Some interpolatory properties of Tchebicheff polynomials
(0, 2) case modified

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1. Recently P. TURÁN and J. SURÁNYI [1] have studied the case of what they call (0, 2) interpolation when the abscissas

$$(1.1) \qquad -1 \leqq x_n < x_{n-1} < \cdots < x_2 < x_1 \leqq +1$$

are the zeros of

$$(1.2) \qquad \Pi_n(x) = (1-x^2) P'_{n-1}(x) = -n(n-1) \int_{-1}^{x} P_{n-1}(t)\,dt$$

where $P_n(x)$ denotes the Legendre polynomial of degree n with the normalisation

$$(1.3) \qquad P_n(1) = 1.$$

By (0, 2) interpolation we seek to find the polynomials $f(x)$ of degree $\leqq 2n-1$, when the values of $f(x)$ and $f''(x)$ are prescribed at the given abscissas. They showed that for n even, these polynomials exist and are unique, but for n odd they are infinitely many. Their explicit forms have been obtained [2] and it has been shown that these polynomials converge uniformly to the given function under certain conditions.

Later FREUD [5] proved the convergence theorem of BALÁZS—TURÁN under different conditions. SAXENA and SHARMA [3] have extended the results of TURÁN to (0, 1, 3) interpolation and SAXENA has further extended them to (0, 1, 2, 4) case. The results of O. KIS [4] deserve mention because he treats the case when the abscissas are the n nth roots of unity.

The object of this note is to treat the case of (0, 2) interpolation when the abscissas are the zeros of the Tchebicheff polynomial $T_n(x)$. It turns out that the explicit forms in this case are not very elegant. We therefore follow the lucky idea of the first author to modify the problem as the explicit forms then come out in a very handy form. In § 9, we also give the corresponding results for the case when we take the Tchebicheff polynomials of the second kind. The proofs are naturally omitted.

2. As usual we denote by

(2. 1) $$T_n(x)=\cos n\theta; \quad \cos\theta=x,$$

the Tchebicheff polynomials of the first kind, and by

(2. 1a) $$U_n(x)=\frac{\sin(n+1)\theta}{\sin\theta}, \quad \cos\theta=x,$$

the Tchebicheff polynomials of the second kind.

Let us consider the set of numbers

(2. 2) $$-1<x_{n+1}<x_n<\cdots<x_2<1$$

by which we shall denote the zeros of $T_n(x)$ or $U_n(x)$ as the case may be. Let

(2. 3) $$R_{n+2}(x)=(1-x^2)T_n(x),$$

where $T_n(x)$ is the Tchebicheff polynomial of the first kind. Let us consider the set of numbers

(2. 4) $$-1=x_{n+2}<x_{n+1}<\cdots<x_2<x_1=1$$

by which we shall denote the zeros of $R_{n+2}(x)$. We shall prove the following theorems:

Theorem 1. *If* $n=2k$ *then to prescribed values* $y_{\nu 0}, y_{\nu 1}$ *there is a uniquely determined polynomial* $f(x)$ *of degree* $\leqq 2n+1$ *such that*

(2. 5) $$f(x_\nu)=y_{\nu 0}, \quad 1\leqq\nu\leqq n+2$$
$$f''(x_\nu)=y_{\nu 1}, \quad 2\leqq\nu\leqq n+1$$

where x_ν*'s are given by* (2. 4).

Theorem 2. *If* $n=2k+1$ *and the points* $x_1, x_2, \ldots, x_{n+2}$ *satisfy* (2. 4), *there is, in general no polynomial* $f(x)$ *of degree* $\leqq 2n+1$ *such that for given* $y_{\nu 0}, y_{\nu 1}$

(2. 6) $$f(x_\nu)=y_{\nu 0}, \qquad \nu=1,2,\ldots,n+2$$
$$f''(x_\nu)=y_{\nu 1}, \qquad \nu=2,3,\ldots,n+1$$

If there exists such a polynomial, then there is an infinity of them.

PROOF. First we shall prove Theorem 1. The first part of the condition requires

(2. 7) $$f(x)=R_{n+2}(x)q_{n-1}(x)$$

where $q_{n-1}(x)$ is a polynomial of degree $\leqq n-1$. But the second part of the condition requires

(2. 8) $$R''_{n+2}(x_\nu)q_{n-1}(x_\nu)-2R'_{n+2}(x_\nu)q'_{n-1}(x_\nu)=0, \quad 2\leqq\nu\leqq n+1.$$

