On the monotone convergence of certain Riemann sums

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1. Let us consider a real valued function $f(x)$, defined in a *finite* interval which might be assumed to be $[0, 1]$. We subdivide it by the points

$$(1.1)\qquad (0 = x_0 <) x_1 < x_2 < \cdots < x_n (< x_{n+1} = 1)$$

and form the sums

$$(1.2)\qquad S_n^{(l)}(f) = \sum_{\nu=0}^{n} (x_{\nu+1} - x_\nu) f(x_\nu),$$

$$(1.3)\qquad S_n^{(r)}(f) = \sum_{\nu=0}^{n} (x_{\nu+1} - x_\nu) f(x_{\nu+1}),$$

which may be called left resp. right Riemann sums. We shall consider sequences of subdivisions, the nth consisting of n points in $(0, 1)$ such that the maximal length of the intervals $(x_\nu, x_{\nu+1})$ tends to 0; then, as $n \to \infty$, both sums tend to the integral $S(f)$ of $f(x)$ in $[0, 1]$ provided that $S(f)$ exists in RIEMANN's sense. Our aim is to find classes of functions $f(x)$ and "interlacing" subdivisions (see (3.3)) so that the sequences $S_n^{(l)}(f)$ or $S_n^{(r)}(f)$ converge *monotonically for* $n = 1, 2, \ldots, \ldots$ to $S(f)$ for the whole class.

It is reasonable to assume that $f(x)$ itself is monotonic. If it is increasing, we have $S_n^{(l)}(f) \leqq S(f)$ for every n so that in the case of monotone convergence the sums $S_n^{(l)}(f)$ are increasing; similarly, the sums $S_n^{(r)}(f)$ must be decreasing. (The behavior is opposite if $f(x)$ is decreasing.) It is easy to see that monotone behavior of $f(x)$ alone does not imply such a regularity, by *any* choice of the points of subdivisions. Indeed let

$$(1.4)\qquad 0 < x_1' < x_1 < x_2' < 1$$

and

$$(1.5)\qquad f_0(0) = 0,\quad f_0(x_1') = p,\quad f_0(x_1) = q,\quad f_0(x_2') = r,\quad f_0(1) = 1$$

with

$$(1.6)\qquad 0 < p < q < r < 1.$$

Then we have

$$S_1^{(l)}(f_0)=(1-x_1)q, \quad S_2^{(l)}(f_0)=(x_2'-x_1')p+(1-x_2')r,$$

but at any choice of the points in (1.4) we can determine p, q, r so that (1.6) is fulfilled and still $S_1^{(l)}(f_0)>S_2^{(l)}(f_0)$. A similar remark holds for the right Riemann sums.[1])

We shall find that assuming, in addition, *convexity* of $f(x)$, the situation changes and for several „classical" systems of subdivisions the convergence will be *monotone from the beginning for the whole class*. (Convexity as well as monotone behavior is throughout meant in the wider sense.) More exactly our results will refer to the following four general classes of functions:

(1.7)

I. $f(x)$ is in $[0, 1]$	decreasing	and convex	from	below
II. „ „ „ „	increasing	„ „	„	above
III. „ „ „ „	„	„ „	„	below
IV. „ „ „ „	decreasing	„ „	„	above.

2. The question of monotone convergence is not an unnatural question in the theory of Riemann integral. If the points of subdivision are chosen so that the points of the kth subdivison belong always to the $(k+1)$th one, the *Darboux* sums converge *monotonically* to $S(f)$ whenever $f(x)$ is R-integrable. This fact and also some investigations concerning the Gibbs phenomenon led L. FEJÉR to raise the question of the monotone convergence of the Riemann sums (1.2) and (1.3). Though he has obtained the necessary and sufficient conditions for the points of subdivisions (see Theorem I below) and also observed that the classes (1.7) are the "proper" classes for monotone convergence, he applied these results exclusively to the equidistant case. In this case he made the further remark that a similar behavior holds for any polynomial $g(x)$ (instead of (1.7)), if we are contented with the monotone character of the Riemann sums for $n>n_0(g)$; for a properly chosen entire function $g(x)$ no such n_0 exists. He did not publish his results; they are announced together with some results of a paper of P. TURÁN[2]) without proofs. The novelty of our results, compared to those of FEJÉR, lies in verifying one of his sufficient conditions for the monotone convergence in certain special cases important in the theory of mechanical quadrature.

