Functional equations for products and compositions of functions

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R. BELLMAN [2] has posed a number of problems on functional equations. One of us [7] has published a solution of the first of these problems. In this note we wish to consider the formal structure of these questions and thereby obtain methods of solution for a larger class of related problems.

1. Multiplicative functionals on an Abelian Banach algebra (BELLMAN'*s Problem 14).* The original problem was that of determining the functionals F_n defined for functions u which are n times differentiable on some set, so that $F_n(u)$ could be expressed as a function $f_n(u, u', \ldots, u^{(n)})$ of u and its first n derivatives in a fixed point and $F_n(uv) = F_n(u) + F_n(v)$ whenever $uv \neq 0$.

The fact that the variables in f_n were derivatives (in a fixed point) is really not relevant to the problem and we could instead ask the question:

Let F be a functional defined for $(n+1)$-dimensional vectors $u = (u_0, u_1, \ldots, u_n)$ with a product uv defined so that

$$(uv)_i = \sum_{j=0}^{i} \binom{i}{j} u_{i-j} v_j \tag{1}$$

and let the values of F lie in a set in which an associative operation $+$ is defined, under what conditions does F satisfy

$$F(uv) = F(u) + F(v) \text{ whenever } u_0 v_0 \neq 0?$$

We subsume these questions in the following.

Problem 1. Let U be an Abelian Banach algebra. What functionals F defined on U satisfy

$$F(uv) = F(u) + F(v) \tag{2}$$

whenever u and v are not zero divisors?

The main problem is that of converting the multiplicative identity (2) into an additive one.

Lemma. *Let U be an Abelian Banach algebra with an identity element e. Then there exists a homomorphism between the additive group of U and the maximal connected multiplicative subgroup of U which contains e.*

PROOF. Define the formal exponential function $\exp u = \Sigma \frac{u^n}{n!}$ which converges for every $u \in U$. Then the mapping $u \to \exp u$ gives

$$u + v \to (\exp u)(\exp v).$$

The fact that every element in the maximal connected subgroup which contains e has the form $\exp u$, — that is, the existence of a formal logarithm for that group, was proved by E. LORCH [5].

Theorem 1. *Let U be as in the Lemma and let $F(u)$ be defined on U so that*

$$F(uv) = F(u) + F(v)$$

whenever u, v are not zero divisors. Then $F(\exp u)$ is an additive function of u.

PROOF. Write $F(\exp u) = g(u)$ then

$$g(u+v) = F((\exp u)(\exp v)) = F(\exp u) + F(\exp v) = g(u) + g(v).$$

REMARK. If U is finite dimensional vector-space then any additive function on U can be written as a sum of additive functions of the components. That is for $u = (u_0, u_1, \dots, u_n)$ we get

$$g(u) = \sum_{i=0}^{n} g(0, \dots, 0, u_i, 0, \dots, 0) = \sum_{i=0}^{n} a_i(u_i)$$

where the a_i are additive.

Under suitable regularity conditions (e. g. measurability) on the functions a_i we get (for real u_i and real valued $F(u)$)

$$g(u) = \sum_{i=0}^{n} a_i u_i.$$

Some discussion may be in order concerning the value of F for units u of U which are not contained in the component E of the identity e. Since E is divisible it is a direct factor of the group U^* of units in U (see e. g. [4, Theorem 2]). Thus we can write

$$u = |u| \arg u$$

for all $u \in U^*$, where $|u| \in E$, $\arg u \in A$ and $U^* = E \otimes A$. Now

$$F(u) = F(|u|) + F(\arg u) \qquad u \in U^*$$

where $F(|u|)$ is an additive functional of $\log |u|$ and $F(\arg u)$ is an arbitrary additive character of A. In particular $F(a) = 0$ for all elements in the torsion group of A.

We now return to BELLMAN's problem as formulated in terms of $(n+1)$-vectors. The formal logarithm of a vector in Lemma is given here by the vector corresponding to the ordinary logarithm of the corresponding function provided $u_0 > 0$. That is

$$(\log u)_i = \frac{d^i \log u(t)}{dt^i},$$

where the right side can be expressed in terms of $u_0, u_1, \ldots, u_i$.

By the theorem and the fact that (for real u) the „argument group" A is $\{e, -e\}$, so that $F(e) = F(-e) = 0$, and $F(u) = F(|u|)$; we now see that the solution to BELLMAN's functional equation

$$F(uv) = F(u) + F(v)$$

is an additive function of the vector $\log|u|$ when $u \neq 0$, or

$$F(u) = \Sigma a_i \left(\frac{d^i \log|u|}{dt^i}\right) \quad a_i \text{ additive.}$$

Under suitable regularity conditions this reduces to

$$F(u) = \Sigma a_i \frac{d^i \log|u|}{dt^i}.$$

2. Problems 15 and 17. We can reformulate these problems as follows. In Problem 15 we associate as before with the n times differentiable function u the vector $(u_0, \ldots, u_n)$. Now the linear differential operator

$$a_0(t)u + a_1(t)u' + \cdots + a_n(t)u^{(n)} = (a, u)$$

is a scalar product. If we use the product uv defined in (1) we get a second product defined by

$$(a, uv) = (a \times u, v).$$

This product satisfies the conditions

(3) $\quad a \times (uv) = (a \times u) \times v$

(4) $\quad a \times e = a$, where e is the multiplicative identity of (1).

