On the maximum terms of an entire function and its derivatives

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1. Introduction. Let $f(z)$ be an entire function represented by the series

$$f(z)=\sum_{0}^{\infty} a_n z^n \tag{1}$$

of order ϱ and lower order $\lambda (0 \leqq \lambda, \varrho \leqq \infty)$. On the circle $|z|=r$, let $M(r)$ be the maximum modulus of $f(z)$, $\mu(r)$ the maximum term of $f(z)$ and $\nu(r)$ the rank of the maximum term. Let $M_j(r)$, $\mu_j(r)$ and $\nu_j(r)$ be defined for the derivative $f^{(j)}(z)$, $j=1,2,3,\dots$, exactly as $M_0(r)\equiv M(r)$, $\mu_0(r)\equiv\mu(r)$ and $\nu_0(r)\equiv\nu(r)$ for $f(z)$. Then it is known that there are asymptotic properties of $\mu_j(r)$, $j\geqq 0$, as $r\to\infty$, analogous to certain properties of $M_j(r)$, $j\geqq 0$. For instance, there is (a) S. K. SINGH's relation between $\mu_1(r)$, $\mu(r)$ and either ϱ or λ ([3] Corollary (i) of Theorem 1, Corollary (ii) of Theorem 2) analogous to S. M. SHAH's relation between $M_1(r)$, $M(r)$ and either ϱ or λ ([2], Theorems A, B), (b) S. K. SINGH's inequality connecting $\mu(r)$ and its derivative $\mu'(r)$ ([4], Theorem 4) which is the pseudo-analogue of VIJAYARAGHAVAN's inequality [7] connecting $M(r)$ and $M_1(r)$. It is the main object of this note, firstly, to simplify the proof of S. K. SINGH's $\mu(r)$-analogue mentioned in (a) and to extend this analogue to $\mu_j(r)$, $j\geqq 2$ (Theorem 1 below), secondly, to obtain an inequality (Theorem 2) connecting $\mu_j(r)$ and $\mu(r)$ which reduces to a precise analogue of VIJAYARAGHAVAN's inequality in the case $j=1$. The remaining results of this note (Theorems 3 and 4) are supplementary.

2. Theorems. The following lemmas, required for the proofs of our theorems, are known results.

Lemma 1. (i) *If $f(z)$ is an entire function then the order (or lower order) of every derivative $f^{(j)}(z)$ is the same as that of $f(z)$.*

(ii) *In* (i) *the order ϱ (or lower order λ) of $f(z)$ and $f^{(j)}(z)$, $j=1,2,3,\dots$, is given, in terms of $\nu_j(r)$ as defined at the outset, by the formulae:*

$$\limsup_{r\to\infty}\frac{\nu_j(r)}{\log r}=\varrho, \qquad j=0,1,2,\dots$$

$$\left(\text{or}\quad \liminf_{r\to\infty}\frac{\nu_j(r)}{\log r}=\lambda, \qquad j=0,1,2,\dots\right)$$

PROOF. (i) The result for the lower order λ, like that for the order ϱ, follows from a familiar relation between $M(r)$ and $M_1(r)$ ([6], p. 35, relation 2.13).

(ii) In the case $j=1$, the results for both ϱ and λ are known ([6], p. 34, and [8], Theorem 1). In the case $j\geqq 2$, the result follows from (i).

Lemma 2. *In the notation explained in the beginning we have, for any entire function,*

$$\nu_j(r)\leqq r\frac{\mu_{j+1}(r)}{\mu_j(r)}\leqq \nu_{j+1}(r), \qquad j=0,1,2,\ldots \tag{2}$$

PROOF. The second half of (2) for $j=0$ is given by VALIRON ([6], p. 35) while (2) in its entirety is indicated by Q. I. RAHMAN ([1], p. 42, relation (7)). The proof requires only definitions. For, writing

$$f^{(j)}(z)=\Sigma A_n z^n, \quad \nu_j(r)=N, \quad \nu_{j+1}(r)=N_1,$$

we get

$$\mu_{j+1}(r)=N_1|A_{N_1}|r^{N_1-1}\leqq\frac{N_1}{r}|A_N|r^N=\frac{\nu_{j+1}(r)}{r}\mu_j(r),$$

$$\mu_j(r)=|A_N|r^N=\frac{r}{N}N|A_N|r^{N-1}\leqq\frac{r}{\nu_j(r)}\mu_{j+1}(r).$$

Lemma 3. With reference to an entire function defined as in (1), let an ordinary value of $|z|=r$, of index α, $0<\alpha<1$, be defined according to VALIRON ([6], p. 96). Then, *for an ordinary value of r common to $f(z)$ and $zf'(z)$,*

$$\nu_0(r)\leqq\nu_1(r)<\nu_0\{1+O(\nu_0^{-\alpha})\} \quad \textit{where} \quad \nu_0(r)=\nu(r),\ r\to\infty.$$

This lemma is given by VALIRON ([6], p. 104, relation (s)).