The case when the interval is no more finite, can be treated after proper modifications; we shall not do it in this paper. The general question in

[1]) A similar remark was made by Prof. G. PÓLYA.

[2]) P. TURÁN, On the zeros of the polynomials of Legendre, Časopis pro pěstováni mat. a fys. 75 (1950), 113—122. In particular see p. 122.

which way the Riemann sums converge to the integral, has contacts with seemingly far-lying questions; this can be illustrated by a problem of HARDY and RAMANUJAN which refers to the distribution of signs of the coefficients a_n in the expansion

$$\zeta(s)-\frac{1}{s-1}=\sum_{n=0}^{\infty} a_n\cdot(s-1)^n$$

where $\zeta(s)$ stands for the Riemann zeta-function. The link between this and FEJÉR's problems is the representation

$$a_n=\frac{(-1)^n}{n!}\lim_{N\to\infty}\left\{\sum_{k=1}^{N}\frac{\log^n k}{k}-\frac{\log^{n+1}N}{n+1}\right\},$$

proved by W. E. BRIGGS and S. CHOWLA.[3])

3. As indicated we shall investigate interlacing point-systems; denoting by

(3. 1) $$0<x_1<x_2<\cdots<x_n<1$$

and

(3. 2) $$0<y_1<y_2<\cdots<y_{n+1}<1$$

the nth resp. $(n+1)$th subdivision-points in $(0, 1)$ we have (see (1. 1))

(3. 3) $$x_\nu<y_{\nu+1}<x_{\nu+1},\quad \nu=0,1,\ldots,n.$$

With the abbreviations

(3. 4) $$y_{\nu+1}-x_\nu=p_\nu,\quad x_{\nu+1}-y_{\nu+1}=q_\nu,\quad \nu=0,1,\ldots,n,$$

and

(3. 5) $$p_{n+1}=0,$$

further

$$S_n^{(l)}(f)=\sum_{\nu=0}^{n}(x_{\nu+1}-x_\nu)f(x_\nu),$$

$$S_{n+1}^{(l)}(f)=\sum_{\nu=0}^{n+1}(y_{\nu+1}-y_\nu)f(y_\nu)$$

and analogously for $S_n^{(r)}(f)$, $S_{n+1}^{(r)}(f)$, we assert

Theorem I (FEJÉR). *In order that either of the inequalities*

(3. 6) $$S_{n+1}^{(l)}(f)\leqq S_n^{(l)}(f),\quad S_{n+1}^{(r)}(f)\geqq S_n^{(r)}(f)$$

[3]) W. E. BRIGGS and S. CHOWLA, The power series coefficients of $\zeta(s)$, *Amer. Math. Monthly*, **62** (1955), 323—325.

should hold for a fixed n for the whole class I it is necessary and sufficient that for $\mu=0,1,\dots,n$ the conditions

$$\sum_{\nu=0}^{\mu} q_\nu(p_{\nu+1}-p_\nu)\leqq 0 \tag{3.7}$$

take place (*note* (3. 5)).

We remark that (3. 7) is certainly satisfied if

$$p_0\geqq p_1\geqq\cdots\geqq p_n. \tag{3.8}$$

A slightly more general condition is the following. The sequence $\{p_\nu\}$ is first decreasing, then increasing and

$$\sum_{\nu=0}^{n} q_\nu(p_{\nu+1}-p_\nu)\leqq 0. \tag{3.9}$$

4. For the necessity of (3. 7) it is enough to show that the validity of the first (or second) inequality of (3. 6) for the special functions

$$f_\lambda(x)=\begin{cases}\lambda-x & \text{for}\quad 0\leqq x\leqq\lambda\\ 0 & \text{for}\quad \lambda\leqq x\leqq 1\end{cases} \tag{4.1}$$

for all $0\leqq\lambda\leqq 1$ implies (3. 7) and conversely. Since we may obviously suppose $f(1)=0$ and all functions $f(x)$ of class I with $f(1)=0$ can be approximated uniformly by a suitable linear combination of the functions $f_\lambda(x)$ with *positive* coefficients, the sufficiency of (3. 7) will follow at once.