(5) $\quad (a \times u)_k = f_k(a_0, \ldots, a_k;\ u_0, \ldots, u_k).$

The problem was to find the most general $\times$-multiplication which satisfies equations (3), (4) and (5).

Problem 17 is essentially the same as Problem 15 except that the coefficients and variables in the differential equation are now matrices and that attention is restricted to first order differential equations.

We subsume both problems in the following

Problem 2. Given a group G, where the group operation for $u, v \in G$ is denoted by uv. What are the possible operations $a \times u$ that can be defined for $u \in G$ and $a \in A$, where A is some system of elements, so that $a \times u \in A$ and

(3) $a \times (uv) = (a \times u) \times v$;

(4) $a \times e = a$, e the identity of G;

for all $a \in A$ and all $u, v \in G$?

This problem was treated in the papers [1] and [3].

The solution of this problem becomes simple if we notice, as was done in [3], that it can be interpreted as saying that G acts as a transformation group on the space A. We can therefore decompose A into disjoint orbits $a \times G = \{a \times u | u \in G\}$. Each of these orbits can be represented by the right cosets of the subgroup S of G which consists of the elements that leave a fixed. Two subgroups S_1, S_2 correspond to the same orbit if and only if they are conjugate subgroups of G.

Conversely, G acts a transitive transformation group on the right cosets of any subgroup S if we define $(Su) \times v = Suv$. We can sum this up as follows.

Theorem 2. *The orbits $a \times G$ in A are determined by a subgroup S of G and a one-to-one mapping h of the right cosets Su into A so that*

$$h(S) = a \text{ and } h(Su) = a \times u.$$

That is

$$b \times u = h(h^{-1}(b)u)$$

for all b in $a \times G$.

Returning now to BELLMAN's Problem 15 we again introduce the formal logarithm so that the multiplicative group G of vectors $(u_0, u_1, \ldots, u_n)$ with $u_0 \neq 0$ and multiplication defined by (1) is mapped onto the additive group A' of all vectors $(a'_0, a'_1, \ldots, a'_n)$.

The set of subgroups of A' is very large; but the only ones, for which the coset representative of a vector can be obtained by continuous functions of its components, are the subspaces (see e. g. [6], p. 127, Example 33). Let S be a k-dimensional subspace. Then it has a basis of the form

$$s^1 = (0, \ldots, 0, 1, s_{1, i_1+1}, \ldots, s_{1n})$$
$$s^2 = (0, \ldots, 0, \ldots, 0, 1, s_{2, i_2+1}, \ldots, s_{2n})$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$s^k = (0, \ldots, 0, \ldots, 0, \ldots, 0, 1, s_{k, i_k+1}, \ldots, s_{kn})$$

and the coset $S+u$ where $u=(u_0,\ldots,u_n)$ can therefore be given the representative

$$\bar{u}=u-u_{i_1}s^1-(u_{i_2}-u_{i_1}s_{1i_2})s^2-[u_{i_3}-u_{i_1}s_{1i_3}-(u_{i_2}-u_{i_1}s_{1i_2})s_{2i_3}]s^3-\cdots$$

so that $\bar{u}_{i_j}=0$ $(j=1,\ldots,k)$ and $\bar{u}_l$ is a (linear) function of $u_0,\ldots,u_l$ for $l=0,\ldots,n$.

Thus, if we write

$$a\times u=a+\overline{\log u}, \tag{6}$$

then conditions (3), (4) and (5) are satisfied. The isomorphic systems are

$$a\times u=h^{-1}(h(a)+\overline{\log u}) \tag{7}$$

where h is a one-to-one mapping of A' onto itself. If we want (7) to satisfy (5) then we must restrict h by the conditions

$$(h(a))_i=h_i(a_0,\ldots,a_i) \qquad (i=0,1,\ldots,n). \tag{8}$$

This completes the solution of BELLMAN's Problem 15 if we restrict ourselves to the case in which his functions f_i are continuous in all variables. For the sake of illustration we consider three examples.

