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Normal flow space forms and their classification

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Abstract. We introduce the normal flow space forms in the class formed by the Riemannian manifolds equipped with a unit Killing vector field. We study their curvature and discuss their classification.

1. Introduction

Real, complex, quaternionic and Sasakian space forms are the simplest examples of Riemannian manifolds in the framework of Riemannian, complex, quaternionic and contact geometry. Their geometric properties have been studied extensively and a lot of characterizations are well-known. At many occasions they are considered as model spaces for comparison purposes. The first three kind of space forms are special cases of symmetric spaces whilst the Sasakian space forms are not symmetric but φ-symmetric. This last notion has been introduced in contact geometry by T. Takahashi [16]: A complete Sasakian manifold is said to be a φ-symmetric space if and only if the reflections with respect to the characteristic flow lines are global isometries. On Sasakian manifolds these flow lines are the integral curves of a unit Killing vector field. This led the authors to the consideration and study of the more general case of Riemannian manifolds equipped with a unit Killing vector field such that the reflections with respect to the flow lines are local or global isometries. These spaces are called (locally or globally) Killing-transversally symmetric spaces (briefly KTS-spaces). We refer to [4]–[10] for more details, results

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and classification theorems. These papers treat some aspects of what we call flow geometry.

The main purpose of this paper is to give the definition and to treat the curvature and classification of a notion of space form in the framework of flow geometry. It will turn out that in the considered cases these space forms are KTS-spaces. For that reason we start in Section 2 with some general information about flow geometry and some particular facts about KTS-spaces. In Section 3 we study the sectional curvature and use the results to introduce the notion of a normal flow space form in Section 4 where we also give some classification theorems.

2. Flow geometry, locally and globally Killing-transversally symmetric spaces

Let (M,g) be an n-dimensional, smooth, connected Riemannian manifold with $n \geq 2$. ∇ denotes its Levi Civita connection and R the corresponding Riemannian curvature tensor with the sign convention

$$R_{UV} = \nabla_{[U,V]} - [\nabla_U, \nabla_V]$$

for all $U, V \in \mathfrak{X}(M)$, the Lie algebra of smooth vector fields on M.

A tangentially oriented foliation of dimension one on (M,g) is called a *flow*. The leaves of this foliation are the integral curves of a non-singular vector field on M and hence, by normalizing length, a flow is also given by a unit vector field with respect to g. In particular, a non-singular Killing vector field defines a *Riemannian flow* and such a flow is said to be an *isometric flow*. See [18] for more information.

In what follows we denote by $\mathfrak{F}_\xi$ an isometric flow generated by a *unit* Killing vector field ξ. The flow lines of $\mathfrak{F}_\xi$ are geodesics and moreover, a geodesic which is orthogonal to ξ at one of its points, is orthogonal to it at all of its points. Such geodesics are called *transversal* or *horizontal* geodesics.

A Riemannian foliation is locally a Riemannian submersion. So, for each $m \in (M,g)$, let $\mathcal{U}$ be a small open neighborhood of m such that ξ is regular on $\mathcal{U}$. Then the mapping $\pi : \mathcal{U} \to \mathcal{U}' = \mathcal{U}/\xi$ is a submersion. Further, let g' denote the induced metric on $\mathcal{U}'$ given by

$$g'(X',Y') = g(X'^*, Y'^*)$$

for $X', Y' \in \mathfrak{X}(\mathcal{U}')$ and where X'^*, Y'^* denote the horizontal lifts of X', Y' with respect to the $(n-1)$-dimensional horizontal distribution on $\mathcal{U}$ determined by $\eta = 0$, η being the dual one-form of ξ with respect to g. Then

$\pi : (\mathcal{U}, g_{|\mathcal{U}}) \to (\mathcal{U}', g')$ is a Riemannian submersion and this allows us to use the tensors A and T introduced by O'Neill in [13] (see also [1], [15], [18]). In our case $T = 0$ since the leaves are geodesics. Further, for the integrability tensor A we have

$$A_U\xi = \nabla_U\xi, \qquad A_\xi U = 0,$$
$$A_X Y = (\nabla_X Y)^{\mathcal{V}} = -A_Y X, \qquad g(A_X Y, \xi) = -g(A_X\xi, Y)$$

where $U \in \mathfrak{X}(M)$, X, Y are horizontal vector fields and $\mathcal{V}$ denotes the vertical component.

Next, put

$$HU = -A_U\xi$$

and define the $(0, 2)$-tensor field h by

$$h(U, V) = g(HU, V),$$

$U, V \in \mathfrak{X}(M)$. Since ξ is a Killing vector field, h is skew-symmetric. Moreover, it is easy to see that

$$A_X Y = h(X, Y)\xi = \frac{1}{2}\eta([X, Y])\xi.$$

This gives

$$h = -d\eta. \tag{2.1}$$

Further, the Levi Civita connection ∇' of g' is determined by

$$\nabla_{X'^*} Y'^* = (\nabla'_{X'} Y')^* + h(X'^*, Y'^*)\xi \tag{2.2}$$

for $X', Y' \in \mathfrak{X}(\mathcal{U}')$. Note that $A = 0$, or equivalently $h = 0$, if and only if the horizontal distribution is integrable. In this case, since $T = 0$, (M, g) is locally a product of an $(n-1)$-dimensional manifold and a line.

From all these formulas one derives by straightforward computations the following

Lemma 2.1 [6]**.** *We have*

$$(\nabla_\xi h)(X, Y) = g((\nabla_\xi A)_X Y, \xi) = 0, \tag{2.3}$$

$$R(X, Y, Z, \xi) = (\nabla_Z h)(X, Y), \tag{2.4}$$

$$R(X, \xi, Y, \xi) = g(HX, HY) = -g(H^2X, Y) \tag{2.5}$$

for horizontal X, Y, Z.

Note that $R(X, Y, Z, W) = g(R_{XY}Z, W)$.

Using this lemma it follows that the sectional curvature $K(X,\xi)$ of the two-plane spanned by X, ξ is non-negative for all horizontal X and since $H\xi = 0$, $K(X,\xi) = 0$ for all horizontal X if and only if $h = 0$. Moreover, $K(X,\xi) > 0$ for each horizontal X if and only if the skew-symmetric endomorphism H is of maximal rank $n-1$. In this case, n is necessarily odd and from (2.1) we see that η is a *contact form* on M. This leads to

Definition 2.1. An isometric flow $\mathfrak{F}_\xi$ on a Riemannian manifold (M,g) is called a *contact flow* if η is a contact form or equivalently, if the endomorphism H is of maximal rank.

In what follows we will need extensively another special type of isometric flow which has been introduced in a natural way in [6]. We recall its definition:

Definition 2.2. An isometric flow $\mathfrak{F}_\xi$ on a Riemannian manifold (M,g) is said to be *normal* if $R(X,Y,X,\xi) = 0$ for all horizontal vector fields X, Y.

From Lemma (2.1) we then get the following useful

Proposition 2.1 [6]. *Let $\mathfrak{F}_\xi$ be an isometric flow on (M,g). Then $\mathfrak{F}_\xi$ is normal if and only if*

$$(\nabla_U H)V = g(HU,HV)\xi + \eta(V)H^2U \tag{2.6}$$

for all $U,V \in \mathfrak{X}(M)$.