By simple computation

(2. 9) $$R'_{n+2}(x_\nu)=(1-x_\nu^2)T'_n(x_\nu), \;\; R''_{n+2}(x_\nu)=-3x_\nu T'_n(x_\nu),$$

which leads to

$$-3x_\nu q_{n-1}(x_\nu)+2(1-x_\nu^2)q'_{n-1}(x_\nu)=0.$$

But this means that

$$2(1-x^2)q'_{n-1}(x)-3xq_{n-1}(x)=c\,T_n(x) \tag{2.10}$$

with a numerical c. If $c\neq 0$ we can express $q_{n-1}(x)$ as

$$q_{n-1}(x)=\sum_{\nu=0}^{n-1} c_\nu \cos \nu\theta. \tag{2.11}$$

Therefore

$$q'_{n-1}(x)=\sum_{\nu=0}^{n-1} \nu c_\nu \frac{\sin \nu\theta}{\sin\theta}$$

so (2.10) becomes

$$2\sin\theta \sum_{\nu=0}^{n-1} \nu c_\nu \sin \nu\theta - 3\cos\theta \sum_{\nu=0}^{n-1} c_\nu \cos\nu\theta = c\cos n\theta.$$

In other words

$$\begin{aligned}&\sum_{\nu=0}^{n-1} \nu c_\nu\{\cos(\nu-1)\theta-\cos(\nu+1)\theta\}-\\ &-\frac{3}{2}\sum_{\nu=0}^{n-1} c_\nu\{\cos(\nu-1)\theta+\cos(\nu+1)\theta\}=c\cos n\theta.\end{aligned} \tag{2.12}$$

Equating coefficients of $\cos n\theta$, $\cos(n-1)\theta, \ldots, \cos\theta$ and constant term we find at once

$$\begin{cases} -\left(n+\frac{1}{2}\right)c_{n-1}=c; \quad -\left(n-\frac{1}{2}\right)c_{n-2}=0; \\ \quad \left(n-\frac{3}{2}\right)c_{n-3}-\left(n-\frac{5}{2}\right)c_{n-1}=0 \\ \cdots \qquad \cdots \\ 3c_3-5c_1=0; \quad c_2-6c_0=0 \\ \text{and} \\ -\frac{1}{2}c_1=0. \end{cases} \tag{2.13}$$

Combining all these equations we have (if n is even)

$$c_0=c_2=c_4=\cdots=c_{n-4}=c_{n-2}=0 \tag{2.14}$$

and

$$c_1=c_3=c_5=\cdots=c_{n-3}=c_{n-1}=c=0. \tag{2.15}$$

This shows that when n is even the only solution is $f(x)\equiv 0$. Hence writing out (2.5) as a linear system, the linear system is always uniquely solvable. Hence Theorem 1 is proved.

For n odd, by a similar argument we have

$$c_1 = c_3 = \cdots = c_{n-1} = c = 0 \qquad \text{(if } n \text{ is odd)}$$

but $c_0, c_2, c_4, \ldots, c_{n-4}, c_{n-2}$ all cannot be determined. This shows clearly that there is in general no polynomial $f(x)$ of degree $\leqq 2n+1$ which satisfies (2.6) and if there exists such a polynomial, then there is an infinity of them.

The other part of the paper will be devoted to obtaining the explicit forms of the polynomials when they are uniquely determined.

Explicit determination of the interpolatory polynomials

3. We now consider the following problem:

Let be given $(n=2k)$ distinct points $x_1, x_2, \ldots, x_{n+2}$, the zeros of $R_{n+2}(x)$, with

$$-1 = x_{n+2} < x_{n+1} < \cdots < x_2 < x_1 = 1, \tag{3.1}$$

and arbitrary numbers

$$a_1, a_2, \ldots, a_{n+2}$$
$$b_1, b_2, \ldots, b.$$

We want to find the explicit form of the polynomials $S_n(x)$ of degree $\leqq 2n+1$ such that

$$\begin{aligned} S_n(x_\nu) &= a_\nu, \qquad \nu = 1, 2, \ldots, n+2 \\ S_n''(x_\nu) &= b_\nu, \qquad \nu = 2, 3, \ldots, n+1. \end{aligned} \tag{3.2}$$

