Some probability problems in inventory control*)

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§ 1. Introduction

We consider the following inventory model which we call the infinite bin system of inventory control.

Demands for an item occur at the instants $t_1, t_2, \ldots, t_n, \ldots$ such that the $\tau_n = t_n - t_{n-1}$, $n = 1, 2, \ldots, t_0 = 0$, are independently and identically distributed non-negative random variables with common distribution function $A(x)$ and finite expectation $\alpha = \int_0^\infty x dA(x)$. Initially N items are held in stock and the demand process $\{t_n\}$ is about to start. After every k-th demand an order for a further k items is placed. There is given a sequence $\{\varkappa_n\}$, $n = 1, 2, \ldots$ of independently and identically distributed non-negative random variables with common distribution function $L(x)$ and finite expectation $l = \int_0^\infty x dL(x)$. $\varkappa_n$ is the lead time of the n-th order, that is the time interval from the instant t_{nk} at which the order is placed until delivery of that order is made. Demands which occur when stock is on hand are satisfied immediately, demands which occur when stock is not on hand are held and are satisfied from subsequent deliveries. Thus no demand is lost and negative inventories can be held.

Define random variables $\xi(t)$ and $\zeta(t)$ as follows: at time t let the number of outstanding orders be $\xi(t)$ and let the number of demands which have occured since the last order was placed be $\zeta(t)$. ($\zeta(t) = 0, 1, \ldots k-1$). Define a random variable $\eta(t)$ by the following equation

$$\eta(t) = \zeta(t) + k\xi(t). \tag{1}$$

The random variable $\eta(t)$ assumes the values $0, 1, 2, \ldots$ and will be called the stock deficit at time t. Let $S(t)$ be the inventory level at time t, then

$$S(t) = N - \eta(t). \tag{2}$$

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If $0 \leqq \eta(t) < N$ then stock is on hand, if $\eta(t) \geqq N$ no stock on hand and a negative inventory is held if $\eta(t) > N$.

In inventory problems interest is often centred on fluctuations in the stock level $S(t)$ and in this paper we study the processes $\{\eta(t)\}$, $\{\xi(t)\}$ and $\{\zeta(t)\}$.

The model will be denoted by the symbol $G(G)k$ where the first letter refers to the distribution of the demand process and the second letter to the distribution of the lead time and k, defined above, is the reorder quantity. The letter G refers to a general distribution. The following symbols will be used. M a negative exponential distribution, E_m an Erlang m-distribution, D the degenerate distribution of a random variable which is constant with probability 1. Thus for example the model $E_m(D)k$ refers to the model described above in which the demand process is an Erlang E_m process and the lead time is constant.

The following is a summary of the contents of this paper.

Section 2. The existence of the limiting distribution for $\eta(t)$ in $G(G)k$ is established when the demand distribution is not a lattice distribution and recurrence relations enabling this distribution to be obtained are given. Expressions for the mean and variance of the distribution are obtained. These expressions are simplified for the special cases $G(E_m)k$ $(m > 1)$ and $G(M)k$, finally the limiting distribution of $\{\zeta(t)\}$ is shewn to be uniform.

Section 3. We consider $G(M)k$ in detail. If the demand distribution is not a lattice distribution an explicit expression for the limiting distribution of $\eta(t)$ is obtained and for the limiting distribution of stock deficit at the instants a demand occurs. An explicit expression for the Laplace transform of the generating function of the distribution of $\eta(t)$ is obtained.

Section 4. We consider $D(G)k$ and obtain the generating function of the distribution of stock deficit on the n-th demand, and obtain the mean and variance of this distribution. For $D(M)k$ we obtain the limiting distribution of stock deficit at a demand.

Section 5. We obtain first the (limiting) mean value of stock deficit at a demand in $G(G)k$. For $G(D)k$ we obtain formulae for the distribution of stock deficit at the n-th demand and the limiting distribution of $\eta(t)$ when the demand distribution is not a lattice distribution. For $E_m(D)k$ we obtain explicit expressions for these distributions.

Some of the results obtained below seem to be the same as results which KARLIN and SCARF [4] and SCARF [6] state can be obtained (but do not do so) in their discussion of the model $G(M)k$. The case $k = 1$ corresponds to the results obtained by TAKÁCS [7]. We remark that the model $M(D)k$ has been considered by PITT [5].

§ 2. The process $\{\eta(t)\}$ in $G(G)k$

The distribution of the random variable $\eta(t)$ does not depend on N the initial stock level. Thus define $f(u, x)$ by

$$f(u,x) = 1 \quad \text{if} \quad 0 \leqq u < x$$
$$= 0 \quad \text{otherwise}$$

then

$$\eta(t) = k \sum_{m=1}^{\infty} f(t-t_{mk}, \varkappa_m) + \sum_{m=0}^{\infty} \sum_{n=1}^{k-1} f(t-t_{mk+n}, t_{(n+1)k+n} - t_{mk+n}). \tag{3}$$

Suppose conditionally that $t_k = x$. Then from (3) we obtain (an can also be seen intuitively),

$$\begin{aligned} \eta(t) &= v(t) && \text{if} \quad t < x \\ &= \eta^*(t-x)+k && \text{if} \quad x \leqq t < x+\varkappa_1 \\ &= \eta^*(t-x) && \text{if} \quad x+\varkappa_1 \leqq t \end{aligned} \tag{4}$$

where $v(t)$ is the number of demands in $(0, t)$ and $\eta^*(t)$ has the same distribution as $\eta(t)$ and is independent of $\varkappa_1$.

Let $A^{(j)}(x)$ be the j-th-fold iterated convolution of the distribution function $A(x)$ with itself. Then for $j = 0, 1, \dots k-1$

$$P(v(t) = j, t_k > t) = P(v(t) = j)$$
$$= A^{(j)}(t) - A^{(j+1)}(t).$$

Write $P_j(t) = P\{\eta(t) = j\}$, $j = 0, 1, 2, \dots$ then from (4) we obtain

$$P_j(t) = A^{(j)}(t) + A^{(j+1)}(t) + \int_0^t P_j(t-x) L(t-x) dA^{(k)}(x) \quad j = 0, 1, \dots k-1 \tag{5}$$

$$P_j(t) = \int_0^t [P_j(t-x) L(t-x) + P_{j-k}(t-x)\{1 - L(t-x)\}] dA^{(k)}(x) \tag{6}$$
$$j = k, k+1, \dots .$$

Write $G(t, z) = E\{z^{\eta(t)}\} = \sum_{m=0}^{\infty} P_m(t) z^m$. From (5) and (6) we obtain

$$G(t,z) = \sum_{j=0}^{k-1} \{A^{(j)}(t) - A^{(j+1)}(t)\} z^j + \int_0^t G(t-x, z)\, [z^k + (1-z^k) L(t-x)] dA^{(k)}(x). \tag{7}$$

Introduce the binomial moments $B_r(t)$ of the distribution $\{P_j(t)\}$ defined by

$$B_r(t) = \frac{1}{r!} \left[\frac{d^r}{dz^r} G(t, z) \right]_{z=1}$$

so that

$$B_r(t)=\sum_{m=r}^{\infty}\binom{m}{r}P_m(t). \tag{8}$$

Introduce the following notation

$$\Phi_r(t)=\sum_{j=r}^{k-1}\binom{j}{r}\{A^{(j)}(t)-A^{(j+1)}(t)\} \tag{9}$$

$$M_k(x)=\sum_{n=1}^{\infty}A^{(nk)}(x).$$

We prove the following theorem.