Further, for a normal flow the curvature tensor satisfies the following identities:

$$R_{UV}\xi = \eta(V)H^2U - \eta(U)H^2V, \tag{2.7}$$

$$R_{U\xi}V = g(HU,HV)\xi + \eta(V)H^2U, \tag{2.8}$$

$U,V \in \mathfrak{X}(M)$. Hence, by using (2.2), the corresponding curvature tensors of ∇ and ∇' are related by (see [6])

$$\begin{aligned}(R'_{X'Y'}Z')^* = R_{X'^*Y'^*}Z'^* &- g(HY'^*,Z'^*)HX'^* \\ &+ g(HX'^*,Z'^*)HY'^* + 2g(HX'^*,Y'^*)HZ'^*\end{aligned} \tag{2.9}$$

for all $X',Y',Z' \in \mathfrak{X}(\mathcal{U}')$. So, from (2.9), one may easily prove the following relation for the sectional curvatures:

$$K'_{m'}(u',v') = K_m(u'^*,v'^*) + 3\{h_m(u'^*,v'^*)\}^2 \tag{2.10}$$

where $\{u', v'\}$ is an orthonormal pair of $T_{m'}\mathcal{U}'$, $m' = \pi(m)$. So

$$K'_{m'}(u', v') \geq K_m(u'^*, v'^*).$$

Here the equality holds on M if and only if the horizontal distribution is integrable.

Now we shall define the Killing-transversally symmetric spaces. Therefore, let $\mathfrak{F}_\xi$ be an isometric flow on (M, g). Let $m \in M$ and denote by σ the geodesic flow line through m. A local diffeomorphism s_m of M defined in a neighborhood $\mathcal{U}$ of m is said to be a (*local*) *reflection with respect to* σ if for every transversal geodesic $\gamma(s)$, where $\gamma(0)$ lies in the intersection of $\mathcal{U}$ and σ, we have

$$(s_m \circ \gamma)(s) = \gamma(-s)$$

for all s with $\gamma(\pm s) \in \mathcal{U}$, s being the arc length of γ. The linear isometry $S_m = s_{m_*}(m)$ on T_mM is given by

$$S_m = (-I + 2\eta \otimes \xi)(m).$$

Since ξ is a Killing vector field, the local reflection s_m satisfies

$$s_m = exp_m \circ S_m \circ exp_m^{-1}.$$

Definition 2.3. A *locally Killing-transversally symmetric space* (briefly, a *locally* KTS-*space*) is a Riemannian manifold (M, g) equipped with an isometric flow $\mathfrak{F}_\xi$ such that the reflection s_m with respect to the flow line through m is an isometry for all $m \in M$.

These spaces may be characterized by using the following result:

Proposition 2.2 [6]. *Let $\mathfrak{F}_\xi$ be an isometric flow on (M, g). Then $(M, g, \mathfrak{F}_\xi)$ is a locally* KTS-*space if and only if $\mathfrak{F}_\xi$ is normal and*

$$(\nabla_X R)(X, Y, X, Y) = 0$$

for all horizontal X, Y.

Now we recall two other useful characterizations.

Proposition 2.3 [8]. *Let $\mathfrak{F}_\xi$ be a contact flow on (M, g). Then $(M, g, \mathfrak{F}_\xi)$ is a locally* KTS-*space if and only if $\mathfrak{F}_\xi$ is normal and*

$$(\nabla_X R)(X, HX, X, HX) = 0$$

for all horizontal X.

Proposition 2.4 [6]. *Let $\mathfrak{F}_\xi$ be a normal flow on (M,g). Then $(M,g,\mathfrak{F}_\xi)$ is a locally* KTS-*space if and only if each base space $\mathcal{U}'$ of a local Riemannian submersion $\pi:\mathcal{U}\to\mathcal{U}'=\mathcal{U}/\xi$ is a locally symmetric space.*

So, according to the terminology used in [19], $(M,g,\mathfrak{F}_\xi)$ is a locally KTS-space if and only if $\mathfrak{F}_\xi$ is a *normal transversally symmetric foliation*.

Next, let (M,g) be equipped with an isometric flow $\mathfrak{F}_\xi$ and define the tensor field T of type $(1,2)$ (unrelated to the O'Neill tensor given before) by

$$T_UV = g(HU,V)\xi + \eta(U)HV - \eta(V)HU$$

for all tangent vectors U,V. Then we have $T_UU=0$. Moreover, $\overline{\nabla}=\nabla-T$ defines a metric connection which is called the *canonical connection* of the isometric flow $\mathfrak{F}_\xi$ [6]. Its torsion $\overline{K}$ is given by $\overline{K}=-2T$ and its geodesics are the same as those of ∇. A direct computation shows that ξ and η are $\overline{\nabla}$-parallel. Further, $\mathfrak{F}_\xi$ is normal if and only if $\overline{\nabla}T=0$ [6] and in this case the curvature tensor $\overline{R}$ of $\overline{\nabla}$ is given by

$$\overline{R}_{UV} = R_{UV} + [T_U,T_V] - 2T_{T_UV}. \tag{2.11}$$

Using the terminology of [20] we have

Proposition 2.5 [6]. *$(M,g,\mathfrak{F}_\xi)$ is a locally* KTS-*space if and only if the tensor field T defines a homogeneous structure on it.*

This implies that a locally KTS-space is a *locally homogeneous* space and T is a *naturally reductive* homogeneous structure on it [20].

Further, we have

Theorem 2.1. *Let $(M_1,g_1,\mathfrak{F}_{\xi_1})$ and $(M_2,g_2,\mathfrak{F}_{\xi_2})$ be locally* KTS-*spaces and $o_1\in M_1$, $o_2\in M_2$. Further, let $L:T_{o_1}M_1\to T_{o_2}M_2$ be a linear isometry satisfying*

(i) $L\xi_1=\xi_2$;

(ii) $L\circ H_1 = H_2\circ L$;

(iii) $LR_{1UV}W = R_{2LULV}LW, \qquad U,V,W\in T_{o_1}M_1.$

Then there exists an isometry f of a neighborhood $\mathcal{U}_1$ of o_1 onto a neighborhood $\mathcal{U}_2$ of o_2 such that $f(o_1)=o_2$, $f_*(o_1)=L$, $f_*\xi_1=\xi_2$. If in addition M_1 and M_2 are complete and simply connected, then f is a global isometry.

Proof. From (i) and (ii) we obtain at once

$$LT_{1U}V = T_{2LU}LV$$

and hence

$$L\overline{K}_1(U,V) = \overline{K}_2(LU, LV)$$

for $U, V \in T_{o_1}M_1$, where $\overline{K}_1$, respectively $\overline{K}_2$, denotes the torsion of the canonical connection $\overline{\nabla}_1$, respectively $\overline{\nabla}_2$. Then (2.11) and (iii) yield

$$L\overline{R}_{1UV}W = \overline{R}_{2LULV}LW.$$

Moreover, by means of Proposition 2.5 we get $\overline{\nabla}_1\overline{K}_1 = \overline{\nabla}_2\overline{K}_2 = \overline{\nabla}_1\overline{R}_1 = \overline{\nabla}_2\overline{R}_2 = 0$. So, from [12, Chapter VI, Theorem 7.4] we may conclude that there exists an affine isomorphism f of a neighborhood $\mathcal{U}_1$ of o_1 onto a neighborhood $\mathcal{U}_2$ of o_2 such that $o_2 = f(o_1)$ and $f_*(o_1) = L$. Since $\overline{\nabla}_1$ and $\overline{\nabla}_2$ are metric and L is an isometry, f is an isometry such that $f_*\xi_1 = \xi_2$ because ξ_1 and ξ_2 are Killing vector fields.