The existence and uniqueness has already been proved in Theorem 1. For $S_n(x)$ we evidently have the form

$$S_{2k}(x) = \sum_{\nu=1}^{2k+2} a_\nu r_\nu(x) + \sum_{\nu=2}^{2k+1} b_\nu \varrho_\nu(x) \tag{3.3}$$

where the polynomials $r_\nu(x)$ and $\varrho_\nu(x)$, the fundamental polynomials of the first and second kind for this interpolation, belonging to the points x_ν, respectively, are polynomials of degree $\leqq 2n+1 = 4k+1$ uniquely determined by the following conditions:

$$r_\nu(x_j) = \begin{matrix} 0 \\ 1 \end{matrix} \Big\} \text{ for } \begin{matrix} j \neq \nu \\ j = \nu \end{matrix} \qquad \nu = 1, 2, \ldots, n+2 \tag{3.4}$$

$$r_\nu''(x_j) = 0 \qquad j = 2, 3, \ldots, n+1$$

$$\varrho_\nu(x_j) = 0 \qquad j = 1, 2, \ldots, n+2$$

$$\varrho_\nu''(x_j) = \begin{matrix} 0 \\ 1 \end{matrix} \Big\} \begin{matrix} j \neq \nu \\ j = \nu \end{matrix} \qquad j = 2, 3, \ldots, n+1 \tag{3.5}$$

respectively. In what follows we shall explicitly determine these fundamental polynomials $r_\nu(x)$, $\varrho_\nu(x)$.

4. We shall denote by $l_\nu(x)$ the fundamental polynomials of the Lagrange interpolation. Here all the formulae (4. 1) to (4. 7) hold for $2 \leqq \nu \leqq n+1$.

$$l_\nu(x_j) = \begin{cases} 0 & j \neq \nu \\ 1 & j = \nu \end{cases} \tag{4. 1}$$

Also

$$l'_\nu(x_\nu) = \frac{x_\nu}{2(1-x_\nu^2)} \tag{4. 2}$$

$$l'_\nu(x_j) = \frac{T'_n(x_j)}{(x_j - x_\nu)\, T'_n(x_\nu)}, \qquad j \neq \nu$$

$$l''_\nu(x_\nu) = \frac{1}{3(1-x_\nu^2)} \left\{ \frac{3x_\nu^2}{1-x_\nu^2} - n^2 + 1 \right\}. \tag{4. 3}$$

From a formula of FEJÉR (see SZEGŐ [6])

$$l_\nu(x) = \frac{T_n(x)}{(x-x_\nu)\, T'_n(x_\nu)} = \frac{1}{n} + \frac{2}{n} \sum_{r=1}^{n-1} T_r(x)\, T_r(x_\nu). \tag{4. 4}$$

We know that the differential equation satisfied by $T_n(x)$ is given by

$$(1-x^2)y'' - xy' + n^2 y = 0, \quad y = T_n(x) \tag{4. 5}$$

$$T''_n(x_j) = \frac{x_j\, T'_n(x_j)}{(1-x_j^2)} \tag{4. 6}$$

$$T'''_n(x_\nu) = 3\, T'_n(x_\nu)\, l''_\nu(x_\nu). \tag{4. 7}$$

5. Theorem 3. *For the fundamental polynomials $r_\nu(x)$ and $\varrho_\nu(x)$ the following explicit forms hold:*

(a)

$$\varrho_\nu(x) = \frac{(1-x^2)^{1/4}\, T_n(x)}{2\, T'_n(x_\nu)} \left\{ A \int_{-1}^{x} \frac{T_n(t)}{(1-t^2)^{1/4}}\, dt + \int_{-1}^{x} \frac{l_\nu(t)}{(1-t^2)^{1/4}}\, dt \right\} \tag{5. 1}$$

where

$$A \int_{-1}^{1} \frac{T_n(t)}{(1-t^2)^{1/4}}\, dt = -\int_{-1}^{1} \frac{l_\nu(t)}{(1-t^2)^{1/4}}\, dt \tag{5. 2}$$

(b)

for $2 \leqq \nu \leqq n+1$

$$r_\nu(x) = \frac{1-x^2}{1-x_\nu^2} \Bigg[l_\nu^2(x) + \frac{T_n(x)}{T'_n(x_\nu)} (1-x^2)^{-3/4} \Big\{ A' \int_{-1}^{x} \frac{T_n(t)}{(1-t^2)^{1/4}}\, dt \tag{5. 3}$$

$$B' \int_{-1}^{x} \frac{l_\nu(t)}{(1-t^2)^{1/4}}\, dt + \int_{-1}^{x} \frac{x_\nu l_\nu(t) - 2(1-t^2)\, l'_\nu(t)}{2(t-x_\nu)(1-t^2)^{1/4}}\, dt$$

where

$$(5.4)\qquad -A'\int_{-1}^{1}\frac{T_n(t)}{(1-t^2)^{1/4}}dt=B'\int_{-1}^{x}\frac{l_\nu(t)}{(1-t^2)^{1/4}}dt+\int_{-1}^{x}\frac{x_\nu l_\nu(t)-2(1-t^2)l'_\nu(t)}{(t-x_\nu)(1-t^2)^{1/4}}dt$$