Theorem 1. *The binomial moments* $B_r(t)$ *exist and can be obtained from the following recurrence relations,* $B_0(t)\equiv 1$ *and*

$$B_r(t)=\Phi_r(t)+\int_0^t \Phi_r(t-x)\,dM_k(x)+$$
$$+\int_0^t[1-L(t-x)]\left[\binom{k}{1}B_{r-1}(t-x)+\cdots+\binom{k}{r}B_0(t-x)\right]dM_k(x) \tag{10}$$
$$r=1,2,\dots k-1$$

$$B_r(t)=\int_0^t[1-L(t-x)]\left[\binom{k}{1}B_{r-1}(t-x)+\cdots+\binom{k}{k}B_{r-k}(t-x)\right]dM_k(x)$$
$$r=k,k+1,\dots. \tag{11}$$

Further

$$P_j(t)=\sum_{r=j}^{\infty}(-)^{r-j}\binom{r}{j}B_r(t). \tag{12}$$

Proof. Differentiation of (7) with respect to z and placing $z=1$ gives

$$B_r(t)=\Phi_r(t)+\int_0^t B_r(t-x)\,dA^{(k)}(x)+$$
$$+\int_0^t[1-L(t-x)]\left[\binom{k}{1}B_{r-1}(t-x)+\cdots+\binom{k}{r}B_0(t-x)\right]dA^{(k)}(x)$$
$$r=1,2,\dots k-1$$

$$B_r(t)=\int_0^t B_r(t-x)\,dA^{(k)}(x)+$$
$$+\int_0^t[1-L(t-x)]\left[\binom{k}{1}B_{r-1}(t-x)+\cdots+\binom{k}{k}B_{r-k}(t-x)\right]dA^{(k)}(x).$$
$$r=k,k+1,\dots$$

These are linear integral equations of the Volterra type and equations (10) and (11) follow in the usual way.

It remains to prove equation (12). To do so we require the following lemma.

Lemma 1. *There exists a positive constant K such that*

$$B_r(t) \leqq \frac{K^{r+1}}{(r+1)!}. \tag{13}$$

Because of (13) *equation* (8) *can be inverted to give* (12).

PROOF. Write $Q_n(t) = P_{nk}(t) + \cdots + P_{nk+k-1}(t)$. From equations (5) and (6) we have

$$\begin{cases} Q_0(t) = 1 - A^{(k)}(t) + \int\limits_0^t Q_0(t-x) L(t-x)\, dA^{(k)}(x) \\ Q_n(t) = \int\limits_0^t [Q_n(t-x) L(t-x) + Q_{n-1}(t-x) 1 - L(t-x)]\, dA^{(k)}(x). \end{cases} \tag{14}$$

Write $F(t, z) = \sum_{n=0}^{\infty} Q_n(t) z^n$ and let $D_r(t)$ defined by

$$D_r(t) = \frac{1}{r!} \left[\frac{d^r}{dz^r} F(t, z) \right]_{z=1}$$

be the r-th binomial moment of the distribution $\{Q_n(t)\}$.

From equations (14) we obtain

$$D_r(t) = \int\limits_0^t D_{r-1}(t-x) [1 - L(t-x)]\, dM_k(x)]. \tag{15}$$

It follows from (15) that there exists a positive constant C such that

$$D_r(t) \leqq C^r / r! \tag{16}$$

There is clearly no loss of generalily in supposing that $C > 1$.

The following proof of (16) is due to TAKÁCS [7]. From (15) we have

$$D_r(t) = \iint\limits_{t_1 + \cdots + t_r \leqq t} \cdots \int [1 - L(t_2 - t_1)] \cdots [1 - L(t - t_r)]\, dM_k(t_1) \cdots dM_k(t_r).$$

Let h be a fixed positive number, write $K(x) = L(x-h)$. Since $M_k(t+h) - M_k(t) \leqq 1 + M_k(h)$ for all $t \geqq 0$ we obtain

$$D_r(t) \leqq \left(\frac{1 + M_k(h)}{h} \right)^r \iint\limits_{x_1 + \cdots + x_r \leqq h} \cdots \int [1 - K(x_1)] \cdots [1 - K(x_r)]\, dx_1 \cdots dx_r$$

$$= \left(\frac{1 + M_k(h)}{h} \right)^r \left\{ \int\limits_0^{t+h} \frac{(1 - K(x))}{r!} \right\}^r \leqq \frac{1}{r!} \left(\frac{1 + M_k(h)}{h} \right)^r (h + l)^r.$$

This proves (16). We have

$$D_r(t)=\sum_{m=r}^{\infty}\binom{m}{r}Q_m(t)$$

and in virtue of (16) this equation can be inverted to give

$$Q_r(t)=\sum_{m=r}^{\infty}(-)^{m-r}\binom{m}{r}D_m(t).$$

Hence using (16) again we obtain

$$Q_r(t)\leqq\frac{C^r}{r!}e^C. \tag{17}$$

From equation (8) we obtain for $u\geqq 0,\ 0\leqq w<k-1$

$$\begin{aligned}B_{uk+w}(t)&=\sum_{m=uk+w}^{\infty}\binom{m}{u\,k+w}P_m(t)\\&\leqq\sum_{s=0}^{\infty}Q_{u+s}(t)\sum_{r=0}^{k-1}\binom{(u+s)k+r}{u\,k+w}\\&=\sum_{s=0}^{\infty}Q_{u+s}(t)\binom{(u+s+1)k}{u\,k+w+1}\end{aligned}$$

in virtue of the identity

$$\sum_{r=0}^{k}\binom{m+r}{m}=\binom{m+k+1}{m+1}.$$

Using (17) we obtain

$$B_{uk+w}(t)\leqq\frac{e^C C^u}{(u\,k+w+1)!}\sum_{s=0}^{\infty}\frac{C^s}{s!}\,\frac{\{(u+s+1)k\}!\,s!}{(u+s)!\,\{(s+1)k-w-1\}!}.$$

Note that for $s>0$

$$\begin{aligned}&\frac{\{(u+s+1)k\}!\,s!}{(u+s)!\,[(s+1)k-w-1]!}\leqq\\&\leqq s^{w+1}k^{uk+w+1}\left(1+\frac{1}{u+s}\right)^k\left(1+\frac{1}{u+s-1}\right)^k\cdots\left(1+\frac{1}{s+1}\right)^k\left(1+\frac{1}{s}\right)^{w+1}\\&\leqq(2k)^{uk+w+1}s^{w+1}\end{aligned}$$

and that

$$\frac{\{(u+1)k\}!}{u!\,(k-w-1)!}\leqq(2k)^{uk+w+1}.$$

Write $L = e^C\left[1+\sum_{s=1}^{\infty}\frac{s^k C^s}{s!}\right] > C$ then we obtain

$$B_{uk+w}(t) \leq \frac{(2k)^{uk+w+1} L C^u}{(uk+w+1)!} < \frac{(2kL)^{uk+w+1}}{(uk+w+1)!}$$

since $L > C > 1$.