The last part of the theorem follows now directly from [12, Chapter VI, Theorem 7.8]. □

Finally we consider some global aspects of flow geometry.

Definition 2.4. Let (M, g) be a Riemannian manifold and ξ a non-vanishing complete Killing vector field on it. Then $(M, g, \mathfrak{F}_\xi)$ is said to be a (*globally*) *Killing-transversally symmetric space* (briefly, a KTS-*space*) if and only if for each $m \in M$ there exists a (unique) global isometry $s_m : M \to M$ such that

$$s_{m*}(m) = (-I + 2\eta \otimes \xi)(m)$$

on T_mM.

The isometry s_m is called the *reflection* of M at m with respect to the flow line of ξ through m. Since it reverses the transversal geodesics through m, s_m is the unique extension of the local reflection at m to the whole of M. So, it follows that a KTS-space is a locally KTS-space. Further, a complete, simply connected locally KTS-space is a KTS-space [4]. Hence, we have

Proposition 2.6 [7]. *Let $(M, g, \mathfrak{F}_\xi)$ be a complete locally* KTS-*space and let $(\tilde{M}, \Psi)$ be the universal covering manifold of M. Then $(\tilde{M}, \tilde{g} = \Psi^* g, \mathfrak{F}_{\tilde{\xi}})$, where $\tilde{\xi}$ is the lift of ξ to $\tilde{M}$, is a* KTS-*space.*

KTS-spaces are homogeneous Riemannian manifolds [7], [10]. Hence ξ is a regular vector field and the orbit space $M' = M/\xi$ admits a unique structure as a differentiable manifold such that the natural projection $\pi : M \to M'$ is a submersion. Further, the existence of reflections s_m, $m \in$

M, implies that M' is a symmetric space [7] and that the projection π intertwines the reflections of M with the geodesic symmetries of M'.

Now, we concentrate on *contact* KTS-*spaces*. Such spaces are always irreducible Riemannian manifolds [6]. Moreover, in [7] we have proved that a simply connected reducible KTS-space is a Riemannian product of a contact KTS-space and a Riemannian symmetric space.

Before starting the next result, we note that a *Sasakian manifold* is a Riemannian manifold equipped with an isometric flow $\mathfrak{F}_\xi$ such that

$$R_{UV}\xi = \eta(U)V - \eta(V)U$$

[2]. Then $(\varphi = H, \xi, \eta, g)$ is called a Sasakian structure on M. Further, $(M, g, \varphi, \xi, \eta)$ is said to be φ-*symmetric* [16], [3] if $(M, g, \mathfrak{F}_\xi)$ is a locally KTS-space.

Relating to the symmetric base space of a contact KTS-space we mention now two useful results.

Proposition 2.7 [7]**.** *Let $(M, g, \mathfrak{F}_\xi)$ be a contact* KTS-*space such that the base space (M', g') is an irreducible symmetric space. Then the sectional curvature $K(X, \xi)$, X horizontal, is a non-vanishing constant c^2. Moreover, $(M, c^2 g, c^{-1}H, c^{-1}\xi, c\eta)$ is a φ-symmetric space which fibers over the Hermitian symmetric space $(M', c^2 g', c^{-1}H')$.*

Note that $H'X' = \pi_*(HX'^*)$, $X' \in \mathfrak{X}(M')$.

Proposition 2.8 [7]**.** *The base space (M', g') of a simply connected contact* KTS-*space $(M, g, \mathfrak{F}_\xi)$ is a (simply connected) Hermitian symmetric space. Moreover, we have*

- (i) *if $M' = M'_0 \times M'_1 \times \cdots \times M'_r$ is its de Rham decomposition and $\mathfrak{H}_i$, $i = 0, 1, \ldots, r$, are the smooth distributions obtained by taking the horizontal lifts of the tangent vectors of M'_i, then $\mathfrak{H}(m) = \mathfrak{H}_0(m) \oplus \mathfrak{H}_1(m) \oplus \cdots \oplus \mathfrak{H}_r(m)$ is an H-invariant orthogonal decomposition of the horizontal subspace $\mathfrak{H}(m)$, for each $m \in M$;*
- (ii) *each sectional curvature $K(\mathfrak{H}_j, \xi)$, $j = 1, \ldots, r$ is a positive constant c_j^2;*
- (iii) *the $(1,1)$-tensor field*

$$J = J_0 \times \frac{1}{c_1}H'_1 \times \cdots \times \frac{1}{c_r}H'_r \tag{2.12}$$

is a Hermitian structure on (M', g') *where* J_0 *denotes the canonical almost complex structure on* $M'_0{=}E^{2p}(x_1, \ldots, x_{2p})$ *and* $H'_j = H' \circ p_j$, $j = 1, \ldots, r$ *where* $p_j : M' \to M'_j$ *denotes the projection of* M' *onto* M'_j;

(iv) $H'_0 = H' \circ p_0$ *on* $E^{2p}(x_1, \ldots, x_{2p})$ *is given by the matrix*

$$\left(\begin{array}{ccc|ccc} & & & -\mu_1 & & \\ & 0 & & & \ddots & \\ & & & & & -\mu_p \\ \hline \mu_1 & & & & & \\ & \ddots & & & 0 & \\ & & \mu_p & & & \end{array}\right)$$

for certain positive real numbers $\mu_1, \ldots, \mu_p$.

Remark 2.1. Proposition 2.6 shows that the universal covering $\tilde{M}$ of a complete contact locally KTS-space $(M, g, \mathfrak{F}_\xi)$ is a simply connected contact KTS-space and then, using Proposition 2.8, we can consider each local submersion $\pi : \mathcal{U} \to \mathcal{U}' = \mathcal{U}/\xi$ as a mapping onto an open subset $\mathcal{U}'$ of the Hermitian symmetric space M' which is the base space of $\tilde{M}$. This means that for the complete contact locally KTS-space $(M, g, \mathfrak{F}_\xi)$ the foliation $\mathfrak{F}_\xi$ is transversally modelled on M' (see [19]). Let $M' = M'_0 \times M'_1 \times \cdots \times M'_r$ be its de Rham decomposition, with $\dim M'_0 = 2p$, and put $\mathcal{U}' = \mathcal{U}/\xi = \mathcal{U}'_0 \times \mathcal{U}'_1 \times \cdots \times \mathcal{U}'_r$ where $\mathcal{U}'_i$, $i = 0, 1, \ldots, r$, is a connected open subset of M'_i. Then it follows again from Proposition 2.8 that there exist $r + p$ real numbers $c_1, \ldots, c_r, \mu_1, \ldots, \mu_p$ and, on each distinguished chart $\mathcal{U} \subset M$, smooth distributions $\mathfrak{H}_0, \mathfrak{H}_1, \ldots, \mathfrak{H}_r$ satisfying, for $\mathcal{U}$ and $\mathcal{U}'$, the conditions (i)–(iv) of Proposition 2.8. Note that the c_j, μ_j and $\dim \mathfrak{H}_j$ are the same for every $\mathcal{U}$ and that $\left\{ \left(\frac{\partial}{\partial x_1}\right)^*, \ldots, \left(\frac{\partial}{\partial x_{2p}}\right)^* \right\}$ is an orthonormal frame field of $\mathfrak{H}_0$ which satisfies $K\left(\left(\frac{\partial}{\partial x_k}\right)^*, \xi\right) = K\left(\left(\frac{\partial}{\partial x_{p+k}}\right)^*, \xi\right) = \mu_k^2$, $k = 1, \ldots, p$.