In order to show that for all $0\leqq\lambda\leqq 1$ the inequality

$$S_{n+1}^{(l)}(f_\lambda)\leqq S_n^{(l)}(f_\lambda)$$

is equivalent to (3. 7), we fix λ and define μ by

$$x_\mu\leqq\lambda\leqq x_{\mu+1}\qquad(\mu=0,1,\dots,n) \tag{4.2}$$

and ξ by

$$\lambda=x_\mu+\xi=p_0+q_0+p_1+q_1+\cdots+p_{\mu-1}+q_{\mu-1}+\xi\quad\text{if}\quad \mu\geqq 1, \tag{4.3}$$
$$\lambda=\xi\quad\text{if}\quad\mu=0$$

where

$$0\leqq\xi\leqq q_\mu+p_\mu\,. \tag{4.4}$$

Then a simple geometrical reasoning (with the aid of the triangular and trapezoidal areas above the curve (4. 1)) shows for $\mu=0$

$$S_n^{(l)}(f_\lambda)=\lambda x_1=\lambda(p_0+q_0) \tag{4.5}$$

and for $1 \leqq \mu \leqq n$

$$(4.6)\quad \begin{aligned} S_n^{(l)}(f_\lambda) &= \frac{\lambda^2}{2} + \frac{1}{2}\sum_{\nu=0}^{\mu-1}(p_\nu+q_\nu)^2 + \left(p_\mu+q_\mu-\frac{\xi}{2}\right)\xi = \\ &= \frac{\lambda^2-\xi^2}{2} + \frac{1}{2}\sum_{\nu=0}^{\mu-1}p_\nu^2 + \sum_{\nu=0}^{\mu-1}p_\nu q_\nu + \frac{1}{2}\sum_{\nu=0}^{\mu-1}q_\nu^2 + (p_\mu+q_\mu)\xi. \end{aligned}$$

Further we assume first $0 \leqq \xi \leqq p_\mu$; we have $\mu = 0$

$$(4.7)\quad S_{n+1}^{(l)}(f_\lambda) = \lambda y_1 = \lambda p_0$$

and for $1 \leqq \mu \leqq n$

$$(4.8)\quad \begin{aligned} S_{n+1}^{(l)}(f_\lambda) &= \frac{\lambda^2}{2} + \frac{1}{2}p_0^2 + \frac{1}{2}\sum_{\nu=1}^{\mu-1}(p_\nu+q_{\nu-1})^2 + \\ &+ \frac{(2p_\mu+q_{\mu-1}-\xi)}{2}(q_{\mu-1}+\xi) = \frac{\lambda^2-\xi^2}{2} + \frac{1}{2}\sum_{\nu=0}^{\mu-1}p_\nu^2 + \\ &+ \sum_{\nu=1}^{\mu}p_\nu q_{\nu-1} + \frac{1}{2}\sum_{\nu=0}^{\mu-1}q_\nu^2 + p_\mu\xi. \end{aligned}$$

Secondedly we assume $p_\mu \leqq \xi \leqq p_\mu + q_\mu$; in the case $\mu = 0$

$$(4.9)\quad S_{n+1}^{(l)}(f_\lambda) = \lambda p_0 + (\lambda - p_0)(q_0 + p_1),$$