$$(5.5)\qquad B'=\frac{2-x_\nu^2}{2(1-x_\nu^2)}$$

$$(5.6)\qquad r_1(x)=\frac{1+x}{2}T_n^2(x)+(1-x^2)^{1/4}T_n(x)\int_{-1}^{x}\frac{a_1T_n(t)-(1+t)T'_n(t)}{2(1-t^2)^{1/4}}dt$$

where

$$(5.7)\qquad a_1\int_{-1}^{1}\frac{T_n(t)}{(1-t^2)^{1/4}}dt=-\int_{-1}^{1}\frac{tT'_n(t)}{(1-t^2)^{1/4}}dt;\ a_{n+2}=a_1$$

and

$$(5.8)\quad r_{n+2}(x)=\frac{1-x}{2}T_n^2(x)+R_{n+2}(x)(1-x^2)^{-3/4}\int_{-1}^{x}\frac{a_{n+2}T_n(t)-(1-t)T'_n(t)}{2(1-t^2)^{1/4}}dt.$$

6. In order to prove the above theorem we shall require the following lemmas:

Lemma 1.

$$(6.1)\qquad \int_{-1}^{x}\frac{T_{2r}(t)}{(1-t^2)^{1/4}}dt=-\frac{\Gamma\left(r-\frac{1}{4}\right)}{\Gamma\left(r+\frac{5}{4}\right)}\left[\frac{1}{4}\frac{\Gamma\left(\frac{5}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}\int_{-1}^{x}\frac{1}{(1-t^2)^{1/4}}dt+\right.$$
$$\left.+(1-x^2)^{3/4}\sum_{k=0}^{r-1}\frac{\Gamma\left(k+\frac{5}{4}\right)}{\Gamma\left(k+\frac{3}{4}\right)}T_{2k+1}(x)\right].$$

PROOF. The lemma follows from the recurrence relation

$$(6.2)\qquad \int_{-1}^{x}\frac{T_{2r}(t)}{(1-t^2)^{1/4}}dt=\frac{4r-5}{4r+1}\int_{-1}^{x}\frac{T_{2r-2}(t)}{(1-t^2)^{1/4}}dt-\frac{4}{4r+1}(1-x^2)^{3/4}T_{2r-1}(x)$$

where $\cos\theta=x$, which is easy to verify.

Lemma 2.

$$\int_{-1}^{x} \frac{T_{2r-1}(t)}{(1-t^2)^{1/4}}\,dt = -\frac{\Gamma\left(r-\frac{3}{4}\right)}{\Gamma\left(r+\frac{3}{4}\right)}(1-x^2)^{3/4}\cdot \tag{6.3}$$

$$\cdot\left[\frac{1}{2}\frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)}+\sum_{k=1}^{r}\frac{\Gamma\left(k+\frac{3}{4}\right)}{\Gamma\left(k+\frac{1}{4}\right)}T_{2k}(x)\right].$$

Proof. The Lemma follows from the recurrence relation

$$\int_{-1}^{x} \frac{T_{2r-1}(t)}{(1-t^2)^{1/4}}\,dt = \frac{4r-7}{4r-1}\int_{-1}^{x}\frac{T_{2r-3}(t)}{(1-t^2)^{1/4}}\,dt - \frac{4}{4r-1}(1-x^2)^{3/4}\,T_{2r-2}(x) \tag{6.4}$$

which is again easy to verify.

Lemma 3.

$$\int_{-1}^{x}\frac{l_\nu(t)}{(1-t^2)^{1/4}}\,dt = k\int_{-1}^{x}\frac{1}{(1-t^2)^{1/4}}\,dt - \frac{2}{n}(1-x^2)^{3/4}\,\alpha_{n-1}(x) \tag{6.5}$$

where

$$k\int_{-1}^{1}\frac{1}{(1-t^2)^{1/4}}\,dt = \int_{-1}^{+1}\frac{l_\nu(t)}{(1-t^2)^{1/4}}\,dt \tag{6.6}$$

and $\alpha_{n-1}(x)$ is a polynomial in x of degree $\leq n-1$.

Proof. We have from (4. 4)

$$\int_{-1}^{x}\frac{l_\nu(t)}{(1-t)^{1/4}}\,dt = \frac{1}{n}\int_{-1}^{x}\frac{1}{(1-t^2)^{1/4}}\,dt + \frac{2}{n}\sum_{r=1}^{\frac{n}{2}-1}T_{2r}(x_\nu)\int_{-1}^{x}\frac{T_{2r}(t)}{(1-t)^{1/4}}\,dt +$$

$$+\frac{2}{n}\sum_{r=1}^{\frac{n}{2}}T_{2r-1}(x_\nu)\int_{-1}^{x}\frac{T_{2r-1}(t)}{(1-t^2)^{1/4}}\,dt.$$

Now applying Lemma 1 and 2 we get the result as stated.