3. H- and ξ-sectional curvatures

In this section we focus our attention on the sectional curvature function on $(M, g, \mathfrak{F}_\xi)$ when we restrict it to some particular planes.

Let $m \in M$. A plane section in T_mM is called an *H-section* if there exists a horizontal X in T_mM such that $\{X, HX\}$ is a basis of this section. The sectional curvature $K(X, HX)$ of an H-section is called the *H-sectional curvature* corresponding to X. In a similar way, a plane section in T_mM spanned by ξ and a horizontal X is called a *ξ-section* and the corresponding sectional curvature $K(X, \xi)$ is called the *ξ-sectional curvature* corresponding to X.

Now, let (M, g) be equipped with a flow $\mathfrak{F}_\xi$ such that the ξ-sectional curvature is globally constant, that is, independent of $X \in T_mM$ and $m \in M$. Then (M, g) is said to be of *constant ξ-sectional curvature*. By using (2.8) we may easily see that a Sasakian manifold is a Riemannian manifold equipped with a normal flow $\mathfrak{F}_\xi$ such that the ξ-sectional curvature equals 1. Moreover, if a normal $\mathfrak{F}_\xi$ on (M, g) is such that $K(X, \xi) = c^2 > 0$, then $(M, c^2g, \varphi = c^{-1}H, c^{-1}\xi, c\eta)$ is a Sasakian manifold. So, [2, Chapter V] yields

Proposition 3.1. *Let $\mathfrak{F}_\xi$ be a contact normal flow on a Riemannian manifold (M, g) with constant ξ-sectional curvature. Then the H-sectional curvatures determine the curvature of (M, g) completely.*

Note that on a Sasakian manifold a plane section π is a φ-section (or equivalently, an H-section) if and only if it is φ-invariant, that is, $\varphi(\pi) = \pi$. In the general case considered above, an H-section is not necessarily H-invariant. In fact, we have

Proposition 3.2. *Let $\mathfrak{F}_\xi$ be a contact normal flow on a Riemannian manifold (M, g). Then all H-sections are H-invariant if and only if (M, g) is of constant ξ-sectional curvature.*

Proof. Because of the remarks made above we have only to prove the "only if" part. Therefore, let $m \in M$ and suppose that all H-sections are H-invariant. Let $\{X_1, \dots, X_{2n}, \xi\}$ be an orthonormal basis of T_mM, $\dim T_mM = 2n + 1$, and $\lambda_1, \dots, \lambda_n$ real non-vanishing numbers such that

$$\begin{gathered} HX_1 = \lambda_1 X_2, \quad HX_2 = -\lambda_1 X_1, \\ \dots \\ HX_{2n-1} = \lambda_n X_{2n}, \quad HX_{2n} = -\lambda_n X_{2n-1}. \end{gathered}$$

Put $X = \sum_{i=1}^{n} (\alpha_{2i-1}X_{2i-1} + \alpha_{2i}X_{2i})$ for X horizontal. Then

$$H^2X = -\sum_{i=1}^{n} \lambda_i^2 (\alpha_{2i-1}X_{2i-1} + \alpha_{2i}X_{2i}).$$

Since $\mathfrak{F}_\xi$ is contact and $g(H^2X, HX) = 0$, we have for an invariant section $\{X, HX\}$, $H^2X = \lambda X$, $\lambda \in \mathbb{R} - \{0\}$. Hence, since X is arbitrary, we get $\lambda = -\lambda_i^2$, $i = 1, \ldots, n$. So, from (2.5) we obtain $K(X, \xi) = -\lambda$ and hence the ξ-sectional curvature does not depend on $X \in T_mM$, that is, λ only depends on m. Further, let γ be an arbitrary unit speed tranversal geodesic. Then (2.5) and (2.6) yield

$$\gamma' K(\gamma', \xi) = \gamma' g(H\gamma', H\gamma') = 2g((\nabla_{\gamma'} H)\gamma', H\gamma') = 0.$$

Further, we also have $\xi K(X, \xi) = 0$ for any horizontal ξ-invariant field X since ξ is a Killing vector field. All this yields the required result. $\square$

An H-section on $(\mathcal{U}, g_{|\mathcal{U}})$ defines an H'-section on the base space $(\mathcal{U}', g')$ of the local fibration $\pi : \mathcal{U} \to \mathcal{U}' = \mathcal{U}/\xi$. Using (2.5) and (2.10) one derives the following relation between the H- and H'-sectional curvatures:

$$(K'(X', H'X'))^* = K(X'^*, HX'^*) + 3K(X'^*, \xi) \tag{3.1}$$

where $X' \in \mathfrak{X}(\mathcal{U}')$. This implies that *Sasakian space forms* $M(k)$, that is, Sasakian manifolds of constant φ-sectional curvature k, fiber locally over Kähler manifolds of constant holomorphic sectional curvature $k + 3$ (see also [14], [21]).

Now, let $(M, g, \mathfrak{F}_\xi)$ be a complete, contact locally KTS-space. From Remark 2.1 we know that $\mathfrak{F}_\xi$ is transversally modelled on a Hermitian symmetric space $M' = M'_0 \times M'_1 \times \cdots \times M'_r$ and each horizontal vector $X \in T_mM$ can be written as $X = \sum_{i=0}^{r} X_i$ where $X_i \in \mathfrak{H}_i(m)$. Hence we have

$$K(X, \xi) = \frac{1}{\|X\|^2} \left\{ \sum_{k=1}^{p} \mu_k^2 \left\{ (X_0^k)^2 + (X_0^{p+k})^2 \right\} + \sum_{j=1}^{r} c_j^2 g(X_j, X_j) \right\}$$

where the X_0^i, $i = 1, \ldots, 2p$, are the components of X_0 with respect to the basis $\left\{ \left(\frac{\partial}{\partial x_i}\right)^*, i = 1, \ldots, 2p \right\}$ of $\mathfrak{H}_0(m)$. Hence we have

Proposition 3.3. *The ξ-sectional curvatures on a complete contact locally* KTS-*space* $(M, g, \mathfrak{F}_\xi)$ *are determined by the scalars* $c_1, \dots, c_r$, $\mu_1, \dots, \mu_p$.