This proves lemma 1.

From Theorem 1 we can deduce the existence of a limiting distribution $P_j = \lim_{t\to\infty} P_j(t)$. In fact we have

Theorem 2. *If $A(x)$ is not a lattice distribution the limiting distribution $P_j = \lim_{n\to\infty} P_j(t)$, $j=0,1,\ldots$ exists, is independent of the initial distribution of $\eta(0)$ and*

$$P_j = \sum_{r=j}^{\infty} (-)^{r-j} B_r \tag{18}$$

where B_r is the r-th binomial moment of distribution $\{P_j\}$ and can be determined as follows: $B_0 = 1$ and

$$B_r = \frac{1}{k}\binom{k}{r+1} + \frac{1}{k\alpha}\int_0^\infty [1-L(x)]\left[\binom{k}{1}B_{r-1}(x) + \cdots + \binom{k}{r}B_0(x)\right]dx \qquad r=1,2,\ldots k-1 \tag{19}$$

$$B_r = \frac{1}{k\alpha}\int_0^\infty [1-L(x)]\left[\binom{k}{1}B_{r-1}(x) + \cdots + \binom{k}{k}B_{r-k}(x)\right]dx \qquad r=k,k+1,\ldots. \tag{20}$$

Theorem 2 follows from Theorem 1 by the application of the following lemma which follows from a theorem of D. BLACKWELL [1].

Lemma 2. *If $g(t)$ is a function of bounded variation in $(0,\infty)$, $\alpha<\infty$, and $A(x)$ is not a lattice distribution, then*

$$\lim_{t\to\infty}\int_0^t g(t-x)\,dM_k(x) = \frac{1}{k\alpha}\int_0^\infty g(x)\,dx. \tag{21}$$

The proof of Theorem 2 is similar to the proof of Theorem 3 of TAKÁCS [7] which corresponds to the case $k=1$ and will be omitted. We remark that the first term of (19) arises from the equations

$$\lim_{t\to\infty} \Phi_r(t) = 0$$

$$\int_0^\infty \Phi_r(u)\,du = \alpha \sum_{j=r}^{k-1} \binom{j}{r} = \alpha \binom{k}{r+1}$$

which follow easily from (9).

Denote by M and D^2 the mean and variance of the distribution $\{P_j\}$ then $M = B_1$ and $D^2 = 2B_2 + B_1 - (B_1)^2$. From (19)

$$M = \frac{k-1}{2} + \frac{l}{\alpha} \tag{22}$$

$$D^2 = \frac{2}{\alpha}\int_0^\infty \{1 - L(u)\} B_1(u)\,du + \frac{l(\alpha - l)}{2} + \frac{(k-1)(k-2)}{12} \tag{23}$$

where

$$B_1(t) = \Phi_1(t) + \int_0^t \Phi_1(t-x)\,dM_k(x) + k\int_0^t \{1 - L(t-x)\}\,dM_k(x). \tag{24}$$

Equation (22) can be derived also by the following heuristic argument. Demands for the item occur at the rate $1/\alpha$ and the average time from the instant a demand occurs until the next order is placed is $\alpha(k-1)/2$; the average time until that order is delivered is l. Thus the mean stock deficit is $\frac{k-1}{2} + \frac{l}{\alpha}$.

In some cases of practical importance it is possible to simplify the expression for D^2. For example this is so when the lead time distribution $L(x)$ is an Erlang E_m distribution, that is, when

$$L(x) = 1 - e^{-\mu x}\left\{1 + \frac{\mu x}{1!} + \cdots + \frac{(\mu x)^{m-1}}{(m-1)!}\right\} \qquad x \geqq 0,\ m \geqq 1. \tag{25}$$

Introduce the following notation:

$$\alpha(s) = \int_0^\infty e^{-sx}\,dA(x)$$

$$\beta_1(s) = \int_0^\infty e^{-sx}\,dB_1(x)$$

$$\gamma_1^{(r)}(s) = \frac{d^r}{ds^r}\,\frac{\beta_1(s)}{s}.$$

We prove the following theorem.

Theorem 3. *If the distribution of lead time is given by* (25) *then*

(26) $$D^2-\frac{l(\alpha-l)}{2}-\frac{(k-1)(k-2)}{12}=\frac{2}{\alpha}\sum_{r=0}^{m-1}\frac{(-)^r}{r!l^r}\gamma_1^{(r)}(l^{-1})$$

and $\gamma_1(s)$ *is given explicitly by*

(27) $$\gamma_1(s)=\frac{\alpha(s)}{1-(s)}-\frac{k}{(1+sl)^m}\frac{\{\alpha(s)\}^k}{1-\alpha\{s)\}^k}.$$

Proof. Since

$$\gamma_1^{(r)}(s)=(-)^r\int_0^\infty x^r B_1(x)e^{-sx}\,dx$$

equation (26) follows from (23) and (25).

Introduce the Laplece transform $f_1(s)$ defined by

$$f_1(s)=\int_0^\infty e^{-sx}\,d\,\Phi_1(x).$$

From (24) we obtain

(28) $$s\gamma_1(s)=\frac{f_1(s)}{1-\{\alpha(s)\}^k}+k\frac{\{\alpha(s)\}^k}{1-\{\alpha(s)\}^k}\left[1-\frac{1}{(1+sl)^m}\right].$$

From (9) we obtain $f_1(s)$, namely

(29) $$f_1(s)=[\alpha(s)-k\{\alpha(s)\}^k+(k-1)\{\alpha(s)\}^{k+1}]/[1-\alpha(s)].$$

Equation (27) follows from (28) and (29).

