definition 2. An element $a \in G$ is said to be *power associative* if and only if all the powers of a are unique.

Lemma 2. *If G is a g-thread which satisfies the cancellation law, then*

(1) $xy \leqq \min(x, y)$

(2) *if* $x \leqq y$ *and* $w \leqq v$, *then* $xw \leqq yv$.

(3) *if* $x \neq u$ *is power ossociative,* $(x^n\colon n \in I)$ *converges to* 0.

(I will denote the positive integers.)

PROOF. (1) and (2) follow immediately from Lemma 1. (3) Since x is power associative, $u \geqq x \geqq x^2 \geqq x^3 \ldots$ by Lemma 1. Since x is power associative $\overline{(x^n\colon n \in I)}$ is a semigroup and $(x^n\colon n \in I)$ must cluster at an idempotent [3]. By the cancellation law, 0 and u are the only idempotents of G. Since $x \neq u$, $(x^n\colon n \in I)$ converges to 0.

Lemma 3. *Let G be a g-thread which satisfies the cancellation law and in which the power associative elements form a dense subset. Then, G is an arc and G is an abelian topological semi-group.*

PROOF. Clearly, each $a \in G$ is power associative. Let Δ^n be the diagonal of the cartesian product of G, n times. For each n define $f_n\colon \Delta^n \to G$ by $f_n(y)=x^n$ where x is the projection of y onto G. Since $f_n(\Delta^n)=G$, every $x \in G$ has an n^{th} root. If $x \neq 0$, x has a unique square root by virtue of Lemma 1. By induction, x has a unique 2^m-th root. If $x, y \in G$ and $x<y$, $x^2<y^2$ by Lemma 1. It follows $x^{2m}<y^{2m}$ for all positive integers m. We call this result (1).

Let us define for $x \in G$, $x \neq 0$

$$x^{p/2^q}=(x^{1/2^q})^p$$

where p and q are positive integers. It is immediate from the fact that the 2^q-th roots are unique, that x^r is well defined for any positive dyadic rational

r, regardless of the representation of r. Clearly,

$$x^r x^s = x^{r+s}. \tag{2}$$

Let r and s be dyadic rationals such that $r<s$. Then, there exists a dyadic rational t such that $r+t=s$. Hence $x^s = x^r x^t \leqq x^r$ by virtue of (2) and Lemma 2. Therefore, if (r_n) is a monotone increasing sequence of dyadic rationals, (x^{r_n}) is a monotone decreasing sequence in G. For $x \in G$, $x \neq 0$ or u, let $D = (x^r\colon r$ is a positive dyadic rational$)$. By (2), D is an abelian submob of G. We claim $\overline{D} = G$. Let $V(0)$ be any neighborhood of zero. Since $x \neq u$, $x^n \to 0$ by Lemma 2. Thus $V(0) \cap D \neq \square$. Let $V(u)$ be any basis element containing u. Let $y \in V(u)$ with $y<u$. Clearly $y^n \to 0$. Thus there exists a positive integer n such that $y^{2^n} < x$. Hence, by (1) $y < x^{1/2^n}$ and $V(u) \cap D \neq \square$. Let t be an arbitrary element of $G, t \neq 0$ and $t \neq u$ and let $B = (s\colon a<s<b)$ be a basis element in the order topology containing t. Without any loss of generality, we may assume $0<a<b<u$. Let

$$R' = (r\colon x^r \geqq b, r \text{ a positive dyadic rational})$$
$$R'' = (r\colon x^r \leqq a, r \text{ a positive dyadic rational}).$$

Let r' be the $1.u.b.$ of R' and r'' be the $g.1.b.$ of R''. Clearly, $r' \leqq r''$. If $r'<r''$, D meets B.

Therefore, we assume $r'=r''$. Let (p_n) be a monotone increasing sequence of dyadic rationals of R' converging to r'. Then (x^{p_n}) is a monotone decreasing sequence converging to its $g.1.b.$ $c \geqq b$. Let W be an open set containing c so that $W \cap [0, a] = \square$. There exists open sets U and V with $u \in U$ and $c \in V$ such that $UV \subset W$. Since $D \cap U \neq \square$, $x^r \in U$ for some r. There exists a p_k in the sequence (p_n) such that $x^{p_k} \in V$ and $r+p_k > r' = r''$. Hence $x^{r+p_k} \in [0, a] \cap W$. From this contradiction, it follows that $\overline{D} = G$. Since the closure of an abelian subsemi-group of a topological groupoid is an abelian semi-group, G is an abelian topological semi-group. Since G is separable, G is an arc by virtue of a theorem of WILDER ([4], pp. 21—39).

We next state a result due to FAUCETT [2].

Lemma 4. *Let S be a compact semi-group with a zero and a unit and no other idempotents. Suppose S is irreducibly connected between* 0 *and u* (2). *Assume further that S contains no non-zero algebraic nilpotents* (2). *Then there exists a function F from S onto the unit interval* $[0, 1]$ *of real numbers with the usual topology and the usual multiplication, that is an isomorphism as well as an order preserving homeomorphism.*

If G is a groupoid with a zero such that G satisfies the cancellation law and every element of G is power associative, then it is easy to see that $x \in G$ and $x^n = 0$ for some n implies $x = 0$.

Therefore by virtue of Lemma 4 we have:

Theorem. *Let G be a g-thread such that G satisfies the cancellation law and the power associative elements of G form a dense subset. Then there exists a function f from G onto the unit interval* $[0, 1]$ *of real numbers under the usual topology and the usual multiplication which is an isomorphism as well as an order preserving homeomorphism.*

We give an example to indicate the effect of the topology on the algebraic structures.

Example 1. An example of a finite groupoid with a zero and a unit which obeys the cancellation law and in which every element has unique powers but which is non-associative and non-abelian is given by $G=(0, 1, 2, 3, 4, 5)$ with the following multiplication table:

	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	1	5	3	4
3	0	3	4	1	5	2
4	0	4	5	2	1	3
5	0	5	3	4	2	1

We next give an example that seems to indicate the necessity of requiring certain elements to be power associative.

Example 2. An example of a continuons multiplication on $[0, 1]$ with the usual topology which is non-associative and non-abelian, which has 0 as a zero, which has 1 as a right unit, and which obeys the cancellation law. We define: $a \circ b = ab^2$ where the last multiplication is that of real numbers. Zero and 1 are the only power associative elements.

Bibliography

[1] A. H. Clifford, Connected ordered topological semigroups with idempotent endpoints I., *Trans. Amer. Math. Soc.* **88** (1958), 80—98.

[2] W. M. Faucett, Compact semigroups irreducibly connected between two idempotents, *Proc. Amer. Math. Soc.* **6** (1955), 741—747.

[3] R. J. Koch, On topological semigroups, *Dissertation, Tulane University* 1953.

[4] R. L. Wilder, Topology of Manifolds (Amer. Math. Soc., Colloquium Publ., Vol. XXXII.), *New York,* 1949, 21—39.

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