Next, we prove

Theorem 3.1. *On a complete contact locally* KTS-*space* $(M, g, \mathfrak{F}_\xi)$, *the H- and ξ-sectional curvatures determine the curvature of* (M, g) *completely.*

Proof. Put

$$\begin{aligned} B(U,V) &= R(U,V,U,V), \quad & B'(X',Y') &= R'(X',Y',X',Y'), \\ D(U) &= B(U,HU), & D'(X') &= B'(X',JX') \end{aligned}$$

for all $U, V \in \mathfrak{X}(\mathcal{U})$ and $X', Y' \in \mathfrak{X}(\mathcal{U}')$.

It is well-known that the curvature of a Riemannian manifold is completely determined by the sectional curvatures and that the curvature of a Kähler manifold is determined by the holomorphic sectional curvatures. More precisely, we have

$$\begin{aligned} 12R(U,V,Z,W) &= B(U+Z,V+W) + B(U-Z,V-W) \\ &\quad -B(U+W,V+Z) - B(U-W,V-Z) \\ &\quad -2B(U,W) - 2B(V,Z) + 2B(U,Z) + 2B(V,W) \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} 32B'(X',Y') &= 3D'(X'+JY') + 3D'(X'-JY') \\ &\quad -D'(X'+Y') - D'(X'-Y') - 4D'(X') - 4D'(Y') \end{aligned} \tag{3.3}$$

for $U, V, Z, W \in \mathfrak{X}(\mathcal{U})$, $X', Y' \in \mathfrak{X}(\mathcal{U}')$.

From (2.9) we get

$$(B'(X',Y'))^* = B(X'^*, Y'^*) + 3(h(X'^*, Y'^*))^2.$$

Hence, using (2.12) and (3.3) we get, for $X' = \sum_{i=0}^{r} X'_i, Y' = \sum_{i=0}^{r} Y'_i, X'_i, Y'_i \in \mathfrak{X}(\mathcal{U}'_i)$,

$$
\begin{aligned}
B(X'^*, Y'^*) = \frac{1}{32}\sum_{j=1}^{r}\Bigg\{ & 3\left(D'(X'_j + \frac{1}{c_j}H'Y'_j)\right)^* + 3\Big(D'(X'_j - \frac{1}{c_j}H'Y'_j)\Big)^* \\
& - (D'(X'_j + Y'_j))^* - (D'(X'_j - Y'_j))^* \\
& - 4(D'(X'_j))^* - 4(D'(Y'_j))^* \Bigg\} - 3(h(X'^*, Y'^*))^2.
\end{aligned} \tag{3.4}
$$

Now, since

$$(D'(X'_j))^* = \frac{1}{c_j^2}D(X'^*_j) + 3c_j^2(g(X'^*_j, X'^*_j))^2,$$

(3.4) becomes

$$
\begin{aligned}
B(X, Y) = \frac{1}{32}\sum_{j=1}^{r}\Bigg\{ & \frac{1}{c_j^2}\Bigg\{3D\left(X_j + \frac{1}{c_j}HY_j\right) + 3D\left(X_j - \frac{1}{c_j}HY_j\right) \\
& - D(X_j + Y_j) - D(X_j - Y_j) - 4D(X_j) - 4D(Y_j)\Bigg\} \\
& + 72(g(X_j, HY_j))^2 + 24c_j^2 g(X_j, X_j)g(Y_j, Y_j) \\
& - 24c_j^2(g(X_j, Y_j))^2\Bigg\} - 3(h(X, Y))^2,
\end{aligned} \tag{3.5}
$$

where now $X = \sum_{i=0}^{r} X_i$, $Y = \sum_{i=0}^{r} Y_i$ and $X_i, Y_i \in \mathfrak{H}_i$.

To obtain the sectional curvature for a plane spanned by two orthonormal vectors U, V, we put

$$U = aX + \eta(U)\xi, \quad V = bY + \eta(V)\xi$$

for unit horizontal X, Y. Then we have $a^2 = 1 - (\eta(U))^2$, $b^2 = 1 - (\eta(V))^2$ and

$$abg(X, Y) + \eta(U)\eta(V) = 0.$$

So, from (2.7) and (2.8) we obtain

$$
\begin{aligned}
K(U, V) = R(U, V, U, V) = a^2b^2R(X, Y, X, Y) + a^2(\eta(V))^2R(X, \xi, X, \xi) \\
+b^2(\eta(U))^2R(Y, \xi, Y, \xi) - 2ab\eta(U)\eta(V)R(X, \xi, Y, \xi)
\end{aligned}
$$

and consequently,

$$
\begin{aligned}
K(U,V) = {}& \Big(1 - (\eta(U))^2 - (\eta(V))^2\Big) K(X,Y) \\
& + \Big(1 - (\eta(U))^2\Big)(\eta(V))^2 K(X,\xi) + \Big(1 - (\eta(V))^2\Big) \\
& \times (\eta(U))^2 K(Y,\xi) - 2\eta(U)\eta(V) g(HU,HV).
\end{aligned} \tag{3.6}
$$

Now, the result follows from (3.2) and (3.5). □

4. Normal flow space forms

In what follows we restrict to contact flows $\mathfrak{F}_\xi$ on Riemannian manifolds and introduce the notion of a flow space form inside the class of such Riemannian manifolds.

Definition 4.1. Let (M,g) be a Riemannian manifold equipped with a contact flow $\mathfrak{F}_\xi$. Then (M,g) is called a *flow space form* if and only if the H-sectional curvature is pointwise constant and (M,g) is called a *normal flow space form* if $\mathfrak{F}_\xi$ is also normal.

Now we shall make a more detailed study of the normal flow space forms and consider two cases.

Case 1. (M,g) is a normal flow space form of constant ξ-sectional curvature c^2.

Suppose that the H-sectional curvature equals k. Then $(M, c^2 g, \varphi = c^{-1}H, c^{-1}\xi, c\eta)$ is a Sasakian manifold of constant φ-sectional curvature kc^{-2} and so, (M,g) is obtained by a homothetic change of metric from Sasakian space forms. Hence, from (3.2) one gets that each base space $(\mathcal{U}', g')$ of a local fibering is a Kähler manifold of constant holomorphic sectional curvature $k + 3c^2$. From this we also obtain that k is globally constant when $\dim M \geq 5$. This follows also from the theory for Sasakian space forms (see [2]). Because of this, one usually supposes k to be globally constant when $\dim M = 3$. Further, as in [2] we get

Proposition 4.1. *Let $\mathfrak{F}_\xi$ be a normal flow on a Riemannian manifold (M,g) with constant ξ-sectional curvature $c^2 > 0$ and* $\dim M \geq 5$. *If the*

H-sectional curvature is pointwise constant, then it is globally constant and the curvature tensor is given by

$$R_{UV}W = \frac{k+3c^2}{4}\{g(U,W)V - g(V,W)U\}$$
$$+\frac{k-c^2}{4}\{\eta(V)\eta(W)U - \eta(U)\eta(W)V + g(V,W)\eta(U)\xi - g(U,W)\eta(V)\xi\}$$
$$+\frac{k-c^2}{4c^2}\{g(W,HU)HV - g(W,HV)HU - 2g(U,HV)HW\}$$

where k is the constant H-sectional curvature.