Example. If $m=1$, Theorem 3 gives

(30) $$D^2=\frac{l(\alpha-l)}{2}+\frac{(k-1)(k-2)}{12}+\frac{l}{\alpha}\frac{2\alpha(l^{-1})}{1-\{\alpha(l^{-1})}-k\frac{\{\alpha(l^{-1})\}^k}{1-\{\alpha(l^{-1})\}^k}.$$

If, in particular, the demand process is Poisson, that is $\alpha(s)=(1+s\alpha)^{-1}$ we obtain

(31) $$D^2=\frac{l(l+\alpha)}{2}+\frac{(k-1)(k-2)}{12}-\frac{kl^{k+1}}{\alpha[(+\alpha)^k-l^k]}.$$

When $k=1$, equation (22) gives $M=l/\alpha$. This is the result obtained by Takács [7]. Applying this result to the input process $\{t_{nk}\}$, $n=0,1,\dots$ we have $\lim_{t\to\infty} M(\xi(t))=l/\alpha k$ where $M(\cdot)$ denotes the expectation. From (1) and (22) it lollows that $\lim_{t\to\infty} M(\zeta(t))=(k-1)/2$. Introduce the following notation: $S_w(t)=\sum_{u=0}^{\infty}P_{ku+w}(t)$, $w=0,1,\dots k-1$, then $S_w(t)=P(\zeta(t)=w)$, we have the following theorem.

Theorem 4. *If $A(x)$ is not a lattice distribution the limiting distribution $S_w = \lim_{t\to\infty} S_w(t)$ exists and is uniform, that is $S_w = 1/k$, $w = 0, 1, \dots k-1$.*

PROOF. From equations (5) and (6) we obtain

$$S_w(t) = A^{(w)}(t) - A^{(w+1)}(t) + \int_0^t S_w(t-x)\, dA^{(k)}(x). \tag{32}$$

Hence

$$S_w(t) = A^{(w)}(t) - A^{(w+1)}(t) + \int_0^t \{A^{w}(t-x) - A^{(w+1)}(t-x)\}\, dM_k(x). \tag{33}$$

The theorem follows from (33) by the application of lemma 2, since

$$\int_0^\infty \{A^{(w)}(x) - A^{(w+1)}(x)\}\, dx = \alpha.$$

Define random variables $\beta_j(t)$ as follows: $\beta_j(t) = 1$ if $\eta(t) = j$, $\beta_j(t) = 0$ if $\eta(t) \neq j$, $j = 0, 1, \dots$. Write $b_j(t) = \int_0^t \beta_j(u)\, du$ then $b_j(t)$ is the total time in $(0, t)$ that the stock deficit is equal to j. Since $M\{b_j(t)\} = \int_0^t M\{p_j(u)\, du\}$ we have

$$M\{b_j(t)\} = \int_0^t P_j(u)\, du$$

and if $A(x)$ is not a lattice distribution then

$$\lim_{t\to\infty} M \frac{\{b_j(t)}{t} = P_j.$$

§ 3. The model $G(M)k$

In this section we suppose that the distribution of lead time is given by (25) with $m = 1$, that is, $L(x) = 1 - e^{-\mu x}$, for $x \geqq 0$. In this case it is possible to derive an explicit solution for the distribution P_j by a method similar to that of TAKÁCS [7] for the case $k = 1$.

Define random variables $\xi_n(w)$, $n = 0, 1, \dots,$ $w = 0, 1, \dots k-1$ by the equations $\xi_n(w) = \xi(t_{kn+w+1} - 0)$, $n = 1, 2, \dots;$ $w = 0, 1, 2, \dots k-1$, Write $R_j^n(\omega) = P\{\xi_n(w) = j\}$, $j = 0, 1, 2, \dots$. It is easy to see that for fixed w the sequence of random variables $\{\xi_n(w)\}$, $n = 1, 2, \dots$ forms a Markov chain. By means of the theorem of FOSTER [3] it is easy to see that this chain is ergodic and that limiting probabilities $R_j(w) = \lim_{n\to\infty} R_j^n(w)$ exist and satisfy the following equations

(34)
$$R_j(0) = \sum_{i=j-1}^{\infty} R_i(k-1) P_{i+1,j} \qquad j = 0, 1, 2, \ldots$$

(35)
$$R_j(w) = \sum_{i=j}^{\infty} R_i(w-1) P_{i,j} \qquad w = 1, 2, \ldots k-1;\ j = 0, 1, 2, \ldots$$

where

$$P_{i,j} = \binom{i}{j} \int_0^\infty e^{-j\mu x}(1-e^{-\mu x})^{i-j} dA(x) \qquad i \geqq j = 0, 1, 2, \ldots$$
$$= 0 \text{ otherwise.}$$

Write $U(w, z) = \sum_{j=0}^{\infty} R_j(w) z^j$ then from (34) and (35) we obtain

(36)
$$U(0, z) = \int_0^\infty (1-e^{-\mu x} + ze^{-\mu x}) U(k-1, 1-e^{-\mu x} + ze^{-\mu x}) dA(x)$$

(37)
$$U(w, z) = \int_0^\infty U(w-1, 1-e^{-\mu x} + ze^{-\mu x}) dA(x) \qquad w = 1, 2, \ldots k-1.$$

Introduce the binomial moments $C_r(w)$ defined by

$$C_r(w) = \frac{1}{r!} \left\{ \frac{d^r}{dz^r} U(w, z) \right\}_{z=1} \qquad w = 0, 1, \ldots k-1.$$

We prowe the following theorem.

Theorem 5. *The limiting distribution $\{R_j(w)\}$ is given by*

(38)
$$R_j(w) = \sum_{r=j}^{\infty} (-)^{r-j} \binom{r}{j} C_r(w) \quad j = 0, 1, \ldots w = 0, 1, \ldots k-1$$

where

(39)
$$C_r(w) = \sum_{r=1}^{r} \frac{(\alpha_j)^k}{1-(\alpha_j)^k} (\alpha_r)^{k-1-w} \qquad w = 0, 1, \ldots k-1$$

and

(40)
$$\alpha_j = \int_0^\infty e^{-j\mu x} dA(x).$$

Proof. From (36) and (37) we obtain

(41)
$$C_r(0) = [C_r(k-1) + C_{r-1}(k-1)]\alpha_r \qquad r = 1, 2, \ldots$$

(42)
$$C_r(w) = \alpha_r C_r(w-1) \qquad w = 1, 2, \ldots k-1.$$

From (41) and (42) together with $C_0(w)\equiv 1$, $w=0,1,\dots k-1$ we obtain (39). To prove (38) we remark that $\lim_{r\to\infty} C_r(w)/C_{r-1}(w)=0$ since $\lim_{r\to\infty}\alpha_r=0$. Thus $U(w,z)=\sum_{r=0}^{\infty} C_r(w)(z-1)^r$ converges for every z and hence

$$R_j(w)=\frac{1}{j!}\left[\frac{d^j}{dz^j}U(w,z)\right]_{z=1}=\sum_{m=j}^{\infty}(-)^{m-j}\binom{m}{j}C_m(w).$$

We shall prove next the following theorem.

