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L_t-Horn sentences and reduced products

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Abstract. It is proved that under continuum hypothesis an L_t-sentence is preserved under reduced products of topological structures iff it is equivalent in basic structures to an L_2-Horn sentence. Specially, each L_t-Horn sentence is preserved under such products.

1. Introduction

The aim of the paper is to prove the topological version of the classical theorem of H. J. Keisler concerning Horn sentences and reduced products (see [2], Theorem 6.2.5 or [6]).

The language L_t, introduced by T. A. McKee in [7] and [8] and M. Ziegler in [9] is a sublanguage of the monadic second-order language L_2 which (using weak structures as models) can be regarded as a two-sorted first-order language.

For the coherence of the text firstly we introduce the notation and recall a few well known facts.

We consider the two-sorted language $L_2 = L \cup \mathrm{CONST} \cup \{\in\}$, where L is a first-order language (with the sets of relations, functions and constants, respectively, Rel, Fnc and Const). CONST is the set of "set constants" ([4]) and $\in$ is a "new" binary relation (not contained in Rel). $\mathrm{Var}^1 = \{v_1, v_2, \dots\}$ and $\mathrm{Var}^2 = \{V_1, V_2, \dots\}$ are the sets of individual and set variables. As usual we use meta variables $x, y, z, \dots$ and $X, Y, Z, \dots$. For the sake of convinience, the sets of terms (Term_{L_2}) and formulas (Form_{L_2}) are defined as follows. The terms of L_2 are exactly the terms of L, i.e.

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$\mathrm{Term}_{L_2} = \mathrm{Term}_L$. The set of atomic L_2-formulas, At_{L_2}, contains atomic L-formulas and formulas of the shape $t \in X$ and $t \in C$, where $t \in \mathrm{Term}_L$, $X \in \mathrm{Var}^2$ and $C \in \mathrm{CONST}$. The set of L_2-formulas is obtained from atomic L_2-formulas by a finite application of connectives $\wedge$, $\vee$ and $\neg$ and quantifiers $\exists x$ and $\exists X$ ($\Longrightarrow$, $\Longleftrightarrow$, $\forall x$ and $\forall X$ are defined in the standard way). The "unofficial" formulas $X = Y$, $X = C$ are to replace the formulas $\forall x(x \in X \Longleftrightarrow x \in Y)$, $\forall x(x \in X \Longleftrightarrow x \in C)$. By $Fv(\varphi)$ we denote the set of free variables of the formula $\varphi \in \mathrm{Form}_{L_2}$. Sent_{L_2} is the set of L_2-sentences.

A model of L_2 is a quadruple $\mathcal{A} = \langle \mathbf{A}, \mathcal{O}, \mathbf{C}, \varrho \rangle$, where $\mathbf{A}$ is a model of (the first-order language) L with domain A, $\mathbf{C} \subseteq \mathcal{O}$ and $\varrho \subseteq A \times \mathcal{O}$ is the interpretation of the relation $\in$. We say that a model $\mathcal{A}$ is *weak* if $\emptyset \neq \mathcal{O} \subseteq P(A)$ and ϱ is the membership relation (we will write again $\in$). Of course, there is no restriction at all if we consider just weak models (any model is isomorphic to some such model). Thus, from now on, a model will mean a weak model and we will simply write $\mathcal{A} = \langle \mathbf{A}, \mathcal{O}, \mathbf{C} \rangle$. A valuation in $\mathcal{A}$ is an union $\tau = \tau^1 \cup \tau^2$, where $\tau^1 : \mathrm{Var}^1 \longrightarrow A$ and $\tau^2 : \mathrm{Var}^2 \longrightarrow \mathcal{O}$. The value of a term and the satisfaction relation (for the given valuation) are defined naturally; the individual variable v_i is interpreted as (an element of A) $\tau^1(v_i)$, the set variable V_j as (an element of $\mathcal{O}$) $\tau^2(V_j)$ (and, we repeat, $\in$ is the set-theoretic membership relation).

Weak L_2-structures $\mathcal{A}$ and $\mathcal{B}$ are L_2-elementary equivalent, in notation $\mathcal{A} \equiv_{L_2} \mathcal{B}$, iff:

$$\mathcal{A} \vDash \varphi \quad \text{iff} \quad \mathcal{B} \vDash \varphi, \quad \text{for all} \quad \varphi \in \mathrm{Sent}_{L_2}.$$

Let $\{\mathcal{A}_i \mid i \in I\}$ be a family of weak L_2-models, Ψ a filter on I, $\sim$ the equivalence relation on $\prod_{i \in I} A_i$ given by: $f \sim g$ iff $\{i \in I \mid f_i = g_i\} \in \Psi$, $[f]$ the equivalence class of the element $f \in \prod_{i \in I} A_i$ and $q : \prod_i A_i \longrightarrow \prod_i A_i/\!\sim$ the natural mapping. By $\prod_\Psi \mathbf{A}_i$ we denote the reduced product of first-order parts of $\mathcal{A}_i$, $i \in I$, by $\prod \mathcal{O}_i$ the family of sets of shape $\prod_{i \in I} U_i$, where, for each $i \in I$, $U_i \in \mathcal{O}_i$ and by $\prod_\Psi \mathcal{O}_i$ the collection of sets $q(\prod_{i \in I} U_i)$, where $\prod_{i \in I} U_i \in \prod \mathcal{O}_i$. For $C \in \mathrm{CONST}$ we define $C^{\mathcal{A}} = q(\prod_{i \in I} C^{\mathcal{A}_i})$. Then

$$\mathcal{A} = \left\langle \prod_\Psi \mathbf{A}_i, \prod_\Psi \mathcal{O}_i, \{C^{\mathcal{A}} \mid C \in \mathrm{CONST}\} \right\rangle$$

is a weak L_2-structure called *the reduced product* of the family $\{\mathcal{A}_i \mid i \in I\}$, in notation $\prod_\Psi \mathcal{A}_i$.

It is easy to imitate the proofs of some of the most important theorems of the classical model theory. More precisely, the logic L_2 satisfies, for instance, the Łoś theorem, the compactness theorem and the Löwenheim-Skolem theorem.

Theorem 1.1 (Łoś). *Let $\{\mathcal{A}_i \mid i \in I\}$ be a family of weak L_2-structures and Ψ an ultrafilter on I. Then for each $\varphi(x^1, \dots, x^p, X^1, \dots, X^q) \in \mathrm{Form}_{L_2}$, each $f^1, \dots, f^p \in \prod A_i$ and each $U^1, \dots, U^q \in \prod \mathcal{O}_i$ it holds:*

$$\prod\nolimits_\Psi \mathcal{A}_i \models \varphi[[f^1], \dots, [f^p], q(U^1), \dots, q(U^q)]$$

iff

$$\{i \in I \mid \mathcal{A}_i \models \varphi[f^1_i, \dots, f^p_i, U^1_i, \dots, U^q_i]\} \in \Psi.$$

Specially, if $\mathcal{A}_i = \mathcal{A}$ for all $i \in I$, then $\mathcal{A} \equiv_{L_2} \prod_\Psi \mathcal{A}$.

Theorem 1.2 (Compactness). *A theory $T \subseteq \mathrm{Sent}_{L_2}$ has a weak model iff each its finite subset has a weak model.*

Theorem 1.3 (Löwenheim-Skolem). *Let κ be an infinite cardinal and $\mathcal{B} = \langle \mathbf{B}, \mathcal{O}_\mathcal{B}, \mathbf{C}_\mathcal{B} \rangle$ a weak L_2-model. If $|L_2| \leq \kappa \leq |B \cup \mathcal{O}_\mathcal{B}|$, $X \subseteq B$, $\mathcal{U} \subseteq \mathcal{O}_\mathcal{B}$ and $|X \cup \mathcal{U}| \leq \kappa$, then there exists a weak model $\mathcal{A} = \langle \mathbf{A}, \mathcal{O}_\mathcal{A}, \mathbf{C}_\mathcal{A} \rangle$ satisfying $X \subseteq A$, $\mathcal{U} \subseteq \mathcal{O}_\mathcal{A}$, $\mathcal{A} \equiv_{L_2} \mathcal{B}$ and $|A \cup \mathcal{O}_\mathcal{A}| \leq \kappa$.*

In particular, if a theory T of a countable language L_2 has a weak model, then it has a countable weak model.