Example 1. $S=\{0\}$. Here $\bar{u}=u$ and (7) becomes

$$a\times u=h^{-1}(h(a)+\log|u|)$$

which, coupled with condition (8), can be written in terms of components as follows

$$f_0(a_0;u)=h_0^{-1}(h_0(a_0)+\log|u_0|) \tag{9}$$

$$f_1(a_0,a_1;u_0,u_1)=h_1^{-1}\left(h_0(a_0)+\log|u_0|,\ h_1(a_0,a_1)+\frac{u_1}{u_0}\right)$$

$$f_2(a_0,a_1,a_2;u_0,u_1,u_2)=$$

$$=h_2^{-1}\left(h_0(a_0)+\log|u_0|,\ h_1(a_0,a_1)+\frac{u_1}{u_0},\ h_2(a_0,a_1,a_2)+\frac{u_2}{u_0}-\frac{u_1^2}{u_0^2}\right)$$

. . . .

$$f_n(a,u)=h_n^{-1}(h_0(a)+(\log|u|)_0,\ldots,h_n(a)+(\log|u|)_n),$$

where

$$h_i^{-1}(h_0(a_0),h_1(a_0,a_1),\ldots,h_i(a_0,a_1,\ldots,a_i))=a_i \qquad (i=0,\ldots,n).$$

Example 2. $S=\{(u_0,u_1,0,\ldots,0)\}$. Here $\bar{u}=(0,0,u_2,\ldots,u_n)$ and (6) becomes

$$f_0(a_0; u_0) = a_0$$

(10) $$f_1(a_0, a_1; u_0, u_1) = a_1$$

$$f_2(a_0, a_1, a_2; u_0, u_1, u_2) = h_2^{-1}\left(h_0(a_0), h_1(a_0, a_1), h_2(a_0, a_1, a_2) + \frac{u_2}{u_0} - \frac{u_1^2}{u_0^2}\right)$$

$$\cdots\cdots\cdots\cdots$$

$$f_n(a, u) = h_n^{-1}(h_0(a), h_1(a), h_2(a) + (\log|u|)_2, \ldots, h_n(a) + (\log|u|)_n),$$

where the h_i are as before.

Finally we consider an example for which S is not a subspace of A.

Example 3. Let A be a vector space over the complex numbers and let S be the subgroup of vectors whose components are pure imaginary. Then to each coset $S+u$ we can associate the representative $\Re u = (\Re u_0, \Re u_1, \ldots, \Re u_n)$. Thus (7) becomes

$$a \times u = h^{-1}(h(a) + \Re \log u)$$

or in more detail

(11) $$f_0(a_0; u_0) = h_0^{-1}(h_0(a_0) + \Re \log u_0)$$

$$f_1(a_0, a_1; u_0, u_1) = h_1^{-1}\left(h_0(a_0) + \Re \log u_0, h_1(a_0, a_1) + \Re \frac{u_1}{u_0}\right)$$

$$\cdots\cdots\cdots\cdots$$

$$f_n(a, u) = h_n^{-1}(h_0(a) + \Re(\log u)_0, \ldots, h_n(a) + \Re(\log u)_n),$$

where the h_i are as before.

The group in Problem 17 is non-Abelian. Thus the analysis of its subgroups — to which the problem has been reduced — is much more complicated.

REMARK. Condition (4), while certainly implicit in BELLMAN's problems, is not necessary for our solution. If we define $A^* = \{a \times e \mid a \in A\}$, then $a^* \times e = a^*$ for every $a^* \in A^*$. The mapping $a \to a^* = a \times e$ is a projection (idempotent mapping) of A onto A^* such that $a \times u = a^* \times u$ for all $u \in G$.

Thus if we omit (4) then we can first determine the mapping $a \to a \times e = a^*$ as an arbitrary projection of A onto a subset A^* and then proceed as before with A replaced by A^*.

3. Problem 16. Here we deal not with multiplication but with composition of functions. We can introduce the notation $u \circ v$ for the vector associated with the function $v(u)$ so that

(12) $$(u \circ v)_i = \frac{d^i}{dt^i} v(u(t)) = \varphi_i(u_1, \ldots, u_i; v_1^u, \ldots, v_i^u), \qquad (i = 1, 2, \ldots, n);$$

$$(u \circ v)_0 = v_0^u,$$

where $v_i^u = v^{(i)}(u(t))$.

If we now define

$$(a, u \circ v) = (a \otimes u, v^u)$$

then the functional equation analogous to (3) is

$$a \otimes (u \circ v) = (a \otimes u) \otimes v^u. \tag{3'}$$

However, since we are only interested in the formal properties of the product $a \otimes u$ we can take notice of the fact that

$$(a \otimes u)_k = f_k(a_k, a_{k+1}, \ldots, a_n; u_1, u_2, \ldots, u_{n-k+1}), \qquad (k = 1, 2, \ldots, n); \tag{13}$$
$$(a \otimes u)_0 = a_0;$$

and that the relation (3′) remains valid with the same function f_k if we replace the product (12) by

$$(u \circ' v)_i = \varphi_i(u_1, \ldots, u_i; v_1, \ldots, v_i), \qquad (i = 1, 2, \ldots, n). \tag{12'}$$

If we ignore the component u_0 and restrict attention to vectors with $u_1 \neq 0$ then (12′) defines a group and the relation

$$a \otimes (u \circ' v) = (a \otimes u) \otimes v, \tag{3''}$$

where we ignore the component a_0, subsumes this problem in our Problem 2. The problem thus reduces to the study of the subgroup of the non-Abelian group defined by (12′).

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