Theorem 6. *If $A(x)$ is not a lattice distribution the limiting distribution $\{P_j\}$ can be obtained from the following equation*

$$P_{ku+w}=\frac{1}{k\alpha\mu u}\sum_{s=0}^{u-1}\{R_s(w)-R_s(w-1)\}\qquad \begin{matrix} u=1,2,\dots \\ w=1,2,\dots k-1.\end{matrix} \tag{43}$$

$$P_{ku}=\frac{R_0(0)}{k\alpha\mu u}-\frac{1}{k\alpha\mu u}\sum_{s=1}^{u-1}\{R_{s-1}(k-1)-R_s(0)\}\qquad u=1,2,\dots. \tag{44}$$

$$P_w=\frac{1}{k}-\sum_{u=1}^{\infty}P_{ku+w}\qquad w=0,1,\dots k-1. \tag{45}$$

PROOF. Let us say that the system is in the state $E_{u,w}$, $u=0,1,2,\dots$, $w=0,1,\dots k-1$ if u orders are outstanding and w demands have occurred since the last order was placed ($E_{u,k}\equiv E_{u+1,0}$). The possible state transitions are $E_{u,w}\to E_{u-1,w}$, $u=1,2,\dots$, $w=0,1,\dots k-1$ and $E_{u,w}\to E_{u,w+1}$, $w=0,1,\dots k-1$, $u=0,1,2,\dots$.

Let $M_{u,w}(t)$ be the expectation of the number of transitions $E_{u,w}\to E_{u,w+1}$ occurring in the time interal $(0,t)$ and $N_{u,w}(t)$ be the expectation of the number of transitions $E_{u,w}\to E_{u-1,w}$ occurring in the time interval $(0,t)$. Suppose for simplicity that $\eta(0)=0$ then for $u=1,2,\dots,w=0,1,\dots k-1$ we have

$$M_{u,w}(t)-[N_{u,w+1}(t)+N_{u,w+2}(t)+\cdots+N_{(u+1),w}(t)]=P(\eta(t)\geqq ku+w) \tag{46}$$

and for $u=0$, $w=0,1,\dots k-1$ we have

$$M_{0,w}(t)-[N_{1,0}(t)+N_{1,1}(t)+\cdots+N_{1,w}(t)]=P(\eta(t)\geqq w). \tag{47}$$

We require the following two lemmas.

Lemma 3. *If $A(x)$ is not a lattice distribution then*

$$\lim_{t\to\infty}\frac{N_{u,w}(t)}{t}=\mu u P_{ku+w}.\qquad j=1,2,\dots\ w=0,1,\dots k-1 \tag{48}$$

Lemma 4. *If $A(x)$ is not a lattice distribution then*

$$\lim_{t\to\infty}\frac{M_{u,w}(t)}{t}=\frac{R_u(w)}{k\alpha}.\qquad u=0,1,\dots\ w=0,1,\dots k-1 \tag{49}$$

In order to prove lemma 3 we note that

$$N_{u,w}(t+\delta t)=N_{u,w}(t)+\mu u P(r_i(t)=ku+w)\delta t+0(\delta t).$$

Hence $N'_{u,w}(t)=\mu u P_{ku+w}(t)$.

By Theorem 2 $\lim\limits_{t\to\infty} N'_{u,w}(t)=\mu u P_{ku+w}$ exists if $A(x)$ is not a lattice distribution and hence we have (48).

In order to prove lemma 4 we remark that the time intervals between consecutive transitions $E_{u,w}\to E_{u,w+1}$ are independent and identically distributed non-negative random variables. Thus by the elementary renewal theorem of DOOB [2] the limit (49) exists and equals the reciprocal of the expectation of the time interval between consecutive transitions $E_{u,w}\to E_{u,w+1}$. These transitions occur at the instants t_{nk+w+1}, $n=1,2,\dots$ and the state $E_{u,w}$ is a recurrent state and the expectation of the time interval between each step is $k\alpha$. Thus (49) follows by the elementary renewal theorem referred to above. From lemmas 3 and 4 and equations (46), (47) we obtain

$$R_u(w)=k\alpha[\mu u\{P_{ku+w+1}+\cdots+P_{ku+k-1}\}+\mu(u+1)\{P_{k(u+1)}+\cdots+P_{k(u+1)+w}\}]$$

(50) $$u=0,1,2,\dots\; w=0,1,\dots k-1.$$

Equations (43) and (44) follow readily from (50). Equation (45) follows from Theorem 4.

When $L(x)=1-e^{-\mu x}, x\geqq 0$ we have from (7)

(51) $$G(t,z)=\sum_{j=0}^{k-1}\{A^{(j)}(t)-A^{(j+1)}(t)\}z^j+\int_0^t G(t-x,z)\{1-(1-z^k)e^{-\mu(t-x)}\}dA^{(k)}(x).$$

Introduce the following notation

$$\psi(s,z)=\int_0^\infty e^{-st}G(t,z)dt$$

$$\alpha(s)=\int_0^\infty e^{-st}dA(t).$$

We shall prove the following theorem.

Theorem 7. *The Laplace transform $\psi(s,z)$ of the generating function of the probabilities $P_j(t)$ is given by*

(52) $$\psi(s,z)=\theta(s,z)+\sum_{j=1}^{\infty}(-)^j(1-z^k)^j\,\theta(s+j\mu,z)\prod_{i=0}^{j-1}\frac{\alpha^k(s+i\mu)}{1-\alpha^k(s+i\mu)}$$

where

(53) $$\theta(s,z)=\frac{1-\alpha(s)}{s(1\alpha^k(s))}\sum_{j=0}^{k-1}z^j\alpha^j(s).$$

PROOF. From (51) we have

$$\psi(s, z) = \theta(s, z) - \frac{(1-z^k)\alpha^k(s)}{1-\alpha^k(s)}\psi(s+\mu, z).$$

Using this formulae successively and noting that $\lim_{n\to\infty} \psi(s+n\mu, z) = 0$ we obtain (51).