A weak L_2-model $\mathcal{A}$ realizes a set of L_2-formulas $\Sigma(x^1, \dots, x^p, X^1, \dots, X^q)$ iff there exist $a^1, \dots, a^p \in A$ and $U^1, \dots, U^q \in \mathcal{O}_\mathcal{A}$ such that

$$\mathcal{A} \models \varphi[a^1, \dots, a^p, U^1, \dots, U^q] \quad \text{for each} \quad \varphi \in \Sigma(x^1, \dots, x^p, X^1, \dots, X^q).$$

$\Sigma(x^1, \dots, x^p, X^1, \dots, X^q)$ is a type over $\mathcal{A}$ iff there exists a weak L_2-model $\mathcal{M}$ satifying: (1) $\mathcal{M} \models \mathrm{Th}(\mathcal{A})$ and (2) $\mathcal{M}$ realizes $\Sigma(x^1, \dots, x^p, X^1, \dots, X^q)$.

A weak L_2-model $\mathcal{A}$ is *saturated* iff for each $X \subseteq A$ and $\mathcal{U} \subseteq \mathcal{O}$ satisfying $|X \cup \mathcal{U}| < |A \cup \mathcal{O}|$, every type $\Sigma(x^1, \dots, x^p, X^1, \dots, X^q)$ of the language $L_{X \cup \mathcal{U}} = L_2 \cup \{c_a \mid a \in X\} \cup \{C_V \mid V \in \mathcal{U}\}$ over $\mathcal{A}_{X \cup \mathcal{U}} = \langle \mathcal{A}, a, V \rangle_{a \in X,\, V \in \mathcal{U}}$ is realized in $\mathcal{A}_{X \cup \mathcal{U}}$.

By the compactness theorem it holds the statement analogous to Theorem 6.1.1 from [2].

Theorem 1.4 (CH). *Let $\{\mathcal{A}_i \mid i \in \omega\}$ be a family of weak models of a countable language L_2 such that, for all $i \in \omega$, $|A_i \cup \mathcal{O}_{\mathcal{A}_i}| \leq \omega_1$ and let Ψ be a nonprincipal ultrafilter on ω. Then the ultraproduct $\prod_\Psi \mathcal{A}_i$ is a saturated weak L_2-model of cardinality $\leq \omega_1$.*

The theorems from this paragraph can also be obtained by "translation" of L_2 in the corresponding (one-sorted) first-order language, but this way is more expensive.

2. Prenex forms of formulas

In the sequel the sequences like $x^1,\dots,x^p$; $X^1,\dots,X^q$; $f^1,\dots,f^p$; $f_i^1,\dots,f_i^p$, $[f^1],\dots,[f^p]$; $U^1,\dots,U^q$; $U_i^1,\dots,U_i^q$; $q(U^1),\dots,q(U^q)$ will be shortly denoted by $\overline{x}$, $\overline{X}$, $\overline{f}$, $\overline{f_i}$, $\overline{[f]}$, $\overline{U}$, $\overline{U_i}$, $\overline{q(U)}$, whenever the confusion is impossible.

For all definitions and facts in connection with L_t-formulas and L_t-language we refer to the book of M. Ziegler and J. Flum ([3]).

Definition 2.1. A weak L_2-structure $\mathcal{A}=\langle \mathbf{A},\mathcal{O},\mathbf{C}\rangle$ is a covering (basic, topological) structure iff $\bigcup\mathcal{O}=A$ ($\mathcal{O}$ is a base for some topology on A, $\mathcal{O}$ is a topology on A).

For an L_2-formula $\varphi(x^1,\dots,x^p,X^1,\dots,X^q)$, shortly denoted by $\varphi\,(\overline{x},\overline{X})$ we write: $\models_w \varphi(\overline{x},\overline{X})$ ($\models_c \varphi(\overline{x},\overline{X})$, $\models_b \varphi(\overline{x},\overline{X})$, $\models_t \varphi(\overline{x},\overline{X})$) iff for each weak (covering, basic, topological) L_2-structure $\mathcal{A}$, each $a^1,\dots,a^p\in A$ and each $U^1,\dots,U^q\in\mathcal{O}$ it holds: $\mathcal{A}\models\varphi[\overline{a},\overline{U}]$.

If $\varphi,\psi\in \mathrm{Form}_{L_2}$, then $\varphi \overset{w}{\Longleftrightarrow} \psi$ ($\varphi \overset{c}{\Longleftrightarrow} \psi$, $\varphi \overset{b}{\Longleftrightarrow} \psi$, $\varphi \overset{t}{\Longleftrightarrow} \psi$) iff $\models_w \varphi\Longleftrightarrow\psi$ ($\models_c \varphi\Longleftrightarrow\psi$, $\models_b \varphi\Longleftrightarrow\psi$, $\models_t \varphi\Longleftrightarrow\psi$).

Clearly, for an L_2-structure $\mathcal{A}$ we have: $\mathcal{A}$ is topological $\longrightarrow$ $\mathcal{A}$ is basic $\longrightarrow$ $\mathcal{A}$ is covering $\longrightarrow$ $\mathcal{A}$ is weak. Thus, for $\varphi,\psi\in\mathrm{Form}_{L_2}$ it holds: $\varphi \overset{w}{\Longleftrightarrow} \psi \longrightarrow \varphi \overset{c}{\Longleftrightarrow} \psi \longrightarrow \varphi \overset{b}{\Longleftrightarrow} \psi \longrightarrow \varphi \overset{t}{\Longleftrightarrow} \psi$.

Definition 2.2. An L_2-formula φ is in L_2-prenex form iff $\varphi\equiv Q_1\dots Q_n\psi$, where Q_i, $i=1,\dots n$, is one of the quantifiers $\exists x$, $\forall x$, $\exists X$, $\forall X$ and ψ is a quantifier free L_2-formula.

An L_t-formula φ is in L_t-prenex form iff $\varphi\equiv Q_1\dots Q_n\psi$, where Q_i, $i=1,\dots,n$, is one of the quantifiers $\exists x$, $\forall x$, $\exists X\ni t$, $\forall X\ni t$, t being a term, and ψ is a quantifier free L_t-formula.

Lemma 2.3. *(A) Let φ and ψ be L_2-formulas and $X\notin Fv(\varphi)$. Then it holds:*

(1) $\neg\exists X\psi \overset{w}{\Longleftrightarrow} \forall X\neg\psi$;

(2) $\neg\forall X\psi \overset{w}{\Longleftrightarrow} \exists X\neg\psi$;

(3) $(\varphi\Longrightarrow\exists X\psi) \overset{w}{\Longleftrightarrow} \exists X(\varphi\Longrightarrow\psi)$;

(4) $(\forall X\psi\Longrightarrow\varphi) \overset{w}{\Longleftrightarrow} \exists X(\psi\Longrightarrow\varphi)$;

(5) $(\varphi\Longrightarrow\forall X\psi) \overset{w}{\Longleftrightarrow} \forall X(\varphi\Longrightarrow\psi)$;

(6) $(\exists X\psi\Longrightarrow\varphi) \overset{w}{\Longleftrightarrow} \forall X(\psi\Longrightarrow\varphi)$.

(B) Let φ and ψ be L_2-formulas and $x\notin Fv(\varphi)$. Then (1)–(6) *holds if we replace X by x.*

Lemma 2.4. *Let* $\varphi(x^1,\dots,x^p,X^1,\dots,X^q)$ *and* $\psi(x^1,\dots,x^p,X^1,\dots,X^q,Y)$ *be* L_2*-formulas and let* $t(x^1,\dots,x^p)\in Term_{L_2}$*. Then it holds:*

(1) $\neg\exists Y\ni t\,\psi \overset{w}{\Longleftrightarrow} \forall Y\ni t\,\neg\psi$;

(2) $\neg\forall Y\ni t\,\psi \overset{w}{\Longleftrightarrow} \exists Y\ni t\,\neg\psi$;

(3) $(\varphi\Longrightarrow\exists Y\ni t\,\psi) \overset{c}{\Longleftrightarrow} \exists Y\ni t\,(\varphi\Longrightarrow\psi)$;

(4) $(\forall Y\ni t\,\psi\Longrightarrow\varphi) \overset{c}{\Longleftrightarrow} \exists Y\ni t\,(\psi\Longrightarrow\varphi)$;

(5) $(\varphi\Longrightarrow\forall Y\ni t\,\psi) \overset{w}{\Longleftrightarrow} \forall Y\ni t\,(\varphi\Longrightarrow\psi)$;

(6) $(\exists Y\ni t\,\psi\Longrightarrow\varphi) \overset{w}{\Longleftrightarrow} \forall Y\ni t\,(\psi\Longrightarrow\varphi)$.