Lemma 4. We have

$$\frac{1}{2}\int_{-1}^{x}\frac{x_\nu l_\nu(t)-2(1-t^2)l'_\nu(t)}{(t-x_\nu)(1-t^2)^{1/4}\,T'_n(x_\nu)}\,dt = K'\int_{-1}^{x}\frac{1}{(1-t^2)^{1/4}}\,dt + (1-x^2)^{3/4}\beta_{n-1}(x) \tag{6.7}$$

where

$$\frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{5}{4}\right)}K' = \frac{1}{2\sqrt{\pi}} \int_{-1}^{1} \frac{x_\nu l_\nu(t) - 2(1-t^2) l'_\nu(t)}{(t-x_\nu)(1-t^2)^{1/4} T'_n(x_\nu)}\, dt \tag{6.8}$$

and $\beta_{n-1}(x)$ is a polynomial in x of degree $\leqq n-1$.

PROOF. It is easy to see that

$$\frac{x_\nu l_\nu(t) - 2(1-t^2) l'_\nu(t)}{2(t-x_\nu)(1-t^2)^{1/4}} = \frac{1}{2\,T'_n(x_\nu)}\left[\frac{x_\nu T_n(t)}{(t-x_\nu)^2(1-t^2)^{1/4}} - \right.$$

$$\left. -2(1-t^2)^{3/4}\left\{\frac{T'_n(t)}{(t-x_\nu)^2} - \frac{T_n(t)}{(t-x_\nu)^3}\right\}\right].$$

From (4. 4) we have on differentiating with respect to t both sides twice:

$$\frac{T'_n(t)}{t-x_\nu} - \frac{T_n(t)}{(t-x_\nu)^2} = \frac{2}{n}\, T'_n(x_\nu) \sum_{r=1}^{n-1} T_r(x_\nu)\, T'_r(t) \tag{6.9}$$

$$\frac{T''_n(t)}{t-x_\nu} - 2\cdot\frac{T'_n(t)}{(t-x_\nu)^2} + 2\cdot\frac{T_n(t)}{(t-x_\nu)^3} = \frac{2}{n}\, T'_n(x_\nu) \sum_{r=1}^{n-1} T_r(x_\nu)\, T''_r(t). \tag{6.10}$$

Multiplying (4. 4), (6. 9), (6. 10) by n^2, $-t$, $(1-t^2)$ respectively and using (4. 5) we have

$$(1-t^2)\frac{T_n(t)}{(t-x_\nu)^3} + \frac{x_\nu}{2}\,\frac{T_n(t)}{(t-x_\nu)^2} - \frac{(1-t^2)\, T'_n(t)}{(t-x_\nu)^2} =$$

$$= + \frac{n^2-1}{2n} + \sum_{r=1}^{n-1} \frac{(n^2-r^2-1)}{n}\, T_r(x_\nu)\, T_r(t).$$

Therefore

$$\int_{-1}^{x} \frac{x_\nu l_\nu(t) - 2(1-t^2) l'_\nu(t)}{2(t-x_\nu)(1-t^2)^{1/4}}\, dt = \frac{n^2-1}{2n}\int_{-1}^{x} \frac{1}{(1-t^2)^{1/4}}\, dt +$$

$$+ \frac{1}{n}\sum_{r=1}^{n-1}(n^2-r^2-1)\, T_r(x_\nu) \int_{-1}^{x} \frac{T_r(t)}{(1-t^2)^{1/4}}\, dt.$$

Now breaking the series into odd and even parts and using Lemma 1 and 2, we get the stated result.

In order to determine $r_1(x)$ and $r_{n+2}(x)$, we shall require the following lemma

Lemma 5.

$$\int_{-1}^{x} \frac{T'_{2r}(t)}{(1-t^2)^{1/4}}\,dt = -4r(1-x^2)^{3/4} \sum_{i=1}^{r} \frac{\Gamma\left(i-\frac{3}{4}\right)}{\Gamma\left(i+\frac{3}{4}\right)} \cdot \tag{6.11}$$

$$\cdot \left\{\frac{1}{2}\frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)} + \sum_{k=1}^{i-1} \frac{\Gamma\left(k+\frac{3}{4}\right)}{\Gamma\left(k+\frac{1}{4}\right)} T_{2k}(x)\right\}$$

and

$$\int_{-1}^{x} \frac{T'_{2r-1}(t)}{(1-t^2)^{1/4}}\,dt = (2r-1)\left[\int_{-1}^{x} \frac{1}{(1-t^2)^{1/4}}\,dt - 2\sum_{i=1}^{r-1} \frac{\Gamma\left(i-\frac{1}{4}\right)}{\Gamma\left(i+\frac{5}{4}\right)} \cdot \right. \tag{6.12}$$

$$\left. \cdot \left\{\frac{1}{4}\frac{\Gamma\left(\frac{5}{4}\right)}{\Gamma\left(\frac{3}{4}\right)} \int_{-1}^{x} \frac{1}{(1-t^2)^{1/4}}\,dt + (1-x^2)^{3/4} \sum_{k=0}^{i-1} \frac{\Gamma\left(k+\frac{5}{4}\right)}{\Gamma\left(k+\frac{3}{4}\right)} T_{2k+1}(x)\right\}\right].$$