§ 4. The model $D(G)k$

In this section we suppose that demands for the item held in stock occur at the instants $t_n = n\alpha$, $n = 1, 2, \ldots$. Suppose that initially $\eta(0) = 0$, and introduce random variables $\xi_n(w, \beta)$, $n = 1, 2, \ldots$; $w = 0, 1, \ldots k-1$; $0 \leqq \beta < 1$ defined by the equation $\xi_n(w, \beta) = \xi\{(kn + w + 1 + \beta)\alpha - 0\}$. Then we have

$$(54) \qquad \xi_1(w, \beta) \begin{cases} = 0 & \text{if} \quad \varkappa_1 < (w+1+\beta)\alpha \\ = 1 & \text{if} \quad \varkappa_1 \geqq (w+1)\beta)\alpha \end{cases} \qquad w = 0, 1, \ldots k-1.$$

More generally for $n > 1$ we have

$$(55) \qquad \begin{aligned} \xi_n(w, \beta) &= \bar{\xi}_{n-1}(w, \beta) && \text{if} \quad \varkappa_1 < [(n-1)k + w + 1 + \beta]\alpha \\ & && w = 0, 1, \ldots k-1 \\ &= \bar{\xi}_{n-1}(w, \beta) + 1 && \text{if} \quad \varkappa_1 \geqq [(n-1)k + w + 1 + \beta]\alpha \end{aligned}$$

where $\bar{\xi}_{n-1}(w, \beta)$ has the same distribution az $\xi_{n-1}(w, \beta)$ and is independent of $\varkappa_1$ the lead time of the first order.

Write $R_j^n(w, \beta) = P\{\xi_n(w, \beta) = j\}$, $n \geqq 1$, $w = 0, 1, \ldots k-1$, $j = 0, 1, 2, \ldots$ and let $U^n(w, \beta, z)$ be the generating function of the distribution $\{R_j^n(w, \beta)\}$

$$(56) \qquad U^n(w, \beta, z) = \sum_{j=0}^{\infty} R_j^n(w, \beta) z^j.$$

We prove the following theorem.

Theorem 8. *The generating function $U^n(w, \beta, z)$ is given explicitly by*

$$(57) \qquad U^n(w, \beta, z) = \prod_{m=1}^{n} [L_{(m-1)k+w+1+\beta} + z(1 - L_{(m-1)k+w+1+\beta})]$$

where $L_m = L(m\alpha)$. Further $U(w, \beta, z) = \lim_{n\to\infty} U^n(w, \beta, z)$ exists is a generating function and is given explicitly by

$$(58) \qquad U^n(w, \beta, z) = \prod_{m=1}^{\infty} [L_{(m-1)k+w+1+\beta} + z(1 - L_{(m-1)k+w+1+\beta})]$$

There exists a limiting distribution $R_j(w,\beta)=\lim_{n\to\infty} R_j^n(w,\beta)$ *and*

$$U(w,\beta,z)=\sum_{j=0}^{\infty} R_j(w,\beta)z^j. \tag{59}$$

PROOF. From (54) we obtain

$$U^1(w,\beta,z)=[L_{w+1+\beta}+z(1-L_{w+1+\beta})] \tag{60}$$

and from (55) for $n>1, j=1,2,\ldots$

$$R_0^n(w,\beta)=L_{(n-1)k+w+1+\beta}R_0^{n-1}(w,\beta)$$

$$R_j^n(w,\beta)=L_{(n-1)k+w+k+\beta}R_j^{n-1}(w,\beta)+[1-L_{(n-1)k+w+1+\beta}R_{j-1}^{n-1}(w,\beta)].$$

Hence

$$U^n(w,\beta,z)=[L_{(n-1)k+w+1+\beta}+z(1-L_{(n-1)k+w+1+\beta})]\,U^{n-1}(w,\beta,z) \tag{61}$$

(57) follows at once from (60) and (61).

Since $\sum_{m=1}^{n}(1-L_m)\leq 1/\alpha\int_0^{n\alpha}\{1-L(u)\}du\leq 1/\alpha\int_0^{\infty}\{1-L(u)\}du=l/\alpha$ the series $\sum_{m=1}^{\infty}(1-L_{k(m-1)+w+1+\beta})$ is convergent and it follows that $U(w,\beta,z)$ exists and is given by the infinite produce expansion (58).

Since $U(w,\beta,1)=1$ it follows from the continuity theorem for generating function that $U(w,\beta,1)$ is a generating function, that the limiting probabilities $R_j(w,\beta)$ exist and that (59) holds.

If $M_n(w,\beta)$, $D_n^2(w,\beta)$ are the mean and variance respectively of the distribution $R_j^n(w,\beta)$ then from (57) we obtain

$$M_n(w,\beta)=\sum_{m=1}^{n}[1-L_{(m-1)k+w+1+\beta}] \tag{62}$$

$$D_n^2(w,\beta)=\sum_{m=1}^{n}L_{(m-1)k+w+1+\beta}[1-L_{(m-1)k+w+1+\beta}]. \tag{63}$$

EXAMPLE. In the case $L(x)=1-e^{-\mu x}$, $x>0$ we have

$$M_n(w,0)=e^{-\alpha\mu(w+1)}\left[\frac{1-e^{-n\alpha\mu k}}{1-e^{-\alpha\mu k}}\right] \tag{64}$$

$$D_n^2(w,0)=e^{-\alpha\mu(w+1)}\left[\frac{1-e^{-n\alpha\mu k}}{1-e^{-\alpha\mu k}}\right]-e^{-2\alpha\mu(w+1)}\left[\frac{1-e^{-2n\alpha\mu k}}{1-e^{-2\alpha\mu k}}\right] \tag{65}$$

and

$$\lim_{n\to\infty} M_n(w,0)=e^{-\alpha\mu(w+1)}[1-e^{-\alpha\mu k}]^{-1} \tag{66}$$

$$\lim_{n\to\infty} D_n^2(w,0)=e^{-\alpha\mu(w+1)}[1-e^{-\alpha\mu k}]^{-1}-e^{-2\alpha\mu(w+1)}[1-e^{-2\alpha\mu k}]^{-1}. \tag{67}$$

It is easily verified that (66) and (67) agree with the mean and variance of the limiting distribution $R_j(w, 0)$ calculated from formula (39).

In order to prove the equivalence of the expressions (59) and (38) for the case $\beta = 0$ of this example we return to equations (36) and (38) which take the form

$$(68) \qquad U(0, 0, z) = (1 - e^{-\mu\alpha} + z e^{-\mu\alpha})\, U(k-1, 0, 1 - e^{-\mu\alpha} + z e^{-\mu\alpha})$$

$$(69) \qquad U(w, 0, z) = U(w-1, 0, 1 - e^{-\mu\alpha} + z e^{-\mu\alpha} \qquad w = 1, 2, \dots k-1.$$

From these equations we obtain

$$(70) \qquad U(k-1, 0, z) = (1 - e^{-k\mu\alpha} + z e^{-k\mu\alpha})\, U(k-1, 0, 1 - e^{-k\mu\alpha} + z e^{-k\mu\alpha})$$

$$(71) \quad U(w, 0, z) = U(k-1, 0, 1 - e^{(k-w-1)\mu\alpha} + z e^{(k-w-1)\mu\alpha}) \qquad w = 1, 2, \dots k-1.$$

From (70) we obtain

$$U(k-1, 0, z) = \prod_{m=1}^{\infty} (1 - e^{-mk\mu\alpha} + z e^{-mk\mu\alpha})$$

and hence from (71)

$$U(w, 0, z) = \prod_{m=1}^{\infty} [1 - e^{-\mu\alpha\{(m-1)k+w+1\}} + z e^{-\mu\alpha\{(m-1)k+w+1\}}].$$

These expressions agree with (58).