Moreover, if the formulas from the left side are L_t*-formulas so are the formulas on the right side.*

Proof. (1) and (2) follow from the previous lemma.

(3) Let $\mathcal{A}$ be a covering structure, $a^1,\dots,a^p\in A$ and $U^1,\dots,U^q\in\mathcal{O}$. ($\Longrightarrow$) Let $\mathcal{A}\vDash(\varphi\Longrightarrow\exists Y\ni t\,\psi)[\overline{a},\overline{U}]$, that is if $\mathcal{A}\vDash\varphi[\overline{a},\overline{U}]$, then there is $V\in\mathcal{O}$ such that $t^{\mathcal{A}}[\overline{a}]\in V$ and $\mathcal{A}\vDash\psi[\overline{a},\overline{U},V]$. Suppose $\mathcal{A}\nvDash\exists Y\ni t\,(\varphi\Longrightarrow\psi)[\overline{a},\overline{U}]$. Then for each $V\in\mathcal{O}$, if $t^{\mathcal{A}}[\overline{a}]\in V$ then $\mathcal{A}\vDash\varphi[\overline{a},\overline{U}]$ and $\mathcal{A}\nvDash\psi[\overline{a},\overline{U},V]$. Since $\bigcup\mathcal{O}=A$ there is V_0 containing $t^{\mathcal{A}}[\overline{a}]$, hence $\mathcal{A}\vDash\varphi[\overline{a},\overline{U}]$ and $\mathcal{A}\nvDash\psi[\overline{a},\overline{U},V_0]$. It follows that there exists some $V_1\in\mathcal{O}$ such that $t^{\mathcal{A}}[\overline{a}]\in V_1$ and $\mathcal{A}\vDash\psi[\overline{a},\overline{U},V_1]$, a contradiction.

($\Longleftarrow$) Let $\mathcal{A}\vDash\exists Y\ni t\,(\varphi\Longrightarrow\psi)[\overline{a},\overline{U}]$. Thus there is $V_0\in\mathcal{O}$ such that $t^{\mathcal{A}}[\overline{a}]\in V_0$ and $\mathcal{A}\nvDash\varphi[\overline{a},\overline{U}]$ or $\mathcal{A}\vDash[\overline{a},\overline{U},V_0]$. Suppose that $\mathcal{A}\nvDash(\varphi\Longrightarrow\exists Y\ni t\,\psi)[\overline{a},\overline{U}]$. Then we have $\mathcal{A}\vDash\varphi[\overline{a},\overline{U}]$ and, for all $V\in\mathcal{O}$, $t^{\mathcal{A}}[\overline{a}]\in V$ implies $\mathcal{A}\nvDash\psi[\overline{a},\overline{U},V]$. But then $\mathcal{A}\vDash\psi[\overline{a},\overline{U},V_0]$ and $\mathcal{A}\nvDash\psi[\overline{a},\overline{U},V_0]$.

(4) follows from (3) and the proofs of (5) and (6) are direct as well. □

Remark. The item (3) of the preceeding theorem does not hold for weak L_2-structures. For example, formulas $(x\neq x\Longrightarrow\exists Y\ni x\ x=x)$ and $\exists Y\ni x(x\neq x\Longrightarrow x=x)$ are not *weak equivalent*.

Corollary 2.5. *For each* L_2*-formula* φ *there exists an* L_2*-formula* ψ *in* L_2*-prenex form such that* $\varphi\overset{w}{\Longleftrightarrow}\psi$*.*

For each L_t*-formula* φ *there exists an* L_t*-formula* ψ *in* L_t*-prenex form such that* $\varphi\overset{c}{\Longleftrightarrow}\psi$*.*

3. Horn sentences and reduced products

Definition 3.1 (Horn L_2-formulas). An L_2-formula φ is a basic Horn L_2-formula iff $\varphi \equiv \vartheta_1 \vee \ldots \vee \vartheta_m$, where at most one of the formulas ϑ_i is an atomic L_2-formula and the rest being negations of atomic L_2-formulas.

Horn L_2-formulas are the formulas obtained from basic Horn L_2-formulas by a finite number of applications of use of conjunction ($\wedge$) and the quantifiers $\exists x$, $\forall x$, $\exists X$ and $\forall X$.

The set of Horn L_2-formulas and the set of Horn L_2-sentences will be denoted, respectively, by HF_{L_2} and HS_{L_2}.

(Horn L_t-formulas)

(1) Basic Horn L_2-formulas are (basic) Horn L_t-formulas.

(2) If φ and ψ are Horn L_t-formulas, so are the formulas $\varphi \wedge \psi$, $\exists x\varphi$, $\forall x\varphi$.

If φ is a Horn L_t-formula, $t \in \mathrm{Term}_{L_2}$ and φ is positive (negative) in X, then the formula $\forall X \ni t\, \varphi$ ($\exists X \ni t\, \varphi$) is a Horn L_t-formula.

(3) An L_t-formula is a Horn L_t-formula iff it is obtained by finite use of (1) and (2).

The set of Horn L_t-formulas and the set of Horn L_t-sentences will be denoted, respectively, by HF_{L_t} and HS_{L_t}.

Lemma 3.2. *(a) For each Horn L_2-formula φ there exists a Horn L_2-formula ψ in L_2-prenex form such that $\varphi \stackrel{w}{\Longleftrightarrow} \psi$.*

(b) For each Horn L_t-formula φ there exists a Horn L_t-formula ψ in L_t-prenex form such that $\varphi \stackrel{c}{\Longleftrightarrow} \psi$ and $Fv(\varphi) = Fv(\psi)$, $Fv^+(\varphi) = Fv^+(\psi)$, $Fv^-(\varphi) = Fv^-(\psi)$, where $Fv^+(\varphi)$ ($Fv^-(\varphi)$) is the set of free set variables of the formula φ in which it is positive (negative).

Proof. (a) The induction follows the construction of φ as a Horn L_2-formula.

(1) If φ is a basic Horn L_2-formula then $\psi \equiv \varphi$;

(2) If $\varphi \equiv \varphi_1 \wedge \varphi_2$, where $\varphi_1, \varphi_2 \in HF_{L_2}$, then, by inductive hypothesis, for some Horn L_2-formulas ψ_1, ψ_2 in L_2-prenex form it holds: $\varphi_i \stackrel{w}{\Longleftrightarrow} \psi_i$, $i = 1, 2$, whence $\varphi \stackrel{w}{\Longleftrightarrow} \psi_1 \wedge \psi_2$. Let $\psi_1 \equiv Q_1^1 \ldots Q_k^1 \eta_1$ and $\psi_2 \equiv Q_1^2 \ldots Q_l^2 \eta_2$. Without loss of generality we can assume that the bounded variables of ψ_1 do not appear in ψ_2 and vice-versa. Then $\varphi \stackrel{w}{\Longleftrightarrow} Q_1^1 \ldots Q_k^1 Q_1^2 \ldots Q_l^2 (\eta_1 \wedge \eta_2)$.

(3) If $\varphi \equiv \exists x \varphi_1$ where $\varphi_1 \in HF_{L_2}$, then, for some Horn L_2-formula ψ in L_2-prenex form $\varphi_1 \stackrel{w}{\Longleftrightarrow} \psi$, thus $\varphi \stackrel{w}{\Longleftrightarrow} \exists x\psi$.

The cases when φ is of the form $\forall x\varphi_1$, $\exists X\varphi_1$ and $\forall X\varphi_1$ are obvious as well.

(b) Still one induction. Let us just consider the case: $\varphi \equiv \exists X \ni t\varphi_1(\dots, X^-, \dots)$, where $\varphi_1 \in HF_{L_t}$. By assumption, there is a Horn L_t-formula $\psi(\dots, X^-, \dots)$ in L_t-prenex form such that $\varphi_1 \overset{c}{\Longleftrightarrow} \psi$, whence $\varphi \overset{c}{\Longleftrightarrow} \exists X \ni t\psi$. □

Lemma 3.3. *For each $\varphi \in HF_{L_t}$ there exists $\vartheta \in HF_{L_2}$ such that $\varphi \overset{c}{\Longleftrightarrow} \vartheta$.*

Proof. By the previous lemma we can consider just Horn L_t-formulas in L_t-prenex form. As usual, the proof is by induction on the number of quantifiers, n, in formula φ.