Note that for $\dim M = 3$ the curvature tensor always takes the form as in Proposition 4.1 but then k is not necessarily constant.

In what follows we shall denote such a normal flow space form by $M^{2n+1}(c^2,k)$. Note that from the classification theorem for Sasakian space forms we deduce that, up to isomorphisms, there exists a unique simply connected complete $M^{2n+1}(c^2,k)$ with given c^2 and k. Model spaces may be constructed directly from the corresponding model spaces of Sasakian space forms. We refer to [2], [17], [21] for more details and further references. One may also use the construction method given in [7], [11]. This leads to the model spaces

(i) $\left(S^{2n+1} = \frac{U(n+1)}{U(n)}\right)(c^2,k)$ for $k+3c^2>0$;

(ii) $H(n,1)(k)$ for $k+3c^2=0$ (here $H(n,1)$ denotes the $(2n+1)$-dimensional Heisenberg group);

(iii) $\left(\left(\frac{U(1,n)}{U(n)}\right)^{\sim} = \frac{SU(1,n)^{\sim}}{SU(n)}\right)(c^2,k)$ for $k+3c^2<0$ (where $^{\sim}$ denotes the universal covering).

Before considering the next case, we prove

Proposition 4.2. *Let $(M,g,\mathfrak{F}_\xi)$ be a normal flow space form with pointwise constant H-sectional curvature k. Then it is a (contact) locally* KTS*-space if and only if k is globally constant.*

Proof. The curvature of (M,g) satisfies

$$R(X,HX,X,HX) = k\|X\|^4 K(X,\xi) \tag{4.1}$$

for all horizontal vector fields X.

Now, let γ be a transversal geodesic with unit tangent $\gamma' = X$. Then, using (2.5), (2.6) and (4.1) we get

$$
\begin{aligned}
&(\nabla_X R)(X, HX, X, HX) \\
&\qquad = XR(X, HX, X, HX) - 2R(X, (\nabla_X H)X, X, HX) \\
&\qquad = X\{kK(X,\xi)\} - 2R(X, (\nabla_X H)X, X, HX) = X(k)K(X,\xi).
\end{aligned}
$$

Further, for a normal flow we have $(\nabla_\xi R)(X, HX, X, HX) = 0$ and hence $\xi(k) = 0$. The result then follows by means of Proposition 2.3. □

Case 2. (M, g) is a normal flow space form with non-constant ξ-sectional curvature and globally constant H-sectional curvature.

We start with

Theorem 4.1. *Let $(M, g, \mathfrak{F}_\xi)$ be a complete normal flow space form of dimension $2n+1$, of constant H-sectional curvature k and with non-constant ξ-sectional curvature. Then we have*

(i) *there exist smooth distributions $\mathfrak{H}_1$ and $\mathfrak{H}_2$ on M such that for each $m \in M$, $\mathfrak{H}(m) = \mathfrak{H}_1(m) \oplus \mathfrak{H}_2(m)$ is an H-invariant decomposition of the horizontal subspace $\mathfrak{H}(m)$ and each sectional curvature $K(\mathfrak{H}_i, \xi)$, $i = 1, 2$, is a positive constant c_i^2 $(c_1^2 > c_2^2)$;*

(ii) *the H-sectional curvature k is a strictly negative constant given by*

$$
k = -6\frac{c_1^2 c_2^2}{c_1^2 + c_2^2}; \tag{4.2}
$$

(iii) *the flow $\mathfrak{F}_\xi$ is locally transversally modelled on $\mathbb{C}P^{n_1}(k_1) \times \mathbb{C}H^{n_2}(k_2)$ where $k_i = k + 3c_i^2$, $2n_i = \dim \mathfrak{H}_i$, $i = 1, 2$.*

Proof. It follows from Proposition 4.2 that $(M, g, \mathfrak{F}_\xi)$ is a complete contact locally KTS-space. So, we may use Remark 2.1. In what follows, we use the same notation as in that remark. First, suppose $\mathfrak{H}_0(m) \neq \emptyset$, $m \in \mathcal{U}$. From (3.1) we get

$$
k = -3K(X_0, \xi) \tag{4.3}
$$

for all $X_0 \in \mathfrak{H}_0(m)$. Using the same (3.1), we also obtain

$$
K'(X'_j, JX'_j) = K'(X'_j, H'X'_j) = k + 3c_j^2 \tag{4.4}
$$

for all $X'_j \in \mathfrak{X}(\mathcal{U}'_j)$, $j = 1, \dots, r$. Put $X = X'^* = X_0 + X_1 + \cdots + X_r$ where $X' \in T_{m'}\mathcal{U}'$, $m' = \pi(m)$ and $X_i \in \mathfrak{H}_i(m)$, $i = 0, 1, \dots, r$. Then, from (2.5) and (4.3) we have

$$K(X,\xi) = \frac{-\dfrac{k}{3}\|X_0\|^2 + \displaystyle\sum_{j=1}^{r} c_j^2 \|X_j\|^2}{\displaystyle\sum_{i=0}^{r} \|X_i\|^2} \tag{4.5}$$

and from (4.4):

$$K'(X', H'X') = \frac{\displaystyle\sum_{j=1}^{r} (k + 3c_j^2) c_j^2 \|X_j\|^4}{\left(\displaystyle\sum_{i=0}^{r} \|X_i\|^2\right)\left(-\dfrac{k}{3}\|X_0\|^2 + \displaystyle\sum_{j=1}^{r} c_j^2 \|X_j\|^2\right)}. \tag{4.6}$$

Using (4.5) and (4.6) we rewrite (3.1) as follows:

$$-\frac{k}{3}\|X_0\|^2 \sum_{j=1}^{r} (k + 3c_j^2)\|X_j\|^2 + \sum_{\substack{i,j=1\\ i\neq j}}^{r} c_i^2 (k + 3c_j^2)\|X_i\|^2 \|X_j\|^2 = 0. \tag{4.7}$$

In particular, putting $X = X_0 + X_h$, $1 \le h \le r$, (4.7) yields

$$-\frac{k}{3}\|X_0\|^2 \|X_h\|^2 (k + 3c_h^2) = 0$$

and hence $k = -3c_h^2$. So (M, g) has constant ξ-sectional curvature which contradicts the hypotheses.

Next, we suppose that $\mathfrak{H}(m) = \sum_{j=1}^{r} \mathfrak{H}_j(m)$. Then (4.7) reduces to

$$\sum_{\substack{i,j=1\\ i\neq j}}^{r} c_i^2 (k + 3c_j^2)\|X_i\|^2 \|X_j\|^2 = 0$$

and we obtain

$$k = -3\,\frac{\displaystyle\sum_{\substack{i,j=1\\ i\neq j}}^{r} c_i^2 c_j^2 \|X_i\|^2 \|X_j\|^2}{\displaystyle\sum_{\substack{i,j=1\\ i\neq j}}^{r} c_i^2 \|X_i\|^2 \|X_j\|^2} < 0.$$

Putting $X = X_h + X_l$, $1 \leq h \neq l \leq r$, this expression leads to

$$k = -6\frac{c_h^2 c_l^2}{c_h^2 + c_l^2}. \tag{4.8}$$

If $r \geq 3$, then it follows that $c_j^2 = c^2$ for $1 \leq j \leq r$ and we get again a contradiction.