Conversely for $D(M)k$ we can obtain expressions for the binomial moments $C_r(w, \beta)$ of the limiting distribution $R_j(w, \beta)$. Thus from (58) we obtain

$$(72) \qquad U(0, \beta, z) = (1 - e^{-\mu\alpha(1+\beta)} + z e^{-\mu\alpha(1+\beta)}\, U(k-1, \beta, 1 - e^{-\mu\alpha} + z e^{-\mu\alpha})$$

$$(73) \qquad U(w, \beta, z) = U(w-1, \beta, 1 - e^{-\mu\alpha} + z e^{-\mu\alpha}). \qquad w = 1, 2, \dots k-1$$

Writing $C_r(w, \beta) = \frac{1}{r!} \left\{ \frac{d^r}{dz^r} U(w, \beta, z) \right\}_{z=1}$ we obtain as in the proof of Theorem 5

$$(74) \qquad R_j(w, \beta) = \sum_{r=j}^{\infty} (-)^{r-j} \binom{r}{j} C_r(w, \beta) \qquad \begin{matrix} w = 0, 1, \dots k-1 \\ j = 0, 1, \dots \end{matrix}$$

where

$$(75) \qquad C_r(w, \beta) = e^{-\mu\alpha k\beta} \left[\prod_{j=1}^{r} \frac{e^{-j\mu\alpha k}}{1 - e^{-j\mu\alpha k}} \right] e^{r\mu\alpha(k-1-w)} \qquad \begin{matrix} w = 0, 1, \dots k-1 \\ r = 1, 2, \dots . \end{matrix}$$

§ 5. The model $G(D)k$

When the demand process is a general renewal process the method of the last section breaks down and the determination of the distribution $\{R_j^n(w)\}$ appears to be very difficulf. It is possible however to obtain an expression for the mean of the distribution in the general case and when the lead time is constant to obtain the distribution $\{R_j^n(w)\}$ explicitly and we shall do so in this section.

We suppose that $\eta(0)=0$. Write $\xi_n(w)=\xi(t_{kn+w+1}-0)$ and $R_j^n(w)=$ $=P(\xi_n(w)=j)$, $j=0, 1, \ldots, n=1, 2, \ldots, w=0, 1, \ldots k-1$. Introduce random variables $\beta_{j,n}(w)$ defined as follows:

$$\beta_{j,n}=0 \quad \text{if} \quad \varkappa_j<t_{nk+w+1}-t_{jk} \qquad w=0, 1, \ldots k-1$$
$$(76) \qquad =1 \quad \text{if} \quad \varkappa_j \geqq t_{nk+w+1}-t_{jk}. \qquad j=1, 2, \ldots n$$
$$n=1, 2, \ldots .$$

Introduce the following notation:

$$(77) \qquad b_m(w)=\int_0^\infty A^{mk+w+1}(x)\,dL(x)$$

where $A^m(x)$ is the m-th-fold iterated convolution of $A(x)$ with itself.

$$(78) \qquad \begin{aligned} N_{k,w}(x) &= \sum_{m=0}^\infty A^{mk+w+1}(x) \\ &= A^{w+1}(x)+\int_0^x A^{w+1}(x-y)\,dM_k(y) \end{aligned}$$

where $M_k(y)=\sum_{n=1}^\infty A^{nk}(y)$ is the renewal function of the renewal process $\{t_{nk}\}$, $n=1, 2, \ldots$.

Finally let us remark that

$$(79) \qquad \xi_n(w)=\sum_{j=1}^n \beta_{j,n}.$$

The following theorem is an immediate consequence of equation (79) and the above definitions.

Theorem 9. *The expectation* $M(\xi_n(w))$ *of* $\xi_n(w)$ *is given by*

$$(80) \qquad M\{\xi_n(w)\}=\sum_{m=0}^{n-1} b_m(w).$$

Further $\lim_{n\to\infty} M\{\xi_n(w)\}$ *exists and is given by*

$$(81) \qquad \lim_{n\to\infty} M\{\xi_n(w)\}=\int_0^\infty N_{k,w}(x)\,dL(x).$$

The variance of $\xi_n(w)$ can be obtained in principle from (79) and the expectations $M(\beta_{j,n}(w)\beta_{j+1,n}(w))$, $j+i \leqq n$ given by

$$(82) \qquad M(\beta_{j,n}(w)\beta_{j+i,n}(w))=\iint_{x\geqq y}[1-L(y)]A^{ik}(x-y)\,dA^{(n-j-i)}(y)^{k+w+1}dL(x).$$

It does not seem possible to obtain a simple formula for the variance of $\xi_n(w)$ in the general case. When the lead time is a contant, l, however we obtain from (82) or direct from (76)

$$M\{\beta_{j,n}(w)\beta_{j+i,n}(w)\} = M\{\beta_{j,n}(w)\} \qquad j \leqq j+i \leqq n. \tag{83}$$

It follows that $M\{\beta^r_{j,n}(w)\beta^s_{j+i,n}(w)\} = M\{\beta_{j,n}(w)\}$, $j+i \leqq n$, for all positive r, s and hence we can in principle obtain all the moment of the distribution $\{R^n_j(w)\}$ from (79). In this case however it is simpler to return to equations (76) and (79). We prove the following theorem.

Theorem 10. *If the lead time is a constant then the distribution* $\{R^n_j(w)\}$, $w = 0, 1, \dots k-1$ *is given explicitly by*

$$\begin{aligned} R^n_0(w) &= 1 - A^{w+1}(l) \\ R^n_j(w) &= A^{(j-1)k+w+1}(l) - A^{jk+w+1}(l) \qquad j = 1, 2, \dots n-1 \\ R^n_n(w) &= A^{(n-1)k+w+1}(l). \end{aligned} \tag{84}$$

The limiting distribution $R_j(w) = \lim_{n\to\infty} R^n_j(w)$ *exists and is given by*

$$\begin{aligned} R_j(w) &= 1 - A^{w+1}(l) \\ R_j(w) &= A^{(j-1)k+w+1}(l) - A^{jk+w+1}(l) \qquad j = 1, 2, \dots . \end{aligned} \tag{85}$$

Proof. *If the lead time is constant then* $\beta_{j,n} = 1$ implies $\beta_{j+1,n} = \cdots = \beta_{n,n} = 1$. Thus from (79)

$$P(\xi_n(w) \geqq m) = P(\beta_{n-m+1}(w) = 1) - A^{(m-1)k+w+1}(l) \qquad m = 1, 2, \dots n. \tag{86}$$

Equations (84) and (85) follow at once from (86).

