A note on entire functions

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1. Let $f(z)$ be an entire function of order $\varrho(0<\varrho<\infty)$; let $\mu(r,f)$ be the maximum term of $f(z)=\sum_{n=0}^{\infty} a_n z^n$ for $|z|=r$ and let as usual $M(r,f)$ and $M(r,f')$ denote the maximum modulii of $f(z)$ and $f'(z)$ on $|z|=r$. G. VALIRON has proved the following results:

$$M(r,f')<r^{\frac{3}{2}\varrho-1+\varepsilon}\mu(r,f) \tag{1}$$

for all sufficiently large values of r exterior to a set of segments in which the total variation of $\log r$ is finite.

$$M(r,f')<r^{2\varrho-1+\varepsilon}\mu(r,f) \quad \text{for all } r\geqq r_0 \tag{2}$$

(see G. Valiron [1], p. 169, 212).

In this note we prove that (1) is a best possible result in the sense that $(3/2)\varrho$ cannot be replaced by a smaller number. Further we provide a simple and alternative proof of (2). We also prove:

$$\nu(r,f)\mu(r,f)\leqq rM(r,f') \tag{3}$$

where $\nu(r,f)$ is the rank of the maximum term.

2. We now establish (1).

Consider

$$f(z)=\cos\sqrt{z}.$$

Then $f(z)$ is an entire function of order $\varrho=1/2$. Let $R_n=(2n-1)2n$. Then for $R_n\leqq r<R_{n+1}$

$$\mu(r,f)=\frac{r^n}{(2n)!}\sim\frac{r^n}{(2n)^{2n+1/2}e^{-2n}\sqrt{2\pi}}\sim\frac{e^{\sqrt{r}}}{\sqrt{2\pi}r^{\frac{1}{4}}}.$$

Now $$f'(z) = -1/2\left(1-\frac{z}{3!}+\frac{z^2}{5!}-\frac{z^3}{7!}+\cdots\right),$$

so $$|f'(-r)| = M(r,f') = 1/2\left(1+\frac{r}{3!}+\frac{r^2}{5!}+\cdots\right) = \frac{e^{\sqrt{r}}-e^{-\sqrt{r}}}{4\sqrt{r}}.$$

Hence

$$\frac{M(r,f')}{\mu(r,f)} \sim \frac{e^{\sqrt{r}}-e^{-\sqrt{r}}}{4\sqrt{r}}\,\frac{\sqrt{2\pi}\,r^{\frac{1}{4}}}{e^{\sqrt{r}}} = \frac{\sqrt{2\pi}}{4}\left\{1-\frac{1}{e^r}\right\} r^{\frac{3}{2}\varrho-1}$$

and the result follows.

Proof of (2):

Let $f(z)=\sum_{n=0}^{\infty} a_n z^n$.

Then

$$rM(r,f') \leqq \sum_{n=1}^{\infty} n|a_n|r^n.$$

Let $r=\left(\dfrac{n}{\varrho+\varepsilon}\right)^{\frac{1}{\varrho+\varepsilon}}$.

Then by CAUCHY's inequality

$$|a_n|r^n \leqq M(r,f) < e^{r^{\varrho+\varepsilon}} = e^{\frac{n}{\varrho+\varepsilon}} = \left\{\frac{e(\varrho+\varepsilon)}{n}\,r^{\varrho+\varepsilon}\right\}^{\frac{n}{\varrho+\varepsilon}} \quad (r \geqq r_0).$$

Put $N=[2e(\varrho+\varepsilon)r^{\varrho+\varepsilon}]$ where $[x]$ denotes the integral part of x. Then

$$rM(r,f') \leqq \sum_{n=1}^{N} n|a_n|r^n + \sum_{N+1}^{\infty} n|a_n|r^n \leqq$$

$$\leqq \mu(r,f)\,\frac{N(N+1)}{2} + \sum_{N+1}^{\infty} n\left\{\frac{e(\varrho+\varepsilon)}{n}\,r^{\varrho+\varepsilon}\right\}^{\frac{n}{\varrho+\varepsilon}} \leqq$$

$$\leqq \mu(r,f)\,\frac{N(N+1)}{2} + \sum_{N+1}^{\infty} n\left\{\frac{e(\varrho+\varepsilon)}{N}\,r^{\varrho+\varepsilon}\right\}^{\frac{n}{\varrho+\varepsilon}} \leqq$$

$$\leqq \mu(r,f)\,\frac{N(N+1)}{2} + \sum_{N+1}^{\infty} n\,2^{\frac{-n}{\varrho+\varepsilon}}.$$

Now for $n \geqq n_0$

$$\sum_{N+1}^{\infty} n\,2^{-\frac{n}{\varrho+\varepsilon}} < \sum_{N}^{\infty} 2^{-\frac{n}{2(\varrho+\varepsilon)}} = 0(1)$$

and the result follows.

To prove (3) we observe that

$$f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}.$$

Hence by CAUCHY's inequality we have

$$M(r, f') \geqq n |a_n| r^{n-1}. \tag{4}$$

If we set $R_n = \left| \frac{a_{n-1}}{a_n} \right|$ then for $R_n \leqq r < R_{n+1}$, we have

$$\mu(r, f) = |a_n| r^n,$$
$$\nu(r, f) = n.$$

Hence from (4) we have

$$M(r, f') \geqq \nu(r, f) \frac{\mu(r, f)}{r}.$$

This proves (3).

Bibliography

[1] G. VALIRON, Fonctions Analytiques, *Paris,* 1954.

(Received October 4, 1960.)