If $n = 0$, φ is a conjunction of basic Horn L_t formulas.

Suppose that the statement holds for formulas with $\leq n$ quantifiers and let $\varphi \equiv Q_1 \dots Q_n Q_{n+1}\varphi_1$. By inductive hypothesis there is $\vartheta_1 \in HF_{L_2}$ such that $Q_2 \dots Q_n Q_{n+1}\varphi_1 \overset{c}{\Longleftrightarrow} \vartheta_1$. We distinguish the following cases:

(a) Q_1 is $\exists X \ni t$. Then $\varphi \overset{c}{\Longleftrightarrow} \exists X(t \in X \wedge \vartheta_1)$ and $\exists X(t \in X \wedge \vartheta_1) \in HF_{L_2}$.

(b) Q_1 is $\forall X \ni t$. According to 2.5, we can assume that ϑ_1 is in L_2-prenex form, let $\vartheta_1 \equiv Q'_1 \dots Q'_k\eta$. Then, by Lemma 2.3, we have: $\varphi \equiv \forall X \ni tQ_2 \dots Q_n Q_{n+1}\varphi_1 \overset{c}{\Longleftrightarrow} \forall X \ni tQ'_1 \dots Q'_k\eta \overset{w}{\Longleftrightarrow} \forall X(t \in X \Longrightarrow Q'_1 \dots Q'_k\eta) \overset{w}{\Longleftrightarrow} \forall XQ'_1 \dots Q'_k(t \in X \Longrightarrow \eta)$. Now $\eta \equiv \eta_1 \wedge \dots \wedge \eta_k$, where $k \geq 1$ and η_i are basic Horn L_2-formulas, so $t \in X \Longrightarrow \eta \overset{w}{\Longleftrightarrow} \neg t \in X \vee (\eta_1 \wedge \dots \wedge \eta_k) \overset{w}{\Longleftrightarrow} (\neg t \in X \vee \eta_1) \wedge \dots \wedge (\neg t \in X \vee \eta_k) \overset{\mathrm{def}}{\equiv} \psi$, but this is a Horn L_2-formula again. Finally, $\varphi \overset{c}{\Longleftrightarrow} \forall XQ'_1 \dots Q'_k\psi \overset{\mathrm{def}}{\equiv} \vartheta \in HF_{L_2}$.

The cases when Q_1 is $\exists x$ or $\forall x$ are still more obvious. □

Definition 3.4. An L_2-formula $\varphi(x^1, \dots, x^p, X^1, \dots, X^q)$ is preserved under reduced products of weak structures iff for each family of weak L_2-structures $\{\mathcal{A}_i \mid i \in I\}$, each filter Ψ on I, each $f^1, \dots, f^p \in \prod A_i$ and each $U^1, \dots, U^q \in \prod \mathcal{O}_i$ there holds:

$$\text{if } \left\{i \in I \mid \mathcal{A}_i \vDash \varphi\left[\,\overline{f_i}, \overline{U_i}\,\right]\right\} \in \Psi \quad \text{then } \prod\nolimits_\Psi \mathcal{A}_i \vDash \varphi\left[\overline{[f]}, \overline{q(U)}\right].$$

The set of such formulas will be denoted by $RPF^w_{L_2}$ and the corresponding set of sentences by $RPS^w_{L_2}$.

In a similar manner we define when an L_t-formula $(\varphi(x^1, \dots, x^p, X^1, \dots, X^q))$ is preserved under reduced product of basic structures, in notation $\varphi \in RPF^b_{L_t}$, that is $\varphi \in RPS^b_{L_t}$ if φ is a sentence. Of course, now only the reduced products of families of basic structures are considered.

Now we prove an L_2-version of Keisler's Lemma 6.2.4. from [2].

Lemma 3.5. *Let α be an infinite cardinal, L_2 the defined language, $\{\mathcal{A}_i \mid i \in I\}$ a family of weak L_2-models and $\mathcal{B}$ a saturated weak model of the language L_2 such that the following conditons are satisfied:*

(1) $2^\alpha = \alpha^+$;

(2) $|L_2| \leq |I| = \alpha$;

(3) *for each* $i \in I$, $|A_i \cup \mathcal{O}_i| \leq \alpha^+$;

(4) $\mathcal{B}$ *is either a finite model or a model of cardinality* α^+;

(5) *For any* $\varphi \in HS_{L_2}$ *it holds: if* $|\{i \in I \mid \mathcal{A}_i \nvDash \varphi\}| < \alpha$ *(in other words, if the Horn sentence* φ *is satisfied in almost all models of the given family) then* $\mathcal{B} \vDash \varphi$.

Then there exists a filter Ψ *on* I *such that* $\mathcal{B} \cong \prod_\Psi \mathcal{A}_i$.

Proof. We follow the proof given in [2] (using the same notation as much as possible). Let $A \stackrel{\text{def}}{=} \prod_{i \in I} A_i$ and $\mathcal{O} \stackrel{\text{def}}{=} \prod \mathcal{O}_i$. Clearly, $|A \cup \mathcal{O}| \leq 2^\alpha = \alpha^+$. Firstly we are to define an onto mapping $h : A \cup \mathcal{O} \longrightarrow B \cup \mathcal{O}_\mathcal{B}$, where $h|_A$ maps A onto B and $h|_\mathcal{O}$ maps $\mathcal{O}$ onto $\mathcal{O}_\mathcal{B}$, which satifies the following:

$(*)$ for any $\varphi(x^1, \dots, x^p, X^1, \dots, X^q) \in HF_{L_2}$, for any $a^1, \dots, a^p \in A$ and any $U^1, \dots, U^q \in \mathcal{O}$ it holds: if $|\{i \in I \mid \mathcal{A}_i \nvDash \varphi[a_i^1, \dots, a_i^p, U_i^1, \dots, U_i^q]\}| < \alpha$ then $\mathcal{B} \vDash \varphi[h(a^1), \dots, h(a^p), h(U^1), \dots, h(U^q)]$.

Let $A = \{a^\xi \mid \xi < \alpha^+\}$, $\mathcal{O} = \{U^\xi \mid \xi < \alpha^+\}$, $B = \{b^\xi \mid \xi < \alpha^+\}$ and $\mathcal{O}_\mathcal{B} = \{V^\xi \mid \xi < \alpha^+\}$. We are looking for the new enumerations of these sets, respectively, $\{\underline{a}^\xi \mid \xi < \alpha^+\}$, $\{\underline{U}^\xi \mid \xi < \alpha^+\}$, $\{\underline{b}^\xi \mid \xi < \alpha^+\}$ and $\{\underline{V}^\xi \mid \xi < \alpha^+\}$ such that there holds:

$(**)$ for any $\nu < \alpha^+$ and any L_2-Horn sentence φ of the expanded L_2-language obtained by adding to the initial language the set of new constants (of both sorts) $\{c^\xi \mid \xi < \nu\} \cup \{C^\xi \mid \xi < \nu\}$ (for this occasion simply denoted by L_2^ν):

$$\text{if } |\{i \in I \mid \langle \mathcal{A}_i, \underline{a}_i^\xi, \underline{U}_i^\xi \rangle_{\xi < \nu} \nvDash \varphi\}| < \alpha \quad \text{then } \langle \mathcal{B}, \underline{b}^\xi, \underline{V}^\xi \rangle_{\xi < \nu} \vDash \varphi.$$

Because of (5), for $\nu = 0$ the condition $(**)$ is automaticaly satisfied.

Let us suppose that we have already defined $\underline{a}^\xi$, $\underline{U}^\xi$, $\underline{b}^\xi$ and $\underline{V}^\xi$ for all $\xi < \nu$. Further we distinguish the next cases.

(I) $\nu = \beta + 2k$, where β is either 0 or a limit ordinal.

