Geometry of automorphisms of the complex field

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We shall use the algebra of the complex field Z to obtain a new proof of a theorem of Euler [1] concerning similar triangles in the plane and also a geometric property of automorphisms of Z which yields a generalization of a theorem of BETH and TARSKI [2] on axiomatizing plane geometry.

1. Euler's theorem. We shall prove that if A, B and A', B' are pairs of distinct points in Z, and $B-A \neq B'-A'$, then there is a unique point z in Z such that the triangles ABz and $A'B'z$ are similar and of the same orientation. If $B-A=B'-A'$ there is no such z unless $A=A'$ and $B=B'$.

PROOF. Triangle ABz is similar to triangle $A'B'z$ and of the same orientation if and only if

$$\frac{z-A}{B-A}=\frac{z-A'}{B'-A'}.$$

This equation has a unique solution for z if $B-A=B'-A'$ and the assertion is proved.

A similar result holds if the words "same orientation" are replaced by "different orientation".

2. Automorphisms of Z preserve regular polygons. We shall prove that if $F: Z \to Z$ is an automorphism of Z then F takes the vertices of regular polygons into vertices of regular polygons.

PROOF. Let $z_1, \dots, z_n$ be the vertices of a regular n-sided polygon and $g=(z_1+\cdots+z_n)/n$ the center of the polygon. Since F is an automorphism of Z

$$F(g)=\bigl(F(z_1)+\cdots+F(z_n)\bigr)/n$$

and thus $F(g)$ is the center of gravity of the n points $F(z_1), \dots, F(z_n)$.

Define h in Z by the equation

$$F(h)=e^{2\pi i/n}.$$

Observe that h is a primitive root of unity and so $h=e^{2\pi mi/n}$ with $(m,n)=1$.

Thus we can assume that the points $z_1, \dots, z_n$ are so indexed that

$$\frac{z_i - g}{z_{i-1} - g} = h,$$

$i = 1, \dots, n$, (where we name z_n also z_0).

Then, since F is an automorphism,

$$\frac{F(z_i) - F(g)}{F(z_{i-1}) - F(g)} = F(h) = e^{2\pi i/n}.$$

Thus the triangles $F(z_{i-1})$, $F(g)$, $F(z_i)$ are isosceles with vertex angle $2\pi/n$. Hence $F(z_1), \dots, F(z_n)$ are vertices of a regular polygon with $F(z_i)$ adjacent to $F(z_{i-1})$ for each $i = 1, \dots, n$, (and with center $F(g)$). This concludes the proof.

It is clear that if the automorphism F leaves all the roots of unity fixed then F takes adjacent vertices of a regular polygon into adjacent vertices of a regular polygon with the same orientation.

Since there are automorphisms of Z which are not Euclidean transformations, it follows that the concept "regular polygon" is inadequate for the axiomatization of plane geometry. This implies in particular that the concepts "equilateral triangle" and "square" are inadequate, a result of BETH and TARSKI [2]. TARSKI in [3] provides another generalization of this theorem.

TARSKI's question, "Can the inadequacy of the concept 'equilateral triangle' for the axiomatization of plane geometry be shown without the use of the axiom of choice?" raises the related question, "Can the existence of automorphisms of Z other than conjugation and the identity be shown without the use of the axiom of choice?".

Bibliography

[1] L. EULER, De centro similitudinis, *Comment. Geom.* (Series Prima) **26** (1953), 276—285.

[2] E. W. BETH and A. TARSKI, Equilaterality as the only primitive notion of Euclidean geometry, *Indag. Math.* **18** (1956), 462—467.

[3] A. TARSKI, A general theorem concerning primitive notions of Euclidean geometry, *Indag. Math.* **18** (1956), 468—474.

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