Hence, we have proved that $\mathfrak{H}(m) = \mathfrak{H}_1(m) \oplus \mathfrak{H}_2(m)$ for $m \in \mathcal{U}$. To prove (i) we also need to check that $\mathfrak{H}_1$ and $\mathfrak{H}_2$ determine global distributions. So, let $\mathfrak{H}_i$ and $\overline{\mathfrak{H}}_i$, $i = 1, 2$, be the corresponding distributions on $\mathcal{U}$ and $\overline{\mathcal{U}}$ with $\mathcal{U} \cap \overline{\mathcal{U}} \neq \emptyset$. Put, $\overline{X}_1 = X_1 + X_2$ where $\overline{X}_1 \in \overline{\mathfrak{H}}_1$ and $X_i \in \mathfrak{H}_i$ for $i = 1, 2$. Then, from (2.12) we have

$$c_1^2 = K(\overline{X}_1, \xi) = \frac{c_1^2\|X_1\|^2 + c_2^2\|X_2\|^2}{\|X_1\|^2 + \|X_2\|^2}$$

and so $X_1 = \overline{X}_1$. Doing the same for $\overline{\mathfrak{H}}_2$ we get $\mathfrak{H}_i = \overline{\mathfrak{H}}_i$, $i = 1, 2$ on $\mathcal{U} \cap \overline{\mathcal{U}}$ and so the $\mathfrak{H}_i$ may be extended to the whole of M.

Finally, (ii) follows from (4.8) and (iii) from (4.4). □

So, we have from the remarks made above and from Theorem 4.1:

Corollary 4.1. *Let $(M, g, \mathfrak{F}_\xi)$ be a complete normal flow space form with non-negative globally constant H-sectional curvature. Then the ξ-sectional curvature is constant and the manifold is locally isometric to a model space $S^{2n+1}(c^2, k)$.*

Further, we have

Theorem 4.2. *Let $\mathfrak{F}_\xi$ be a normal contact flow on a complete $(2n+1)$-dimensional Riemannian manifold (M, g) with non-constant ξ-sectional curvature. Then the H-sectional curvature is globally constant if and only if $\mathfrak{F}_\xi$ is transversally modelled on $\mathbb{C}P^{n_1}(k_1) \times \mathbb{C}H^{n_2}(k_2)$ where $|k_2| < k_1$, $n_1 + n_2 = n$ and the ξ-sectional curvatures $c_i^2 = K(X_i, \xi)$, $i = 1, 2$, satisfy*

$$c_i^2 = (-1)^{i+1} k_i \frac{k_1 - k_2}{3(k_1 + k_2)}. \tag{4.9}$$

Further, in this case we have

$$k = 2\frac{k_1 k_2}{k_1 + k_2} \tag{4.10}$$

and the curvature tensor is given by

$$
\begin{aligned}
(4.11) \qquad R_{UV}W = \sum_{i=1}^{2} \Bigg\{ & \frac{k_i}{4}\Big(g(X_i, Z_i)Y_i - g(Y_i, Z_i)X_i\Big) \\
& +(-1)^i \frac{3(k_1+k_2)}{4(k_1-k_2)}\Big(g(HY_i, Z_i)HX_i - g(HX_i, Z_i)HY_i - 2g(HX_i, Y_i)HZ_i\Big) \\
& +(-1)^i \frac{k_i(k_1-k_2)}{3(k_1+k_2)}\Big\{\big(g(Y_i, Z_i)\eta(U) - g(X_i, Z_i)\eta(V)\big)\xi \\
& +\eta(W)\big(\eta(V)X_i - \eta(U)Y_i\big)\Big\}\Bigg\} \\
& +g(HV, W)HU - g(HU, W)HV - 2g(HU, V)HW
\end{aligned}
$$

for vector fields $U = \sum_{i=1}^{2} X_i + \eta(U)\xi$, $V = \sum_{i=1}^{2} Y_i + \eta(V)\xi$, $W = \sum_{i=1}^{2} Z_i + \eta(W)\xi$ *on* M.

PROOF. For a normal flow space form of globally constant H-sectional curvature the result follows from Theorem 4.1.

So, we prove the converse. First, Proposition 2.4 implies that $(M, g, \mathfrak{F}_\xi)$ is a locally KTS-space and then, using Remark 2.1, we have $H' = c_1 J_1 \times c_2 J_2$. Hence, taking into account (3.1), (4.9) and (4.10) we get the required result by explicit computations.

Next, we prove (4.11). Using the well-known expression for the curvature tensor of a Kähler manifold of constant holomorphic sectional curvature, (2.9) and (2.12), we have

$$
\begin{aligned}
R_{XY}Z = \sum_{i=1}^{2} \frac{k_i}{4}\Big\{ & g(X_i, Z_i)Y_i - g(Y_i, Z_i)X_i \\
& +\frac{1}{c_i^2}\Big(g(HX_i, Z_i)HY_i - g(HY_i, Z_i)HX_i + 2g(HX_i, Y_i)HZ_i\Big)\Big\} \\
& +g(HY, Z)HX - g(HX, Z)HY - 2g(HX, Y)HZ
\end{aligned}
$$

for horizontal $X = \sum_{i=1}^{2} X_i$, $Y = \sum_{i=1}^{2} Y_i$, $Z = \sum_{i=1}^{2} Z_i$. Now, the required result follows from (4.9), (2.7) and (2.8). $\square$

Remark 4.1. A. Note that under the same hypotheses as in Theorem 4.2 and if M is simply connected, then the base space $M'=(M/\xi, g', J)$ is holomorphically isometric to $\mathbb{C}P^{n_1}(k_1) \times \mathbb{C}H^{n_2}(k_2)$.

B. In the considered Case 2 we supposed the H-sectional curvature to be globally constant. We do not know what happens when it is only pointwise constant because we do not know if there exists here a Schur-like theorem or, following Proposition 4.2, if such a normal flow space form is a locally KTS-space.

With respect to the classification of normal flow space forms inside the class of KTS-spaces, we have

Theorem 4.3. *Let $(M_1, g_1, \mathfrak{F}_{\xi_1})$ and $(M_2, g_2, \mathfrak{F}_{\xi_2})$ be simply connected, complete normal flow space forms fibering over the same $\mathbb{C}P^{n_1}(k_1) \times \mathbb{C}H^{n_2}(k_2)$. Then the space forms are isomorphic.*