We put $\underline{a}^\nu = a^{\beta+k}$, $\underline{U}^\nu = U^{\beta+k}$. Let

$$\Sigma(x, X) \stackrel{\text{def}}{=} \{\varphi(x, X) \in HF_{L_2^\nu} \mid \left|\{i \in I \mid \langle \mathcal{A}_i, \underline{a}_i^\xi, \underline{U}_i^\xi \rangle_{\xi<\nu} \nvDash \varphi[\underline{a}_i^\nu, \underline{U}_i^\nu]\}\right| < \alpha\}.$$

This set of formulas is a type over $\langle \mathcal{B}, \underline{b}^\xi, \underline{V}^\xi \rangle_{\xi<\nu}$. For let $\varphi_1, \dots, \varphi_n \in \Sigma(x, X)$ and

$$I_k = \{i \in I \mid \langle \mathcal{A}_i, \underline{a}_i^\xi, \underline{U}_i^\xi \rangle_{\xi<\nu} \nvDash \varphi_k[\underline{a}_i^\nu, \underline{U}_i^\nu]\}, \quad k = 1, \dots, n.$$

Then $|\bigcup_{k=1}^n I_k| < \alpha$ and for $i \notin \bigcup_{k=1}^n I_k$ it holds $\langle \mathcal{A}_i, \underline{a}_i^\xi, \underline{U}_i^\xi \rangle_{\xi<\nu} \vDash \bigwedge_{k=1}^n \varphi_k[\underline{a}_i^\nu, \underline{U}_i^\nu]$, that is $\langle \mathcal{A}_i, \underline{a}_i^\xi, \underline{U}_i^\xi \rangle_{\xi<\nu} \vDash \exists x \exists X \bigwedge_{k=1}^n \varphi_k(x, X)$. By $(**)$ (for ν), $\langle \mathcal{B}, \underline{b}^\xi, \underline{V}^\xi \rangle_{\xi<\nu} \vDash \exists x \exists X \bigwedge_{k=1}^n \varphi_k(x, X)$. Now, since $\mathcal{B}$ is a saturated model, $\langle \mathcal{B}, \underline{b}^\xi, \underline{V}^\xi \rangle_{\xi<\nu}$ realizes the type $\Sigma(x, X)$; let $\langle \mathcal{B}, \underline{b}^\xi, \underline{V}^\xi \rangle_{\xi<\nu} \vDash \Sigma[b, V]$. We define: $\underline{b}^\nu = b$, $\underline{V}^\nu = V$ and check that $(**)$ holds for $\nu + 1$. Let $\varphi \in HS_{L_2^{\nu+1}}$, $|\{i \in I \mid \langle \mathcal{A}_i, \underline{a}_i^\xi, \underline{U}_i^\xi \rangle_{\xi\leq\nu} \nvDash \varphi\}| < \alpha$ and let $\varphi(x, X)$ be the formula obtained from φ by replacing the constants c^ν, C^ν by the suitable variables, respectively, x, X (clearly, if these constants do not appear in the sentence φ, the case is trivial). By the assumption, $|\{i \in I \mid \langle \mathcal{A}_i, \underline{a}_i^\xi, \underline{U}_i^\xi \rangle_{\xi<\nu} \nvDash \varphi(x, X)[\underline{a}_i^\nu, \underline{U}_i^\nu]\}| < \alpha$, whence $\varphi(x, X) \in \Sigma(x, X)$ and, furthermore, $\langle \mathcal{B}, \underline{b}^\xi, \underline{V}^\xi \rangle_{\xi<\nu} \vDash \varphi(x, X)[\underline{b}^\nu, \underline{V}^\nu]$, i.e. $\langle \mathcal{B}, \underline{b}^\xi, \underline{V}^\xi \rangle_{\xi\leq\nu} \vDash \varphi$.

(II) $\nu = \beta + 2k + 1$, where, again, β is either 0 or a limit ordinal.

We put $\underline{b}^\nu = b^{\beta+k}$, $\underline{V}^\nu = V^{\beta+k}$. Let

$$\Sigma(x, X) \stackrel{\text{def}}{=} \{\varphi(x, X) \in HF_{L_2^\nu} \mid \langle \mathcal{B}, \underline{b}^\xi, \underline{V}^\xi \rangle_{\xi<\nu} \vDash \neg\varphi[\underline{b}^\nu, \underline{V}^\nu]\}.$$

For any $\varphi(x, X) \in \Sigma(x, X)$ it holds: the set $I_\varphi \stackrel{\text{def}}{=} \{i \in I \mid \langle \mathcal{A}_i, \underline{a}_i^\xi, \underline{U}_i^\xi \rangle_{\xi<\nu} \nvDash \forall x \forall X \varphi(x, X)\}$ is of cardinality α; otherwise, by $(**)$ it would follow $\langle \mathcal{B}, \underline{b}^\xi, \underline{V}^\xi \rangle_{\xi<\nu} \vDash \forall x \forall X \varphi(x, X)$, a contradiction. By the known result from set theory, the sets I_φ, $\varphi(x, X) \in \Sigma(x, X)$, contain subsets J_φ of cardinality α which are mutually disjoint. Now, for all $i \in I$, we pick $\underline{a}_i^\nu$, $\underline{U}_i^\nu$ in the following way: if $i \in J_\varphi$ we choose elements, $\underline{a}_i^\nu, \underline{U}_i^\nu$, such that $\langle \mathcal{A}_i, \underline{a}_i^\xi, \underline{U}_i^\xi \rangle_{\xi<\nu} \models \neg\varphi[\underline{a}_i^\nu, \underline{U}_i^\nu]$; if $i \notin \bigcup_{\varphi\in\Sigma} J_\varphi$, we choose elements $\underline{a}_i^\nu, \underline{U}_i^\nu$ arbitrarily. So we obtain the "wanted" elements: $\underline{a}^\nu = \langle \underline{a}_i^\nu \mid i \in I \rangle$, $\underline{U}^\nu = \prod_{i\in I} \underline{U}_i^\nu$.

Again the validity of the condition $(**)$ for $\nu + 1$ must be checked. For the sentence $\varphi \in HS_{L_2^{\nu+1}}$ let $|\{i \in I \mid \langle \mathcal{A}_i, \underline{a}_i^\xi, \underline{U}_i^\xi \rangle_{\xi\leq\nu} \not\models \varphi\}| < \alpha$. Suppose $\langle \mathcal{B}, \underline{b}^\xi, \underline{V}^\xi \rangle_{\xi\leq\nu} \models \neg\varphi$. Then the formula $\varphi(x, X)$ (obtained from the sentence φ as above) is in $\Sigma(x, X)$, hence, for all $i \in J_\varphi$, $\langle \mathcal{A}_i, \underline{a}_i^\xi, \underline{U}_i^\xi \rangle_{\xi<\nu} \models \neg\varphi[\underline{a}_i^\nu, \underline{U}_i^\nu]$, that is $\langle \mathcal{A}_i, \underline{a}_i^\xi, \underline{U}_i^\xi \rangle_{\xi\leq\nu} \not\models \varphi$, a contradiction ($|J_\varphi| = \alpha$).

By the very construction of the new enumeration we have: $A = \{\underline{a}^\xi \mid \xi < \alpha^+\}$, $B = \{\underline{b}^\xi \mid \xi < \alpha^+\}$, $\mathcal{O} = \{\underline{U}^\xi \mid \xi < \alpha^+\}$ and $\mathcal{O}_\mathcal{B} = \{\underline{V}^\xi \mid \xi < \alpha^+\}$.

Finally we are able to define h: let, for all $\xi < \alpha^+$, $h(\underline{a}^\xi) = \underline{b}^\xi$ and $h(\underline{U}^\xi) = \underline{V}^\xi$. The mapping h is well-defined; for if, for instance, $\underline{U}^\beta = \underline{U}^\gamma$ and $\beta < \gamma < \delta$ $(< \alpha^+)$, then $\{i \in I \mid \langle \mathcal{A}_i, \underline{a}_i^\xi, \underline{U}_i^\xi \rangle_{\xi<\delta} \models \forall x(x \in C^\beta \iff x \in C^\gamma)\} = I$, thus, by $(**)$, $\langle \mathcal{B}, \underline{b}^\xi, \underline{V}^\xi \rangle_{\xi<\delta} \models \forall x(x \in C^\beta \iff x \in C^\gamma)$ (for it is a Horn sentence in question) and so $h(\underline{U}^\beta) = \underline{V}^\beta = \underline{V}^\gamma = h(\underline{U}^\gamma)$.