and in the case $1 \leqq \mu \leqq n$

$$(4.10)\quad \begin{aligned} S_{n+1}^{(l)}(f_\lambda) &= \frac{\lambda^2}{2} + \frac{1}{2}p_0^2 + \frac{1}{2}\sum_{\nu=1}^{\mu}(p_\nu+q_{\nu-1})^2 + \\ &+ \left(q_\mu + p_{\mu+1} - \frac{\xi-p_\mu}{2}\right)(\xi - p_\mu) = \frac{\lambda^2-\xi^2}{2} + \frac{1}{2}\sum_{\nu=0}^{\mu-1}p_\nu^2 + \\ &+ \sum_{\nu=1}^{\mu}p_\nu q_{\nu-1} + \frac{1}{2}\sum_{\nu=0}^{\mu-1}q_\nu^2 + (q_\mu+p_\mu+p_{\mu+1})\xi - p_\mu(q_\mu+p_{\mu+1}). \end{aligned}$$

This yields for $\mu = 0$, $0 \leqq \lambda \leqq p_0$ (cf. (4.5), (4.7))

$$(4.11)\quad S_{n+1}^{(l)}(f_\lambda) - S_n^{(l)}(f_\lambda) = -\lambda q_0,$$

and for $\mu = 0$, $p_0 \leqq \lambda \leqq p_0 + q_0$ (cf. (4.5), (4.9))

$$(4.12)\quad S_{n+1}^{(l)}(f_\lambda) - S_n^{(l)}(f_\lambda) = -p_0q_0 + p_1(\xi - p_0) = q_0(p_1 - p_0) + p_1(\xi - p_0 - q_0).$$

Now let $1 \leqq \mu \leqq n$, $0 \leqq \xi \leqq p_\mu$; from (4.6) and (4.8)

$$(4.13)\quad S_{n+1}^{(l)}(f_\lambda) - S_n^{(l)}(f_\lambda) = \sum_{\nu=0}^{\mu-1}q_\nu(p_{\nu+1} - p_\nu) - q_\mu\xi,$$

and finally for $1 \leqq \mu \leqq n$, $p_\mu \leqq \xi \leqq p_\mu + q_\mu$, from (4. 6) and (4. 10)

$$S_{n+1}^{(l)}(f_\lambda) - S_n^{(l)}(f_\lambda) = \sum_{\nu=0}^{\mu} q_\nu (p_{\nu+1} - p_\nu) + p_{\mu+1}(\xi - p_\mu - q_\mu). \tag{4. 14}$$

From (4. 14) and (4. 12) the necessity of (3. 7) follows at once; owing to (4. 11), (4. 12), (4. 13) and (4. 14) the monotone convergence for the special functions in (4. 1) follows conversely and as remarked at the beginning of **4.**, also the sufficiency of (3. 7).

Since the proof of the second assertion in (3. 6) goes along the same lines, we omit the details.

5. Since $f(x) \in$ II implies that $-f(x)$ belongs to the class I we obtain the

Corollary I. The inequalities (3. 7) represent the necessary and sufficient condition that either of the inequalities

$$S_{n+1}^{(l)}(f) \geqq S_n^{(l)}(f), \quad S_{n+1}^{(r)}(f) \leqq S_n^{(r)}(f)$$

should hold for a fixed n for the whole class II.

Since $f(x) \in$ III implies that $\psi(x) = f(1-x)$ belongs to the class I, we find with

$$x_\nu^* = 1 - x_{n-\nu+1}, \qquad \nu = 0, 1, \ldots, n+1,$$
$$0 = x_0^* < x_1^* < \cdots < x_n^* < x_{n+1}^* = 1$$
$$y_\nu^* = 1 - y_{n+2-\nu}, \qquad \nu = 0, 1, \ldots, n+2$$
$$0 = y_0^* < y_1^* < \cdots < y_{n+1}^* < y_{n+2}^* = 1,$$

that the interlacing condition (3. 3) holds. Further we have for $\mu = 0, 1, \ldots, n$

$$p_\mu^* = y_{\mu+1}^* - x_\mu^* = x_{n+1-\mu} - y_{n+1-\mu} = q_{n-\mu}$$

and similarly

$$q_\mu^* = p_{n-\mu}.$$

Now the inequality

$$\sum_{\nu=0}^{\mu} q_\nu^* (p_{\nu+1}^* - p_\nu^*) \leqq 0$$

amounts to

$$\sum_{\nu=\mu}^{n} p_\nu (q_{\nu+1} - q_\nu) \leqq 0. \tag{5. 1}$$

Hence we can conclude

Corollary II. The inequalities (5. 1) yield the necessary and sufficient condition that either of the inequalities

$$S_{n+1}^{(l)}(f) \leqq S_n^{(l)}(f), \quad S_{n+1}^{(r)}(f) \geqq S_n^{(r)}(f)$$

should hold for a fixed n for the whole class III.