PROOF. We know that

$$T'_{2r}(t) = 2r\frac{\sin 2r\theta}{\sin\theta} = 4r\sum_{i=1}^{r} T_{2i-1}(t)$$

and

$$T'_{2r-1}(t) = (2r-1)\left[1 + 2\sum_{i=1}^{r-1} T_{2i}(t)\right].$$

Dividing both sides by $(1-t^2)^{1/4}$ and integrating and again applying Lemma 1 and 2 we have the required result.

Lemma 6.

$$\int_{-1}^{x} \frac{a\,T_n(t)-(1+t)\,T'_n(t)}{2(1-t^2)^{1/4}}\,dt = M\int_{-1}^{x} \frac{1}{(1-t^2)^{1/4}}\,dt + (1-x^2)^{3/4}\gamma_{n-1}(x) \tag{6.13}$$

where

$$M\frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{5}{4}\right)} = \int_{-1}^{1} \frac{a_1\,T_n(t)-(1+t)\,T'_n(t)}{2(1-t^2)^{1/4}}\,dt$$

and $\gamma_{n-1}(x)$ is a polynomial in x of degree $\leqq n-1$.

PROOF. As the recurrence relation for $T_n(x)$ is

$$T_{n+1}(x) - T_{n-1}(x) = 2x\,T_n(x),$$

we have

$$T'_{n+1}(x) + T'_{n-1}(x) - 2\,T_n(x) = 2x\,T'_n(x).$$

Now by the help of the above relation and Lemma 1, 2 and 5 we get the stated result.

7. PROOF OF THEOREM 3. Consider the function

$$\psi_\nu(x) = \frac{R_{n+2}(x)\,q_{n-1}(x)}{2\,T'_n(x)} \tag{7.1}$$

where

$$q_{n-1}(x) = (1-x^2)^{-3/4} \cdot A \int_{-1}^{x} \frac{T_n(t)}{(1-t^2)^{1/4}}\,dt + \int_{-1}^{x} \frac{l_\nu(t)}{(1-t^2)^{1/4}}\,dt. \tag{7.2}$$

Then obviously

$$\psi_\nu(x_j) = 0, \qquad j = 1, 2, \ldots, n+2.$$

$$2\,T'_n(x_\nu)\,\psi''_\nu(x_j) = T'_n(x_j)[-3x_j q_{n-1}(x_j) + 2(1-x_j^2)q'_{n-1}(x_j)]. \tag{7.3}$$

But from (7. 2) we have

$$2(1-x_j^2)q'_{n-1}(x_j) - 3x_j q_{n-1}(x_j) = 0 \text{ for } j \neq \nu.$$

Therefore

$$\psi''_\nu(x_j) = 0 \qquad j \neq \nu,\ j = 2, 3, \ldots, n-1. \tag{7.4}$$

Again

$$\psi''_\nu(x_\nu) = -\frac{3}{2}\,x_\nu q_{n-1}(x_\nu) + (1-x_\nu^2)q'_{n-1}(x_\nu) = 1, \qquad \nu = 2, 3, \ldots, n+1.$$

Therefore all the conditions of (3. 5) are satisfied. In order to make it a polynomial we equate to zero the coefficient of

$$\int_{-1}^{x} \frac{1}{(1-t^2)^{1/4}}\,dt$$

which determines A as required.

Therefore by Lemma 1, 2 and 3 it is now easy to see that $\psi_\nu(x)$ is a polynomial of degree $\leqq 2n-1$ satisfying all the conditions of [3. 5]. So by the uniqueness theorem

$$\psi_\nu(x) \equiv \varrho_\nu(x). \tag{7.5}$$

8. *Determination of* $r_\nu(x)$. For $2 \leqq \nu \leqq n+1$ consider the function

$$\mu_\nu(x) = (1-x^2)\left[l_\nu^2(x) + \frac{T_n(x) q_{n-1}(x)}{T_n'(x_\nu)} \right] \tag{8.1}$$

where

$$q_{n-1}(x) = (1-x^2)^{-3/4} \left\{ A' \int_{-1}^{x} \frac{T_n(t)}{(1-t^2)^{1/4}} dt + \frac{B'}{2} \int_{-1}^{x} \frac{l_\nu(t)}{(1-t^2)^{1/4}} dt + \right.$$
$$\left. + \int_{-1}^{x} \frac{x_\nu l_\nu(t) - 2(1-t^2) l_\nu'(t)}{2(t-x_\nu)(1-t^2)^{1/4}} dt. \right. \tag{8.2}$$