Example. Suppose that $A(x)$ is an Erlang E_m distribution, $m \geqq 1$, that is, $A(x) = 1 - e^{-x/\alpha}\left[1 + \frac{x}{\alpha} + \cdots + \frac{1}{(m-1)!}\left(\frac{x}{\alpha}\right)^{m-1}\right]$. Write $\varrho = l/\alpha$ then we obtain

$$\begin{aligned} R^n_0(w) &= e^{-\varrho} \sum_{s=0}^{(w+1)m-1} \varrho^s/s! \\ R^n_j(w) &= e^{-\varrho} \sum_{s=\{(j-1)k+w+1\}m}^{(jk+w+1)m-1} \varrho^s/s! \qquad j = 1, 2, \dots n-1 \\ R^n_n(w) &= e^{-\varrho} \sum_{s=0}^{\{(n-1)k+w+1\}m-1} \varrho^s/s! \end{aligned} \tag{87}$$

and

$$
(88)\qquad
\begin{aligned}
R_0(w) &= e^{-\varrho} \sum_{s=0}^{(w+1)m-1} \varrho^s/s! \\
R_j(w) &= e^{-\varrho} \sum_{s=\{(j-1)k+w+1\}m}^{(jk+w+1)m-1} \varrho^s/s! \qquad j \geqq 1.
\end{aligned}
$$

When the lead time is constant it is possible to derive explicit expressions for the limiting probabilities $P_j = \lim\limits_{t \to} P_j(t)$ of the random variable defined by equation (1). Namely we have

Theorem 11. *If the lead time is constant, then*

$$
P_w(t) = \begin{cases} A^w(t) - A^{w+1}(t) & \text{if } t < l \\ A^w(t) - A^{w+1}(t) + \int\limits_0^{t-l} [A^w(t-y) - A^{w+1}(t-y)]\, dM_k(y) & \text{if } t > l \end{cases}
$$

$$
w = 0, 1, \ldots, k-1
$$

and for $j = 1, 2, \ldots;\ w = 0, 1, \ldots, k-1$

$$
(90)\qquad P_{jk+w}(t) = \begin{cases} A^{jk+w}(t) - A^{jk+w+1}(t) & \text{if } t < l \\ \int\limits_{t-l}^{t} [A^{(j-1)k+w}(t-y) - A^{(j-1)k+w+1}(t-y) - A^{jk+w}(t-y) + \\ \qquad + A^{jk+w+1}(t-y)]\, dM_k(y) & \text{if } t > l. \end{cases}
$$

Where $M_k(y) = \sum\limits_{n=1}^{\infty} A^{nk}(y)$ *is the renewal function of the renewal process* $\{t_{nk}\}$, $n = 1, 2, \ldots$ *. Further, if* $A(x)$ *is not a lattice distribution then there exist limiting probabilities* $P_j = \lim\limits_{t \to \infty} P_j(t)$, $j = 0, 1, 2, \ldots$ *given explicitly by*

$$
(91)\qquad
\begin{aligned}
P_w &= \frac{1}{k} - \frac{1}{k\alpha} \int\limits_0^l [A^w(u) - A^{w+1}(u)]\, du \qquad w = 0, 1, \ldots k-1 \\
P_{jk+w} &= \frac{1}{k\alpha} \int\limits_0^l [A^{(j-1)k+w}(u) - A^{(j-1)k+w+1}(u) - A^{jk+w}(u) + A^{jk+w+1}(u)]\, du \\
&\qquad j = 1, 2, \ldots \\
&\qquad w = 0, 1, \ldots, k-1.
\end{aligned}
$$

PROOF. Suppose first that $t < l$, then $\eta(t) = n$ if and only if exactly n demands occur in $(0, t)$ and the probability of this is just $\{A^n(t) - A^{n+1}(t)\}$. Suppose next that $t > l$, then $\eta(t) = w$, $w = 0, 1, \ldots k-1$ if and only if either exactly w demands occur in $(0, t)$ or the nk-th demand $n = 1, 2, \ldots$ occurs at $t_{nk}=y$ in $(0, t-l)$ and exactly w demands occur in (y, t). Thus for $t>l$

$$P_w(t)=A^w(t)-A^{w+1}(t)+\sum_{n=1}\int_0^{t-l}[A^w(t-y)-A^{w+1}(t-y)]\,dA^{nk}(y)$$

and this establishes the second of equations (89).

To establish the second of equations (90) when $t>l$ let $\varepsilon_m(t)$ denote the event $[\eta(t)=mk+w]$ and write $\varepsilon_j^*(t)=\bigcup_{m=j}^{\infty}\varepsilon_m(t)$. The event $\varepsilon_j^*(t)$ can occur if and only if an (nk)-th demand $n=1,2,\ldots$ occurs at $t_{nk}=y$ in $(t-l, t)$ and exactly $(j-1)k+w$ further demands occur in (y, t). Thus

$$\sum_{m=j}^{\infty}P_{mk+w}(t)=\int_t^{t-l}[A^{(j-1)k+w}(t-y)-A^{(j-1)k+w+1}(t-y)]\,dM_k(y)$$

and this establishes the second of equations (90).

If $A(x)$ is not a lattice distribution the existence of the limiting distribution $\{P_j\}$ has been proved in Theorem 1 when the lead time is general. The existence of the distribution can be proved in the special case of this theorem by a direct application of lemma 2 to equations (89) and (90). By means of lemma 2 we obtain the limiting distribution explicitly in the equations (91).

EXAMPLE. Suppose that the d. f. $A(x)$ is an Erlang E_m distribution as in the example to Theorem 10. Then we obtain

(92)
$$P_w=\frac{1}{k}-\frac{1}{k}\sum_{s=wm}^{(w+1)m-1}\left[1-\left(1+\frac{\varrho}{1!}+\cdots+\frac{\varrho^s}{s!}\right)e^{-\varrho}\right]$$
$$\sum_{i=j}^{\infty}P_{ik+w}=\frac{1}{k}\sum_{s=\{(j-1)k+w\}m}^{\{(j-1)k+w+1\}m-1}\left[1-\left(1+\frac{\varrho}{1!}+\cdots+\frac{\varrho^s}{s!}e^{-\varrho}\right]\right..$$

In the particular case the demand process is Poisson, that is $m=1$ we obtain

(93)
$$P_w=\frac{1}{k}e^{-\varrho}\sum_{s=0}^{w}\varrho^s/s! \qquad w=0,1,\ldots k-1$$
$$P_{jk+w}=\frac{1}{k}e^{-\varrho}\sum_{s=(j-1)k+w+1}^{jk+w}\varrho^s/s! \qquad j=1,2,\ldots;\ w=0,1,\ldots k-1.$$

Equation (93) is equivalent to result of PITT's lemma 1, PITT [5].

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