Theorem 1. *For an entire function $f(z)$ defined as in* (1),

$$\lim_{r\to\infty}\begin{matrix}\sup\\ \inf\end{matrix}\frac{\log\left\{r\left[\dfrac{\mu_j(r)}{\mu(r)}\right]^{1/j}\right\}}{\log r}=\begin{matrix}\varrho\\ \lambda\end{matrix} \qquad (j=1,2,3,\ldots). \tag{3}$$

(Here and elsewhere $[\cdots]^{1/j}$ is the positive j-th root of $[\cdots]$.)

PROOF. We first prove the case $j=1$ of (3). Putting $j=0$ in (2), then taking logarithms of all three members of (2), finally, dividing all three members by $\log r$ and letting $r\to\infty$, we get

$$\limsup_{r\to\infty}\frac{\log\nu_0(r)}{\log r}\leqq\limsup_{r\to\infty}\frac{\log\left\{r\dfrac{\mu_1(r)}{\mu_0(r)}\right\}}{\log r}\leqq\limsup_{r\to\infty}\frac{\log\nu_1(r)}{\log r},$$

along with a similar inequality in which 'lim inf' replaces 'lim sup'. The case $j=1$ of (3) then follows immediately from Lemma 1 (ii).

In the case $j \geqq 2$, we observe that (2) can be written:

$$\nu_0(r) \leqq r\frac{\mu_1(r)}{\mu_0(r)} \leqq \nu_1(r) \leqq \cdots \leqq \nu_{j-1}(r) \leqq r\frac{\mu_j(r)}{\mu_{j-1}(r)} \leqq \nu_j(r). \tag{4}$$

Multiplying together the ratios involving the μ's, we obtain

$$\nu_0(r) \leqq r\left[\frac{\mu_j(r)}{\mu_0(r)}\right]^{1/j} \leqq \nu_j(r). \tag{5}$$

Treating (5) exactly as we have already treated its case $j=1$, we complete the proof of the case $j \geqq 2$.

(2) shows that, given any small $\varepsilon>0$, we can find $r_0(\varepsilon)$ such that

$$r^{(\lambda-1)j+\varepsilon} < \frac{\mu_j(r)}{\mu(r)} < r^{(\varrho-1)j+\varepsilon} \quad \text{for} \quad r>r_0.$$

Hence the following result of S. K. SINGH ([3], Theorems 1, 2) is a consequence of Theorem 1.

Corollary 1a. (i) *For an entire function of order $\varrho<1$ and p such that $p<(1-\varrho)j$ where j is a positive integer,*

$$\frac{r^p\mu_j(r)}{\mu(r)} \to 0 \quad as \quad r\to\infty.$$

(ii) *For an entire function of lower order $\lambda>1$ and p such that $p<(\lambda-1)j$ where j is a positive integer,*

$$r^{-p}\frac{\mu_j(r)}{\mu(r)} \to \infty \quad as \quad r\to\infty.$$

The case $j=1$ of Theorem 1 is true with the pair $\mu(r)$ and $\mu_1(r)$ replaced by $\mu_1(r)$ and $\mu_2(r)$ or the pair $\mu_2(r)$ and $\mu_3(r)$, etc. Hence the following is a type of result included in Theorem 1 and its basic inequality (4).

Corollary 1b. (i) *$\lambda \geqq 1$ is a necessary condition for*

$$\mu(r)<\mu_1(r)<\cdots<\mu_j(r),\ r>r_0. \tag{6}$$

(ii) *Either $\lambda>1$ or $\liminf_{r\to\infty}\frac{\nu(r)}{r}>1$ (and $\lambda=1$) is sufficient for* (6).