Proof. (2.12) implies that the possible tensor fields H' on $\mathbb{C}P^{n_1}(k_1) \times \mathbb{C}H^{n_2}(k_2)$ may be written as $H'_{ab} = aJ_1 \times bJ_2$ where $a = \pm c_1$, $b = \pm c_2$, $c_1, c_2 > 0$ and where c_1, c_2 are given by (4.9). Fixing a point $o' \in M'$, we can define a linear isometry L of $T_{o_1}M_1$ onto $T_{o_2}M_2$ where $\pi_1(o_1) = \pi_2(o_2) = o'$ satisfying the conditions of Theorem 2.1. In fact, if $\{e'_{2i-1}, e'_{2i} = J_1 e'_{2i-1}, v'_{2j-1}, v'_{2j} = J_2 v'_{2j-1}, 1 \le i \le n_1, 1 \le j \le n_2\}$ is an orthonormal basis of $T_{o'}M'$, then with respect to the lifted bases $\{(e'_{2i-1})^*, (e'_{2i})^*, (v'_{2j-1})^*, (v'_{2j})^*, \xi_l\}$ of $T_{o_l}M_l$, $l = 1, 2$, the linear map L given by

$$L(e'_{2i-1})^* = -\mathrm{sign}(a_1 a_2)(e'_{2i})^*, \quad L(e'_{2i})^* = (e'_{2i-1})^*,$$
$$L(v'_{2j-1})^* = -\mathrm{sign}(b_1 b_2)(v'_{2j})^*, \quad L(v'_{2j})^* = (v'_{2j-1})^*, \quad L\xi_1 = \xi_2,$$

where $H'_{a_l b_l}$, $l = 1, 2$, is the corresponding tensor on the base space of $(M_l, g_l, \mathfrak{F}_{\xi_l})$, satisfies (i) and (ii) in Theorem 2.1. Moreover, the expression (4.11) shows that (iii) in Theorem 2.1 also holds. $\square$

We finish this paper by indicating briefly a construction of explicit models for these normal flow space forms corresponding to the coefficients $(n_1, n_2; k_1, k_2)$. We use the technique developed in [7]. First, consider the model spaces of Case 1: $S^{2n_1+1}(c_1^2, k) = \left(\dfrac{U(n_1+1)}{U(n_1)}, g_1, \xi_1\right)$ and $\left(\dfrac{U(1,n_2)}{U(n_2)}\right)^{\sim}(c_2^2, k) = \left(\left(\dfrac{U(1,n_2)}{U(n_2)}\right)^{\sim}, g_2, \xi_2\right)$ where

$$c_i^2 = (-1)^{i+1} k_i \frac{k_1 - k_2}{3(k_1 + k_2)} \quad \text{and } k = 2\frac{k_1 k_2}{k_1 + k_2}.$$

Using [7, Remark 4.1], the length of the integral curves of ξ_1 and so, of any great circle of $S^{2n_1+1}(c_1^2,k)$, is

$$l=\Big|\frac{8\pi c_1}{k_1}\Big|=8\pi\Big(\frac{k_1-k_2}{3k_1(k_1+k_2)}\Big)^{1/2}.$$

$S^{2n_1+1}(c_1^2,k)$ is a principal S^1-bundle over $\mathbb{C}P^{n_1}(k_1)$ and, if we identify the circle group S^1 with the set $\{e^{2\pi it},t\in\mathbb{R}\}$, the action of S^1 on $S^{2n_1+1}(c_1^2,k)$ is given by

$$m\circ e^{2\pi it}=\psi^1_{lt}(m)$$

where $\{\psi^1_t\}_{t\in\mathbb{R}}$ is the one-parameter group of global transformations generated by ξ_1. The corresponding fundamental vector field ς_1 generated by $\frac{d}{dt}$ is given by $\varsigma_1=l\xi_1$ and $l^{-1}\eta_1$ defines a connection form on $S^{2n_1+1}(c_1^2,k)$.

On the other hand, $\left(\dfrac{U(1,n_2)}{U(n_2)}\right)^{\sim}(c_2^2,k)$ is a principal $\mathbb{R}$-bundle over $\mathbb{C}H^{n_2}(k_2)$ where the action of $t\in\mathbb{R}$ on this bundle is identified with that of ψ^2_t where $\{\psi^2_t\}_{t\in\mathbb{R}}$ is the one-parameter group generated by ξ_2. Let $\mathbb{Z}[l]$ denote the subgroup of $\mathbb{R}$ generated by l. Then

$$M_2=\left(\frac{U(1,n_2)}{U(n_2)}\right)^{\sim}\Big/\mathbb{Z}[l]$$

is a principal S^1-bundle over $\mathbb{C}H^{n_2}(k_2)$ with l as length of the fibres. We use the same notation (g_2,ξ_2) for the induced structure.

Next, let M be the quotient space

$$M=\frac{S^{2n_1+1}\times\left(\dfrac{U(1,n_2)}{U(n_2)}\right)^{\sim}\Big/\mathbb{Z}[l]}{S^1}$$

where the action of S^1 on $S^{2n_1+1}\times\left(\dfrac{U(1,n_2)}{U(n_2)}\right)^{\sim}\Big/\mathbb{Z}[l]$ is defined by

$$(m_1,m_2)\circ e^{2\pi it}=\Big(\psi^1_{lt}(m_1),\psi^2_{-lt}(m_2)\Big).$$

Let $\rho: S^{2n_1+1}\times\left(\dfrac{U(1,n_2)}{U(n_2)}\right)^{\sim}\Big/\mathbb{Z}[l]\to M$ be the canonical projection and put $[(m_1,m_2)]=\rho(m_1,m_2)$. Then M is a principal circle bundle over $\mathbb{C}P^{n_1}(k_1)\times\mathbb{C}H^{n_2}(k_2)$ and the action of S^1 on M is given by

$$[(m_1,m_2)]e^{2\pi it}=\left[\big(\psi^1_{lt}(m_1),m_2\big)\right]=\left[\big(m_1,\psi^2_{lt}(m_2)\big)\right].$$

Hence, the fundamental vector field generated by $\frac{d}{dt}$ is

$$\varsigma = l\rho_*\xi_i, \quad i = 1, 2$$

and $l^{-1}\eta$, where η is the unique differential form on M such that

$$\rho^*\eta = \eta_1 \times \eta_2,$$

defines a connection form on M (see [7]).

Put $\xi = \rho_*\xi_i$, $i = 1, 2$. Then $\eta(\xi) = 1$. Finally, we denote by g the unique Riemannian metric on M such that $g(\xi, \xi) = 1$, ξ is orthogonal to $\ker \eta$ and $\pi : M \to \mathbb{C}P^{n_1}(k_1) \times \mathbb{C}H^{n_2}(k_2)$ becomes a Riemannian submersion. With respect to g, ξ defines a contact flow which is moreover normal. Further, from Theorem 4.2 we deduce that $(M, g, \mathfrak{F}_\xi)$ is a flow space form with H-sectional curvature k. We denote this by $M(n_1, n_2; k_1, k_2)$. Note that the length of the integral curves of ξ is precisely l.

Then we have

Theorem 4.4. *Let (M, g) be a complete, simply connected, $(2n+1)$-dimensional Riemannian manifold equipped with a normal flow $\mathfrak{F}_\xi$ such that the H-sectional curvature is globally constant and which fibers over $\mathbb{C}P^{n_1}(k_1) \times \mathbb{C}H^{n_2}(k_2)$, $|k_2| < k_1$ and $n_1 + n_2 = n$. Then $(M, g, \mathfrak{F}_\xi)$ is isomorphic to the universal covering of $M(n_1, n_2; k_1, k_2)$ where* $l = 8\pi \left(\dfrac{k_1 - k_2}{3k_1(k_1 + k_2)} \right)^{1/2}$.

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