The condition $(*)$ also holds. For let $\varphi(x^1, \ldots, x^p, X^1, \ldots, X^q) \in HF_{L_2}$, $\underline{a}^{\xi_1}, \ldots, \underline{a}^{\xi_p} \in A$ and $\underline{U}^{\nu_1}, \ldots, \underline{U}^{\nu_q} \in \mathcal{O}$ and let us suppose that the set $I_0 \stackrel{\text{def}}{=} \{i \in I \mid \mathcal{A}_i \not\models \varphi[\underline{a}_i^{\xi_1}, \ldots, \underline{a}_i^{\xi_p}, \underline{U}_i^{\nu_1}, \ldots, \underline{U}_i^{\nu_q}]\}$ is of cardinality less than α. Then, according to $(**)$, if $\xi_1, \ldots, \xi_p, \nu_1, \ldots, \nu_q < \delta$ $(< \alpha^+)$,

$$\langle \mathcal{B}, \underline{b}^\xi, \underline{V}^\xi \rangle_{\xi<\delta} \models \varphi(c^{\xi_1}, \ldots, c^{\xi_p}, C^{\nu_1}, \ldots, C^{\nu_q}),$$

that is

$$\mathcal{B} \models \varphi[\underline{b}^{\xi_1}, \ldots, \underline{b}^{\xi_p}, \underline{V}^{\nu_1}, \ldots, \underline{V}^{\nu_q}],$$

that is

$$\mathcal{B} \models \varphi[h(\underline{a}^{\xi_1}), \ldots, h(\underline{a}^{\xi_p}), h(\underline{U}^{\nu_1}), \ldots, h(\underline{V}^{\nu_q})].$$

In addition, for any atomic formula $\varphi(x_1, \ldots, x_p, X_1)$ of the language L_2 and any valuation τ in the model $\mathcal{A}$ we define:

$$K_{\varphi,\tau} \stackrel{\text{def}}{=} \{i \in I \mid \mathcal{A}_i \models \varphi[(\tau^1(x_1))_i, \ldots, (\tau^1(x_p))_i, (\tau^2(X_1))_i]\}$$

and, as well:

$$E \stackrel{\text{def}}{=} \{K_{\varphi,\tau} \mid \mathcal{B} \models \varphi[h(\tau^1(x_1)), \ldots, h(\tau^1(x_p)), h(\tau^2(X_1))]\}.$$

From the above it follows that if $K_{\varphi,\tau} \in E$ then $|K_{\varphi,\tau}| = \alpha$ (basic formulas are Horn formulas). In fact, we have more: every finite intersection of elements of E is of cardinality α (thus, in particular, E has the finite intersection property). For any given finite set of atomic formulas $\varphi_1, \dots, \varphi_n$ we can assume, without loss of generality, that they do not have common variables, consequently that just one valuation is in question (let it be τ). The assumption that $|\bigcap_{k=1}^n K_{\varphi_k,\tau}| < \alpha$ would imply that for the corresponding valuation in $\mathcal{B}$ this model satisfies the disjunction of the negations of given atomic formulas (for it is a Horn formula) and if, for example, $\mathcal{B}$ satisfies the formula $\neg\varphi_j$, $1 \leq j \leq n$, it follows $K_{\varphi_j,\tau} \notin E$, a contradiction.

Let Ψ be the (proper) filter generated by E. In the end we claim that one isomorphic mapping of the model $\prod_\Psi \mathcal{A}_i$ onto the model $\mathcal{B}$ is given by: $f([a]) = h(a)$, $f(q(U)) = h(U)$ (clearly, $a \in A$, $[a] = \{b \in A \mid \{i \in I \mid a_i = b_i\} \in \Psi\}$, $U \in \mathcal{O}$ and $q(U) = \{[a] \mid a \in U\}$).

Firstly we show that f is well-defined. Let us suppose that for $a, b \in A$, $[a] = [b]$. Then if τ is the valuation mapping x onto a and y onto b we have $K_{x=y,\tau} \in \Psi$, whence for some finite family of elements from E, let it be $K_{\varphi_1,\tau_1}, \dots, K_{\varphi_n,\tau_n}$, it holds: $\bigcap_{k=1}^n K_{\varphi_k,\tau_k} \subseteq K_{x=y,\tau}$. As in the previous consideration we can assume that the formulas φ_k, $k = 1, \dots, n$, and $x = y$ do not have common variables and that all valuations τ_k, $k = 1, \dots, n$, are equal to τ. Let $\psi \equiv \varphi_1 \wedge \dots \wedge \varphi_n \Longrightarrow x = y$. Obviosly, the formula ψ is satisfied in all models $\mathcal{A}_i$, $i \in I$, for the corresponding valuations determined by τ. If $i \in \bigcap_{k=1}^n K_{\varphi_k,\tau}$ then both the antecedent and consequence of ψ are satisfied, and if $i \notin \bigcap_{k=1}^n K_{\varphi_k,\tau}$, then the antecedent is not satisfied. By $(*)$ and the definition of E both the formula ψ and its antecedent are satisfied in the model $\mathcal{B}$ for the valuation $h \circ \tau$. Thus the consequence is satisfied as well (for the same valuation) which just means that $f([a]) = h(a) = h(b) = f([b])$.

Suppose now that $q(U) = q(V)$, i.e that $\{i \in I \mid U_i = V_i\} \in \Psi$, or, in other words that $K \overset{\text{def}}{=} \{i \in I \mid \mathcal{A}_i \vDash \forall x(x \in X \iff x \in Y)[U_i, V_i]\} \in \Psi$. Again, with the same notation and assumptions as a moment ago, we have $\bigcap_{k=1}^n K_{\varphi_k,\tau} \subseteq K$. The formula $\psi \equiv \varphi_1 \wedge \dots \wedge \varphi_n \Longrightarrow \forall x(x \in X \iff x \in Y)$ is equivalent to the Horn formula $\vartheta \equiv \forall x((\bigvee_{k=1}^n \neg\varphi_k \vee \neg x \in X \vee x \in Y) \wedge (\bigvee_{k=1}^n \neg\varphi_k \vee x \in X \vee \neg x \in Y))$. Again ψ holds in all models $\mathcal{A}_i$, $i \in I$, and again it and its antecedent are satisfied in $\mathcal{B}$ for the valuation $h \circ \tau$. Thus $f(q(U)) = h(U) = h(V) = f(q(V))$.

Obviously, f is a surjection. But f is an injection too. For let $h(a) = f([a]) = f([b]) = h(b)$. Then $\mathcal{B} \vDash (x = y)[h(a), h(b)]$ and, consequently, $K_{x=y,\tau} \in E$ ($\subseteq \Psi$), where, of course, we assume: $\tau(x) = a$, $\tau(y) = b$. Thus $\{i \in I \mid \mathcal{A}_i \vDash (x = y)[a_i, b_i]\} \in \Psi$ and $[a] = [b]$.

Further we prove: for any $U \in \mathcal{O}$, it holds: $f''(q(U)) \stackrel{\text{def}}{=} \{f([a]) \mid [a] \in q(U)\} = f(q(U))$.

($\subseteq$) Let $[a] \in q(U)$. Then $\{i \in I \mid a_i \in U_i\} = \{i \in I \mid \mathcal{A}_i \models (x \in X)[a_i, U_i]\} \in \Psi$. As in the proof of well-definability of f, we obtain $\mathcal{B} \models (x \in X)[h(a), h(U)]$, that is $h(a) = f([a]) \in f(q(U)) = h(U)$.

($\supseteq$) Let $b \in f(q(U)) = h(U)$ and $b = h(a) = f([a])$. Thus, by definition of E, $K_{x\in X,\tau} \in E$ ($\subseteq \Psi$) (surely, $\tau(x) = a$, $\tau(X) = U$) and so $\{i \in I \mid \mathcal{A}_i \models (x \in X)[a_i, U_i]\} \in \Psi$, that is $[a] \in q(U)$, which proves: $b = f([a]) \in f''(q(U))$.

Now if $f(q(U)) = f(q(V))$, i.e. $f''(q(U)) = f''(q(V))$, then
$f^{-1}(f''\ (q(U))) = f^{-1}(f''(q(V)))$ and, since the restriction of f on the set $\prod A_i/\sim$ is a bijection, we have $q(U) = q(V)$.