Theorem 2. *For an entire function $f(z)$ defined as in* (1),

$$r\left[\frac{\mu_j(r)}{\mu(r)}\right]^{1/j} > \frac{\log\mu(r)}{\log r} \quad \text{for} \quad r>r_0 \qquad (j=1,2,3,\ldots). \tag{7}$$

PROOF. From (2) with $j=0$ and (5),

$$r\left[\frac{\mu_j(r)}{\mu(r)}\right]^{1/j} \geqq \nu(r) \qquad (j=1, 2, 3, \ldots). \tag{8}$$

Also,

$$\log \mu(r) = \log |a_\nu| + \nu \log r, \qquad \nu = \nu(r), \tag{9}$$

where $\limsup |a_\nu|^{\frac{1}{\nu}} = 0$ as r or $\nu \to \infty$ and consequently $|a_\nu| < 1$ for $r > r_0$. Hence (9) gives

$$\frac{\log \mu(r)}{\log r} < \nu(r) \quad \text{for} \quad r > r_0. \tag{10}$$

(10) in conjunction with (8) establishes the required result (7).

For functions of finite non-zero order, the lower estimate for $\nu(r)$ in (10), and hence also Theorem 2, can be improved by using the first half of (11) of the next theorem.

Theorem 3. *For an entire function of order ϱ, $0 < \varrho < \infty$, defined as in* (1),

$$\limsup_{r\to\infty} \frac{\log \mu(r)}{\nu(r)\log r} \leqq 1 - \frac{\lambda}{\varrho}, \qquad \liminf_{r\to\infty} \frac{\log \mu(r)}{\nu(r)\log r} = 0. \tag{11}$$

Both the results in (11) are known. The first result in (11) is obtained ([5] section I, Theorem 1) by using, in (9), the known formula ([6], p. 40, Theorem 14):

$$\liminf_{n\to\infty} \frac{\log |a_n|^{-1}}{n \log n} = \frac{1}{\varrho},$$

instead of the inequality $|a_\nu| < 1$ for $\nu > \nu_0$, in conjunction with the definition of λ in Lemma 1 (ii). The second result in (11) is known in a more general form with $\log r$ replaced by any function of r which tends to infinity with r ([4], Theorem 1 (2)).

The first result in (11) is best-possible in view of the example of $f(z)$ given by S. K. SINGH ([3], proof of Theorem 4 (i)), where $\varrho = 1$, $\lambda = 0$ and

$$\limsup_{r\to\infty} \frac{\log \mu(r)}{\nu(r)\log r} = 1.$$

In a study of formal similarities in behaviour of $\mu(r)$ and $M(r)$, the theorem which follows may be compared with a theorem for $M(r)$ given by VALIRON ([6], p. 103).

Theorem 4. *Let $f(z)$ be an entire function defined as in* (1). *Let $|z| = r \to \infty$ through ordinary values of index α which are common to $f(z)$*

and $zf'(z)$ understood in the sense of Lemma 3. Then

$$r\mu_1(r) \sim \nu(r)\mu(r).$$

The theorem follows at once from (2) with $j=0$ and Lemma 3.

3. Remarks. (i) R. P. SRIVASTAV states Theorem 2 with the superfluous hypothesis $\lambda > 1$ ([5], p. 32, result (i)).

(ii) R. P. SRIVASTAV also states Theorem 2 for $j \geqq 2$ without proof ([5], p. 32, result (iv)). But there is no clue to the proof either in his paper or in any of the references which he gives.

(iii) S. K. SINGH observes ([3], Corollary (ii) of Theorem 1) that, if $f(z)$ is a function of finite order, taking note of the fact $\log \mu(r) \sim \log M(r)$ as $r \to \infty$, we can replace $\mu_j(r)$ by $M_j(r)$ $(j=0, 1)$ in the case $j=1$ of Theorem 1 and obtain S. M. SHAH's $M(r)$-analogue already referred to ([2], Theorems A, B). This observation assumes without justification that

$$\limsup_{r\to\infty} (\text{or inf}) \{f(r)-g(r)\} = \limsup_{r\to\infty} (\text{or inf}) \{F(r)-G(r)\}$$

if
$$0 < f(r) \sim G(r), \ 0 < g(r) \sim G(r).$$

(iv) Exact $M(r)$-analogues of all these results are known with two exceptions: (1) the $M(r)$-analogue of Corollary 1 b (ii) which involves a condition on $\nu(r)$, (2) the case $j \geqq 2$ of Theorem 2.

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