Since $f(x) \in \mathrm{IV}$ implies that $-f(x) \in \mathrm{III}$, we can formulate

Corollary III. The inequalities (5. 1) give the necessary and sufficient condition that either of the inequalities

$$S_{n+1}^{(l)} \geqq S_n^{(l)}(f), \quad S_{n+1}^{(r)}(f) \leqq S_n^{(r)}(f)$$

should hold for a fixed n for the whole class IV.

We remark again that

$$q_0 \geqq q_1 \geqq \cdots \geqq q_{n+1} (\geqq 0) \tag{5. 2}$$

is sufficient to (5. 1).

6. It follows at once from (3. 8) that in the equidistant case, as FEJÉR remarked,

$$x_\nu = \frac{\nu}{n+1}, \quad y_\nu = \frac{\nu}{n+2}$$

$$\nu = 0, 1, \ldots, n+1, \tag{6. 1}$$

$$p_\nu = \frac{\nu+1}{n+2} - \frac{\nu}{n+1} = \frac{n+1-\nu}{(n+1)(n+2)},$$

i. e. (3. 8) is fulfilled for $n = 1, 2, \ldots$. Hence we proved

Corollary IV. (FEJÉR). The sequence of the left/right Riemann sums based on the equidistant system (6. 1) tends monotonically decreasingly/increasingly resp. increasingly/decreasingly to $S(f)$ whenever f belongs to class I resp. II.

7. In the theory of mechanical quadrature an important role is played by the zeros of the ultraspherical polynomials[4]) $P_n^{(\alpha)}(x)$ where $0 < \alpha < 1$. This motivates the classical investigations of CHEBYSHEV, HEINE, A. MARKOV, BRUNS, STIELTJES, FEJÉR and others concerning the finer distribution of these zeros; a brief account of this theory can be found in *OP*, chapter 6. It is well-known that the zeros of $P_n^{(\alpha)}(x)$ are in $(-1, +1)$ and interlace with those of $P_{n+1}^{(\alpha)}(x)$; in what follows we shall prove the more informative

Theorem II. *Denoting the zeros of $P_n^{(\alpha)}(x)$ for fixed $0 < \alpha < 1$ in decreasing order by $x_\nu^{(n)}$, the chain of inequalities*

$$x_1^{(n+1)} - x_1^{(n)} < x_2^{(n+1)} - x_2^{(n)} < \cdots < x_{\left[\frac{n+1}{2}\right]}^{(n+1)} - x_{\left[\frac{n+1}{2}\right]}^{(n)} \tag{7. 1}$$

holds $(n = 2, 3, \ldots)$.

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[4]) We shall follow the notation of G. SZEGŐ, „Orthogonal polynomials" revised edition, 1959, p. 81; this book will be quoted in the sequel by *OP*.

We note that if n is even, the last difference in (7. 1) is

$$x^{(n+1)}_{\frac{n}{2}} - x^{(n)}_{\frac{n}{2}},$$

$x^{(n+1)}_{\frac{n}{2}}$ and $x^{(n)}_{\frac{n}{2}}$ being the smallest positive zeros of $P^{(\alpha)}_{n+1}(x)$ resp. $P^{(\alpha)}_{n}(x)$; if n is odd, the last difference in (7. 1) is

$$x^{(n+1)}_{\frac{n+1}{2}} - x^{(n)}_{\frac{n+1}{2}} = x^{(n+1)}_{\frac{n+1}{2}}$$

the smallest positve zero of $P^{(\alpha)}_{n+1}(x)$.