Obviously

$$\mu_\nu(x_j) = \begin{matrix} 1 \\ 0 \end{matrix} \begin{cases} j = \nu \\ j \neq \nu \end{cases} \qquad \begin{matrix} j = 1, 2, \ldots, n+2 \\ \nu = 2, 3, \ldots, n+1 \end{matrix} \tag{8.3}$$

is satisfied. Also for $j \neq \nu$, $j, \nu = 2, 3, \ldots, n+1$,

$$\mu_\nu''(x_j) = 2(1-x_j^2) l_\nu'(x_j) + \frac{T_n'(x_j)}{T_n'(x_\nu)} [2(1-x_j^2) q_{n-1}'(x_j) - 3 x_j q_{n-1}(x_j)].$$

But from (8. 2)

$$2(1-x_j^2) q_{n-1}'(x_j) - 3 x_j q_{n-1}(x_j) = -\frac{2(1-x_j^2) l_\nu'(x_j)}{(x_j - x_\nu)}.$$

Therefore

$$\mu_\nu''(x_j) = 0 \qquad j \neq \nu.$$

Similarly

$$\mu_\nu''(x_\nu) = 0 \qquad \nu = 2, 3, \ldots, n+1.$$

For $2 \leqq \nu \leqq n+1$, $\mu_\nu(x)$ satisfies all the conditions of (3. 4). As before in order to make it a polynomial of degree $\leqq 2n+1$ we equate to zero the coefficients of $\int_{-1}^{x} \frac{1}{(1-t^2)^{1/4}} dt$ which determines A'.

Now by Lemma 1, 2, 3 and 4 it is easy to see that $\mu_\nu(x)$ is a polynomial of degree $\leqq 2n+1$ satisfying all the conditions (3. 4), therefore by the uniqueness theorem

$$\mu_\nu(x) \equiv r_\nu(x) \qquad 2 \leqq \nu \leqq n+1.$$

Now we will prove the last part of Theorem 3, namely the determination of $r_1(x)$ and $r_{n+2}(x)$. For this consider the function

$$\mu_1(x) = \frac{1+x}{2} T_n^2(x) + R_{n+2}(x) q_{n-1}(x) \tag{8.4}$$

where

$$(8.5)\qquad (1-x^2)^{3/4} q_{n-1}(x) = \int_{-1}^{x} \frac{a_1 T_n(t) - (1+t) T_n'(t)}{2(1-t^2)^{1/4}}\, dt.$$

Obviously

$$\mu_1(x_j) = \begin{matrix} 1 \\ 0 \end{matrix} \left\{ \begin{matrix} j = 1 \\ j \neq 1 \end{matrix} \right. , \qquad j = 1, 2, \ldots, n+2.$$

$$(8.6)\quad \mu_1''(x_j) = (1+x_j) T_n'(x_j) + 2 T_n'(x_j) \{ -3 x_j q_{n-1}(x_j) + 2(1-x_j^2) q_{n-1}'(x_j) \}.$$

But from (8.5) it is easy to see that

$$2(1-x_j^2) q_{n-1}'(x_j) - 3 x_j q_{n-1}(x_j) = -(1+x_j) T_n'(x_j),$$

hence

$$\mu_1''(x_j) = 0, \qquad j = 2, 3, \ldots, n+1.$$

Therefore $\mu_1(x)$ satisfies all the conditions of (3.4); in order to make it a polynomial, we equate to zero the coefficient of

$$\int_{-1}^{x} \frac{1}{(1-t^2)^{1/4}}\, dt,$$

which determines a_1 as stated. Hence by the uniqueness theorem

$$\mu_1(x) \equiv r_1(x).$$

Similarly we can determine $r_{n+2}(x)$. Theorem 3 is then completely proved.

9. Here we shall consider the case of (0, 2) modified interpolation with regard to Tchebicheff abscissas of the second kind, namely the zeros of $U_n(x)$, where

$$(9.1)\qquad U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta}, \qquad \cos\theta = x.$$

Let

$$(9.2)\qquad S_{n+2}(x) = (1-x^2) U_n(x),$$

and let us consider the set of numbers

$$(9.3)\qquad -1 = x_{n+2} < x_{n+1} < \cdots < x_1 = 1,$$

by which we shall denote the zeros of $S_{n+2}(x)$. We shall state without proof the following theorems as their proofs can be given on the same lines as in the previous case.