The homomorphic property of f follows from the analogous result for first order logic, while it is already proved for the relation $\in$. $\square$

Theorem 3.6 (CH). *An L_2-sentence φ is preserved under reduced products of weak structures iff there is a Horn L_2-sentence ϑ such that $\varphi \stackrel{w}{\Longleftrightarrow} \vartheta$.*

Proof. ($\Longleftarrow$) We show that any Horn L_2-formula is preserved under reduced products of weak structures. Practically, there is no difference from the proof of the analogous statement of the first order logic. Let us consider just the case $\varphi \equiv \exists Y \psi(\overline{x}, \overline{X}, Y)$ (naturally, we use induction). Fix $\overline{f}$, $\overline{U}$ and suppose $I_\varphi = \{i \in I \mid \mathcal{A}_i \models \exists X \psi \left[\overline{f_i}, \overline{U_i}\right]\} \in \Psi$. For $i \in I_\varphi$ let $V_i \in \mathcal{O}_i$ be such that $\mathcal{A}_i \models \psi \left[\overline{f_i}, \overline{U_i}, V_i\right]$, otherwise choose V_i arbitrary. Let $V = \prod V_i$. Then $I_\varphi = I_\psi = \{i \in I \mid \mathcal{A}_i \models \psi \left[\overline{f_i}, \overline{U_i}, V_i\right]\}$, thus, by inductive hypothesis $\prod_\Psi \mathcal{A}_i \models \psi \left[\overline{[f]}, \overline{q(U)}, \overline{q(V)}\right]$ and furthermore $\prod_\Psi \mathcal{A}_i \models \exists Y \psi \left[\overline{[f]}, \overline{q(U)}\right]$, i.e. $\prod_\Psi \mathcal{A}_i \models \varphi \left[\overline{[f]}, \overline{q(U)}\right]$.

($\Longrightarrow$) Let φ be an L_2-sentence preserved under reduced products of weak structures. If φ is inconsistent we simply put $\varphi \stackrel{w}{\Longleftrightarrow} \exists x \neg(x = x)$. So let φ be consistent. Without loss of generality we can assume that the language L_2 is countable. Let $\Sigma \stackrel{\text{def}}{=} \{\psi \in HS_{L_2} \mid \models_w \varphi \Longrightarrow \psi\}$. Clearly, Σ is a nonempty set ($\exists x\,(x = x) \in \Sigma$), closed under conjunction. We show $\Sigma \models_w \varphi$ (for then, certainly, we have for some finite subset of Σ, let us say Σ_1, $\Sigma_1 \models \varphi$ and $\bigwedge \Sigma_1$ is the formula we are looking for). Let $\mathcal{M}$ be a weak model of Σ. If it is a finite model ($|M \cup \mathcal{O}_\mathcal{M}| < \omega$), thus saturated, we put

$\mathcal{B} \stackrel{\text{def}}{=} \mathcal{M}$. Otherwise, keeping in mind Löwenheim-Skolem theorem, we can assume that $\mathcal{M}$ is (infinitely) countable. Then, if $\mathcal{B} \stackrel{\text{def}}{=} \prod_\Psi \mathcal{M}$, where Ψ is some nonprincipal ultrafilter over ω, it holds: $\mathcal{B} \equiv_{L_2} \mathcal{M}$ and $\mathcal{B}$ is saturated model of cardinality ω_1. Let us now define $\Delta \stackrel{\text{def}}{=} \{\psi \in HS_{L_2} \mid \varphi \wedge \neg\psi$ has a weak model$\}$. For any $\psi \in \Delta$ we choose a countable weak model $\mathcal{A}_\psi$ of $\varphi \wedge \neg\psi$. Let $I \stackrel{\text{def}}{=} \omega \times \Delta$ and $\mathcal{A}_{(n,\psi)} \stackrel{\text{def}}{=} \mathcal{A}_\psi$. Now the conditions of the previous lemma are satisfied. For, if $\eta \in HS_{L_2}$ and $|\{i \in I \mid \mathcal{A}_i \nvDash \eta\}| < \omega$, then $\eta \in \Sigma$ (since $\eta \in \Delta$ would imply $\omega \times \{\eta\} \subseteq \{i \in I \mid \mathcal{A}_i \nvDash \eta\}$), thus, in particular, $\mathcal{B} \vDash \eta$. By the lemma, there exists a filter Φ on I such that $\mathcal{B} \cong \prod_\Phi \mathcal{A}_i$. But $\varphi \in RPS^w_{L_2}$, whence $\prod_\Phi \mathcal{A}_i \vDash \varphi$, consequently, $\mathcal{B} \vDash \varphi$ and $\mathcal{M} \vDash \varphi$. □

Theorem 3.7. *Each Horn L_t-formula is preserved under reduced products of basic structures ($HF_{L_t} \subseteq RPF^b_{L_t}$).*

Proof. Let $\varphi \in HF_{L_t}$ and let $\{\mathcal{A}_i \mid i \in I\}$, Ψ, $\overline{f}$ and $\overline{U}$ be as in Definition 3.4. By Lemma 3.3, there is $\vartheta \in HF_{L_2}$ such that $\varphi \stackrel{c}{\Longleftrightarrow} \vartheta$, thus also $\varphi \stackrel{b}{\Longleftrightarrow} \vartheta$. Let $J = \{i \in I \mid \mathcal{A}_i \vDash \varphi\left[\overline{f_i}, \overline{U_i}\right]\} \in \Psi$. By the previous theorem, $\prod_\Psi \mathcal{A}_i \vDash \vartheta\left[\overline{[f]}, \overline{q(U)}\right]$. But $\prod_\Psi \mathcal{A}_i$ is a basic structure, so $\prod_\Psi \mathcal{A}_i \vDash \varphi\left[\overline{[f]}, \overline{q(U)}\right]$. □

Lemma 3.8. *There is a sentence $\vartheta_{\text{bas}} \in HS_{L_2}$ such that for each weak L_2-structure $\mathcal{A}$ there holds:*

$$\mathcal{A} \vDash \vartheta_{\text{bas}} \quad \textit{iff } \mathcal{A} \textit{ is a basic structure.}$$

Proof. If $\varphi_{\text{bas}} \equiv \varphi_1 \wedge \varphi_2$, where $\varphi_1 \equiv \forall x \exists X (x \in X)$ and $\varphi_2 \equiv \forall X \forall Y \forall x (x \in X \wedge x \in Y \Longrightarrow \exists Z (x \in Z \wedge \forall z (z \in Z \Longrightarrow z \in X \wedge z \in Y)))$, then for each weak structure $\mathcal{A}$ we have: $\mathcal{A} \vDash \varphi_{\text{bas}}$ iff $\mathcal{A}$ is a basic structure. Obviously, $\varphi_1 \in HS_{L_2}$, while, by Lemma 2.3 and necessary tautologies, $\varphi_2 \stackrel{w}{\Longleftrightarrow} \varphi'_2$ where

$$\begin{aligned} \varphi'_2 \equiv & \forall X \forall Y \forall x \exists Z \forall z \big((\neg x \in X \vee \neg x \in Y \vee x \in Z) \\ & \wedge (\neg x \in X \vee \neg x \in Y \vee \neg z \in Z \vee z \in X) \\ & \wedge (\neg x \in X \vee \neg x \in Y \vee \neg z \in Z \vee z \in Y) \big). \end{aligned}$$

So we can put: $\vartheta_{\text{bas}} \equiv \varphi_1 \wedge \varphi'_2 \in HS_{L_2}$, for, surely, $\vartheta_{\text{bas}} \stackrel{w}{\Longleftrightarrow} \varphi_{\text{bas}}$. □

Lemma 3.9. *Let $\{\mathcal{A}_i \mid i \in I\}$ be a family of weak L_2-structures, Ψ a filter on I and $I_1 \in \Psi$. If $\Psi_1 = \{F \cap I_1 \mid F \in \Psi\}$, then we have:*

(a) for each formula $\varphi(\overline{x}, \overline{X}) \in \mathrm{Form}_{L_2}$, each $\overline{f} \in \prod A_i$ and each $\overline{U} \in \prod \mathcal{O}_i$ it holds

$$\prod\nolimits_{\Psi} \mathcal{A}_i \models \varphi\left[\overline{[f]}, \overline{q(U)}\right] \quad \text{iff} \prod\nolimits_{\Psi_1} \mathcal{A}_i \models \varphi\left[\overline{[f|_{I_1}]}, \overline{q_1(U|_{I_1})}\right],$$

of course, the index set in the second product is I_1;

(b) $\prod_{\Psi} \mathcal{A}_i \equiv_{L_2} \prod_{\Psi_1} \mathcal{A}_i$.