Since $-1 < x^{(n)}_{\nu} < 1$, there is a unique $\vartheta^{(n)}_{\nu}$ so that

$$x^{(n)}_{\nu} = \cos \vartheta^{(n)}_{\nu}, \quad 0 < \vartheta_{\nu} < \pi. \tag{7. 2}$$

We have evidently

$$0 < \vartheta^{(n+1)}_1 < \vartheta^{(n)}_1 < \vartheta^{(n+1)}_2 < \cdots < \vartheta^{(n+1)}_{\left[\frac{n+1}{2}\right]} < \vartheta^{(n)}_{\left[\frac{n+1}{2}\right]} \leq \frac{\pi}{2}. \tag{7. 3}$$

We assert the following stronger

Theorem III. *At a fixed* $0 < \alpha < 1$ *for the above* $\vartheta^{(n)}_{\nu}$ *'s the chain of inequalities*

$$\vartheta^{(n)}_1 - \vartheta^{(n+1)}_1 < \vartheta^{(n)}_2 - \vartheta^{(n+1)}_2 < \cdots < \vartheta^{(n)}_{\left[\frac{n+1}{2}\right]} - \vartheta^{(n+1)}_{\left[\frac{n+1}{2}\right]}$$

holds $(n = 2, 3, \ldots)$.

Since the proof of this theorem is essentially the same as that of Theorem II, we shall confine ourselves to the proof of Theorem II.

8. For the proof of this theorem we employ STURM's method (cf. OP, p. 19). The function

$$u = (1-x^2)^{\frac{\alpha}{2}+\frac{1}{4}} P^{(\alpha)}_n(x)$$

satisfies the differential equation (OP, p. 82, (4. 7. 10))

$$\frac{d^2u}{dx^2} + \varphi_n(x)u = 0 \tag{8. 1}$$

where

$$\varphi_n(x) = \frac{(n+\alpha)^2}{1-x^2} + \frac{\frac{1}{2} + \alpha - \alpha^2 + \frac{x^2}{4}}{(1-x^2)^2}.$$

In order to show

$$x^{(n+1)}_{\nu-1} - x^{(n)}_{\nu-1} < x^{(n+1)}_{\nu} - x^{(n)}_{\nu} = h, \quad 2 \leq \nu \leq \left[\frac{n+1}{2}\right] \tag{8. 2}$$

we consider the two equations:

$$\frac{d^2u}{dx^2}+\varphi_n(x-h)u=0 \qquad u=P_n^{(\alpha)}(x-h)$$

$$\frac{d^2v}{dx^2}+\varphi_{n+1}(x)v=0 \qquad v=P_{n+1}^{(\alpha)}(x)$$

in the interval $[x_\nu^{(n+1)}, x_{\nu-1}^{(n+1)}]$. Visibly $\varphi_n(x)$ is increasing with n and x as long as $0\leqq x\leqq 1$ so that

$$\varphi_n(x-h)<\varphi_{n+1}(x);$$

indeed

$$x-h\geqq x_\nu^{(n+1)}-h=x_\nu^{(n)}\geqq 0.$$

Hence v is "stronger oscillating" than u. Now u vanishes at $x_\nu^{(n+1)}$ (since $x_\nu^{(n+1)}-h=x_\nu^{(n)}$) and again at $x_{\nu-1}^{(n)}+h$, since v vanishes at $x_\nu^{(n+1)}$, its next zero, that is $x_{\nu-1}^{(n+1)}$, must be $<x_{\nu-1}^{(n)}+h$, which is the assertion.

We conclude from (8. 2) that

$$x_{\nu-1}^{(n+2)}-x_{\nu-1}^{(n)}<x_\nu^{(n+2)}-x_\nu^{(n)}, \qquad 2\leqq\nu\leqq\left[\frac{n+1}{2}\right]. \tag{8. 3}$$

Also we observe that the positive zeros of the ultraspherical polynomials $P_{2n+1}^{(\alpha)}(x)$ and $P_{2n+3}^{(\alpha)}(x)$ interlace; similarly the positive zeros of $P_{2n+2}^{(\alpha)}(x)$ and $P_{2n+4}^{(\alpha)}(x)$. (Cf. OP, p. 59, formulas (4. 1. 5).)