Theorem 4. *If $n=2k$, then to prescribed values $y_{\nu 0}$, $y_{\nu 1}$, there is a uniquely determined polynomial $f(x)$ of degree $\leq 2n+1$ such that*

$$f(x_\nu)=y_{\nu 0}, \qquad \nu=1,2,3,\ldots,n+2$$
$$f''(x_\nu)=y_{\nu 1}, \qquad \nu=2,3,4,\ldots,n+1$$

where x_ν's are given by (9.3).

For $n=2k+1$ a result similar to theorem 2 can be stated.

For the fundamental polynomials $r_\nu(x)$ and $\varrho_\nu(x)$ the following explicit forms hold.

Theorem 5. *As before we have the following representation:*

$$\varrho_\nu(x)=\frac{(1-x^2)^{3/4}U_n(x)}{2U_n'(x_\nu)}\left[a\int_{-1}^{x}\frac{U_n(t)}{(1-t^2)^{3/4}}\,dt+\int_{-1}^{x}\frac{l_\nu(t)}{(1-t^2)^{3/4}}\,dt,\right.$$

where

$$a\int_{-1}^{1}\frac{U_n(t)}{(1-t^2)^{3/4}}\,dt=-\int_{-1}^{1}\frac{l_\nu(t)}{(1-t^2)^{3/4}}\,dt,$$

and for $2\leq\nu\leq n+1$

$$r_\nu(x)=\frac{1-x^2}{1-x_\nu^2}\left[l_\nu^2(x)+\frac{(1-x^2)^{-1/4}U_n(x)}{2U_n'(x_\nu)}\left\{a_\nu'\int_{-1}^{x}\frac{U_n(t)}{(1-t^2)^{3/4}}\,dt+b_\nu'\int_{-1}^{x}\frac{l_\nu(t)}{(1-t^2)^{3/4}}\,dt+\right.\right.$$

$$\left.\left.+\int_{-1}^{x}\frac{3x_\nu l_\nu(t)-2(1-t^2)l_\nu'(t)}{(t-x_\nu)(1-t^2)^{3/4}}\,dt\right\}\right],$$

where

$$a_\nu'\int_{-1}^{1}\frac{U_n(t)}{(1-t^2)^{3/4}}\,dt=b_\nu'\int_{-1}^{1}\frac{l_\nu(t)}{(1-t^2)^{3/4}}\,dt+\int_{-1}^{1}\frac{3x_\nu l_\nu(t)-2(1-t^2)l_\nu'(t)}{(t-x_\nu)(1-t^2)^{3/4}}\,dt$$

and

$$b_\nu'=\frac{19x_\nu^2-4}{1-x_\nu^2},$$

$$r_1(x)=\frac{1+x}{2}\,U_n^2(x)+S_{n+2}(x)(1-x^2)^{-1/4}\int_{-1}^{x}\frac{a_1U_n(t)-(1+t)U_n'(t)}{2(1-t^2)^{3/4}}\,dt$$

where

$$a_1\int_{-1}^{1}\frac{U_n(t)}{(1-t^2)^{3/4}}\,dt=4\int_{-1}^{1}\frac{tU_n'(t)}{(1-t^2)^{3/4}},$$

and lastly

$$r_{n+2}(x) = \frac{1-x}{2} U_n^2(x) + S_{n+2}(x)(1-x^2)^{-1/4} \int_{-1}^{x} \frac{a_n U_n(t) - (1-t) U_n'(t)}{2(1-t^2)^{3/4}}$$

where $a_n = a_1$.

Bibliography

[1] J. Surányi and P. Turán, Notes on interpolation I, *Acta Math. Acad. Sci. Hungar.* **6** (1955), 67—79.

[2] J. Balázs and P. Turán, Notes on interpolation II, *Acta Math. Acad. Sci. Hungar.* **8** (1957), 201—215.

[3] R. B. Saxena and A. Sharma, On Some Interpolatory properties of Legendre polynomials, *Acta Math. Acad. Sci. Hungar.* **9** (1958), 345—358.

[4] O. Kis, Remarks on interpolation, *Acta Math. Acad. Sci. Hungar.* **11** (1960), 49—64, (Russian).

[5] G. Freud, Bemerkung über die Konvergenz eines Interpolationsverfahrens von P. Turán, *Acta Math. Acad. Sci. Hungar.* **9** (1958), 337—341.

[6] G. Szegő, Orthogonal Polynomials, second ed., *Providence*, 1959.

(Received April 26, 1961.)