Proof. The proof of (a) is by the usual induction; it is a consequence of the classical result (for first-order parts) and the result concerning the reduced ideal products given in [5]. □

Theorem 3.10 (CH). *An L_t-sentence φ is preserved under reduced products of basic L_2-structures iff there exists a Horn L_2-sentence η satisfying $\varphi \stackrel{b}{\Longleftrightarrow} \eta$.*

Proof. ($\Longrightarrow$) Suppose $\varphi \in RPS^b_{L_t}$. Let us prove

$$\varphi \wedge \vartheta_{\mathrm{bas}} \in RPS^w_{L_2}. \tag{1}$$

Let $\{\mathcal{A}_i \mid i \in I\}$ be a family of weak L_2-structures, Ψ a filter on I and let $J \stackrel{\mathrm{def}}{=} \{i \in I \mid \mathcal{A}_i \models \varphi \wedge \vartheta_{\mathrm{bas}}\} \in \Psi$ and $I_1 \stackrel{\mathrm{def}}{=} \{i \in I \mid \mathcal{A}_i \models \vartheta_{\mathrm{bas}}\}$. Since $J \subseteq I_1$, we have $I_1 \in \Psi$ (and $J = J \cap I_1 = \{i \in I_1 \mid \mathcal{A}_i \models \varphi\} \in \Psi_1$). Now, $\{\mathcal{A}_i \mid i \in I_1\}$ is a family of basic structures and because of $\varphi \in RPS^b_{L_t}$, it follows:

$$\prod\nolimits_{\Psi_1} \mathcal{A}_i \models \varphi. \tag{2}$$

By Lemma 3.8, $\vartheta_{\mathrm{bas}} \in HS_{L_2}$, thus $\vartheta_{\mathrm{bas}} \in RPS^w_{L_2}$ and $\prod_{\Psi_1} \mathcal{A}_i \models \vartheta_{\mathrm{bas}}$. By (2), $\prod_{\Psi_1} \mathcal{A}_i \models \varphi \wedge \vartheta_{\mathrm{bas}}$ and by the previous lemma $\prod_{\Psi} \mathcal{A}_i \models \varphi \wedge \vartheta_{\mathrm{bas}}$ which proves (1). By Theorem 3.6, there is $\eta \in HS_{L_2}$ such that $\varphi \wedge \vartheta_{\mathrm{bas}} \stackrel{w}{\Longleftrightarrow} \eta$, thus also $\varphi \wedge \vartheta_{\mathrm{bas}} \stackrel{b}{\Longleftrightarrow} \eta$. But clearly, $\varphi \wedge \vartheta_{\mathrm{bas}} \stackrel{b}{\Longleftrightarrow} \varphi$ and so $\varphi \stackrel{b}{\Longleftrightarrow} \eta$.

($\Longleftarrow$) Let $\varphi \in \mathrm{Sent}_{L_t}$, $\eta \in HS_{L_2}$, $\varphi \stackrel{b}{\Longleftrightarrow} \eta$ and let $\{\mathcal{A}_i \mid i \in I\}$ be a family of basic structures. If Ψ is a filter on I and $I_\varphi = \{i \in I \mid \mathcal{A}_i \models \varphi\} \in \Psi$, then, because of $\varphi \stackrel{b}{\Longleftrightarrow} \eta$, we have $I_\varphi = I_\eta$ ($= \{i \in I \mid \mathcal{A}_i \models \eta\}$). Again by Theorem 3.6, $\prod_{\Psi} \mathcal{A}_i \models \eta$ and being $\prod_{\Psi} \mathcal{A}_i$ a basic structure too we obtain $\prod_{\Psi} \mathcal{A}_i \models \varphi$. □

Lemma 3.11. *By the invariance of L_t-sentences, for an L_t-sentence φ holds:*

$$\varphi \in RPS^b_{L_t} \quad \textit{iff} \quad \varphi \in RPS^t_{L_t}.$$

Example 3.12. The separation axioms T_0, T_1, T_2 and the *regular* property of topologies are expressed by the formulas, respectively:

$$\begin{aligned}
\varphi_{T_0} &\equiv \forall x \forall y\, (x = y \;\vee\; \exists X \ni x \,\neg y \in X \;\vee\; \exists Y \ni y \,\neg x \in Y);\\
\varphi_{T_1} &\equiv \forall x \forall y\, (x = y \;\vee\; \exists Y \ni y \,\neg x \in Y);\\
\varphi_{T_2} &\equiv \forall x \forall y\, (x = y \;\vee\; \exists X \ni x \,\exists Y \ni y \,\forall z (\neg z \in X \;\vee\; \neg z \in Y));\\
\varphi_{\mathrm{reg}} &\equiv \forall x \forall X \ni x \,\exists Y \ni x \,\forall y\, (y \in X \;\vee\; \exists Z \ni y \,\forall z (\neg z \in Z \;\vee\; \neg z \in Y))\\
&\quad \wedge \varphi_{T_1}.
\end{aligned}$$

By Lemma 2.4 we can find the prenex forms of these formulas:

$$\begin{aligned}
\varphi_{T_0} &\overset{c}{\Longleftrightarrow} \forall x \forall y \exists X \ni x \,\exists Y \ni y\, (x = y \;\vee\; \neg y \in X \;\vee\; \neg x \in Y);\\
\varphi_{T_1} &\overset{c}{\Longleftrightarrow} \forall x \forall y \exists Y \ni y\, (x = y \;\vee\; \neg x \in Y);\\
\varphi_{T_2} &\overset{c}{\Longleftrightarrow} \forall x \forall y \exists X \ni x \,\exists Y \ni y \forall z\, (x = y \;\vee\; \neg z \in X \;\vee\; \neg z \in Y);\\
\varphi_{\mathrm{reg}} &\overset{c}{\Longleftrightarrow} \forall x \forall X \ni x \,\exists Y \ni x \,\forall y \exists Z \ni y \,\forall z\, (y \in X \;\vee\; \neg \in Y \;\vee\; \neg \in Z).
\end{aligned}$$

All sentences on the right side are Horn L_t-sentences, thus preserved under reduced products of topological spaces. It holds as well for the separation axiom T_3 (for $\varphi_{T_3} \equiv \varphi_{T_1} \wedge \varphi_{\mathrm{reg}}$ and a conjunction of Horn formulas is a Horn formula). More general result considering separation axioms and reduced ideal-products can be found in [5]. In connection with it let us note that a Horn L_t-sentence does not have to be preserved under reduced ideal-products, even if the condition $(\Lambda\Psi)$ is satisfied. For instance, the property *discrete* of topologies is expressed by Horn L_t-sentence:

$$\varphi_{\mathrm{disc}} \equiv \forall x \exists X \ni x \,\forall y \;(y = x \;\vee\; \neg u \in X),$$

which, however, is not preserved under Tychonoff products.

Following one part of the proof of Proposition 6.2.6. from [2] we obtain:

Lemma 3.13. *A disjunction of Horn L_t-sentences is preserved under reduced powers of basic structures.*

The above lemma does not hold for reduced products. One simple example gives the reduced product of the family of topological spaces $\{\mathcal{A}_i = \langle A_i, \mathcal{O}_i \rangle \mid i \in \omega\}$, $\prod_\Psi \mathcal{A}_i$, where all spaces have the same "ground" set

$\{0,1\}$, and, for k even, topological space $\mathcal{A}_k$ is discrete, for k odd, indiscrete while the filter Ψ is the Fréchet filter. If $\varphi_{\rm indisc} \equiv \forall x \forall X \ni x \, \forall y \, (y \in X)$, then

$$\{i \in \omega \mid \mathcal{A}_i \vDash \varphi_{\rm disc} \vee \varphi_{\rm indisc}\} = \omega \in \Psi,$$

but, obviously, $\prod_\Psi \mathcal{A}_i$ is neither Hausdorff nor indiscrete.

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