9. Now the n positive zeros of

$$P_{2n+1}^{(\alpha)}(x)=0 \tag{9. 1}$$

for fixed $0<\alpha<1$ and $n=1,2,\ldots$ resp. the n greatest positive zeros of

$$P_{2n+2}^{(\alpha)}(x)=0 \tag{9. 2}$$

for fixed $0<\alpha<1$ and $n=1,2,\ldots$ form the points of subdivision to which our monotone convergence theorems refer. With the notation of Theorem II. and in addition with the convention

$$x_0^{(2n+1)}=1, \quad x_{n+1}^{(2n+1)}=0 \tag{9. 3}$$

we assert

Theorem IV. *The Riemann sums*

$$\sum_{\nu=0}^{n}(x_{n-\nu}^{(2n+1)}-x_{n-\nu+1}^{(2n+1)})f(x_{n-\nu+1}^{(2n+1)})$$

and

$$\sum_{\nu=0}^{n}(x_{n-\nu}^{(2n+1)}-x_{n-\nu+1}^{(2n+1)})f(x_{n-\nu}^{(2n+1)})$$

based on the points (9. 1)—(9. 3) *tend for* $n=1,2,\ldots$ *in a monotonically decreasing resp. increasing way to* $S(f)$ *whenever* f *belongs to class I.*

The proof follows at once from (3. 8) and (8. 3). We assert further with the convention

(9. 4) $$x_0^{2n+2}=1, \quad x_{n+1}^{(2n+2)}=0,$$

the following

Theorem V. *The Riemann sums*

$$\sum_{\nu=0}^{n}(x_{n-\nu}^{(2n+2)}-x_{n-\nu+1}^{(2n+2)})f(x_{n-\nu+1}^{(2n+2)})$$

and

$$\sum_{\nu=0}^{n}(x_{n-\nu}^{(2n+2)}-x_{n-\nu+1}^{(2n+2)})f(x_{n-\nu}^{(2n+2)}),$$

based on the points (9. 2)—(9. 4) *tend for* $n=1,2,\ldots$ *in a monotonically decreasing resp. increasing way to* $S(f)$ *whenever* f *belongs to class I.*

The proof follows at once from (3. 8) and (8. 3) except for the inequality $p_1<p_0$. But in our case we have indeed

$$p_1=x_n^{(2n+4)}-x_n^{(2n+2)}<x_{n+1}^{(2n+4)}-x_{n+1}^{(2n+2)}<x_{n+1}^{(2n+1)}=p_0.$$

10. Finally we have to motivate why in the case of the points (9. 2) of subdivision the points $x_{n+1}^{(2n+2)}$ have been omitted. Indeed, in the contrary case the condition $p_1\leqq p_0$ would assume the form

(10. 1) $$x_{n+1}^{(2n+4)}-x_{n+1}^{(2n+2)}\leqq x_{n+2}^{(2n+4)}.$$

Using the first term of the asymptotic expansion of OP, p. 195, (8. 21. 15), we find that for even n and $n\to\infty$ the smallest positive zeros

$$x_{n-\varrho}^{(2n)}=\cos\vartheta_{n-\varrho}^{(2n)}$$

$\varrho=0,1,2,\ldots$ but fixed, can be approximated as follows

$$x_{n-\varrho}^{(2n)}=\frac{\left(\varrho+\frac{1}{2}\right)\pi}{2n+\alpha}+o\left(\frac{1}{n}\right).$$

Hence (10. 1) would imply for large n

$$\frac{\frac{3\pi}{2}}{2n+4+\alpha}-\frac{\frac{\pi}{2}}{2n+2+\alpha}\leqq\frac{\frac{\pi}{2}}{2n+4+\alpha}+o\left(\frac{1}{n}\right)$$

which is not the case.

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