Matrix-valued stochastical processes

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1. The theory of stochastical processes is one of the central parts of probability theory. In the literature there are known several generalizations of the concept of stochastical process. It is indeed possible to replace the linear parameter t by a variable element of a more general space, and also to consider instead of the numerical-valued random variables x_t random variables with values taken from a more general space. The n-dimensional vector-valued process, resp. the process in which the parameter t is being replaced by a vector of finite dimension, are to be considered today as classical.

The present paper considers stochastical processes, in which t runs through the set of reals, and the random variable $\mathbf{X}_t$ is a functional matrix of order r, the elements of which are random variables (throughout the paper r stands for a fixed natural number). This motivates the terminology matrix-valued stochastical process. We remark, that on defining stationarity, as well as in investigating the fundamental properties of the covariant functional matrix, we suppose about t only that it is the element of an abelian group, or, where this is necessary, of a topological abelian group.

To the best of the author's knowledge, a detailed investigation of matrix-valued stochastical processes has not been effected so far, although the concept of matrix-valued process has come up repeatedly during the last 5-6 years ([1], [2]).

The process to be considered in the present paper is neither a special case, nor a generalization of the vector-valued process; in fact, it is a differently directed generalization of the ordinary stochastical process. The investigation of matrix-valued processes is a rather natural but by no means trivial generalization of the theory of stochastical processes, and it is also possible to think of its practical applications.

The investigations of the present paper are based on the concept of quasi-Hilbert space, as elaborated by BÉLA GYIRES ([3]). The vectors of this space are arbitrary abstract elements, to which there are ordered as inner product, norm and distance respectively, matrices of order r, having constant elements. As a matter of fact, we make use only of the special case of this space, in which the elements of the space are functional matrices of order r, defined on some abstract set, and satisfying some conditions. In the first part of the paper we are treating this space. Then we define the concept of matrix-valued stochastical process, and investigate some of its properties. The concept of stationarity makes it clear that the investigation of matrix-valued processes in not simply equivalent to the simultaneous investigation of r^2 ordinary stationary processes, but we have a generalization also of the concept of ordinary stationary process, since we require only that the integrals of the elements of the matrix product $(\mathbf{X}_t-\mathbf{m}_t)(\mathbf{X}_s-\mathbf{m}_s)^*$ should depend on the difference $t-s$. Theorems 4 and 5 are preparatory to Theorems 6 and 7 respectively. The latter two are generalizations of the spectral representation known for the ordinary case. Theorem 8 is about the decomposition of the spectral functional matrix of distribution, whereas Theorem 9 treats properties of the spectral density.

2. Here we give a summary exposition of the notation used throughout the paper, and of well-known matrix-theoretical concepts. The zero- and the unit matrix of order r are denoted by $(\mathbf{O})_r$ and by $\mathbf{E}_r$ respectively. The conjugate transpose of the matrix $\mathbf{A}$ will be denoted by $\mathbf{A}^*$.

The matrix $\mathbf{A}$ will be called positive definite, positive semidefinite and Hermitian respectively, if for any row-vector $\mathbf{z}=(z_1,\ldots,z_r)$, $\mathbf{z}\mathbf{z}^*\neq 0$ the condition

$$\mathbf{z}\mathbf{A}\mathbf{z}^*>0, \quad \mathbf{z}\mathbf{A}\mathbf{z}^*\geqq 0 \qquad \text{resp. } \mathbf{A}=\mathbf{A}^*$$

is fulfilled.

A matrix $\mathbf{A}(x)$ will be called bounded, measurable and continuous respectively, if all its elements are bounded, measurable resp. continuous functions.

By the convergence of a sequence $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_n, \ldots$ of matrices we understand the convergence of the sequences formed from corresponding elements. By the integral $\int_a^b \mathbf{A}(x)\,dx$, and by the differential quotient $\frac{d}{dx}\mathbf{A}(x)=\mathbf{A}'(x)$ of a matrix $\mathbf{A}(x)$ we understand the matrix formed from the integrals, resp. from the differential quotients of the elements of the given matrix.

By the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_r$ of the matrix $\mathbf{A}$ of order r we understand the roots of the equation

$$\mathrm{Det}(\lambda\mathbf{E}_r-\mathbf{A})=0.$$

By the spur of the matrix $\mathbf{A}=(a_{ik})$ $(i,k=1,2,\ldots,r)$ we understand the sum of the elements standing in the main diagonal, i. e.

$$\mathrm{Sp}\,\mathbf{A}=\sum_{k=1}^{r} a_{kk}.$$

As is known $\sum_{k=1}^{r} a_{kk} = \sum_{k=1}^{r} \lambda_k$. The following relations clearly hold:

a) $$\mathrm{Sp}\,(a\mathbf{A}+b\mathbf{B}) = a\,\mathrm{Sp}\,\mathbf{A} + b\,\mathrm{Sp}\,\mathbf{B},$$

where a and b are arbitrary complex numbers,

b) $$\mathrm{Sp}\,\mathbf{AB} = \mathrm{Sp}\,\mathbf{BA},$$

c) $$\mathrm{Sp}\,\mathbf{U}^{-1}\mathbf{A}\mathbf{U} = \mathrm{Sp}\,\mathbf{A},$$

where $\mathbf{U}$ is an arbitrary regular matrix of order r.

Any Hermitian matrix $\mathbf{A}$ of order r can be represented in the form

(1) $$\mathbf{A} = \mathbf{U}\Lambda\mathbf{U}^*,$$

where $\mathbf{U}\mathbf{U}^* = \mathbf{E}_r$, and Λ is the diagonal matrix containing the eigenvalues of $\mathbf{A}$. The representation (1) will be called the canonical form of $\mathbf{A}$. Let $\mathbf{A}$ be a positive semidefinite Hermitian matrix. By the square root of this matrix we understand the matrix, also positive semidefinite and Hermitian, which can be obtained from the canonical representation of $\mathbf{A}$ by replacing the diagonal matrix Λ, containing the eigenvalues of $\mathbf{A}$, by the diagonal matrix containing the positive square roots of these eigenvalues.

3. Let $(\Omega, \mathfrak{F}, P)$ be a probability field, and consider the totality $L_2(\Omega)$ of functional matrices of order r, defined on the base set Ω, measurable with respect to the probability measure P, and having integrable square. Let $\mathfrak{M}$ denote the set of all matrices of order r, having constant complex elements. If we define on $L_2(\Omega)$ addition in the usual way, while by multiplication with elements from $\mathfrak{M}$ we understand matrix multiplication, $L_2(\Omega)$ will be closed with respect to these operations, and these operations satisfy conditions 1a) and b) of § 2. in [3], i. e. $L_2(\Omega)$ will be a linear space.

The correspondence

$$(\mathbf{f}, \mathbf{g}) = \int_{\Omega} \mathbf{f}(\omega)\mathbf{g}^*(\omega)\,dP(\omega) \in \mathfrak{M}, \quad \mathbf{f}, \mathbf{g} \in L_2(\Omega)$$

satisfies the conditions (8) from [3], and so the integral $(\mathbf{f}, \mathbf{g})$ is the generalized inner product of the elements $\mathbf{f}$ and $\mathbf{g}$.

To the element $\mathbf{f} \in L_2(\Omega)$ we make correspond as its norm the square root of the positive semidefinite Hermitian matrix $(\mathbf{f}, \mathbf{f}) = \|\mathbf{f}\|^2$. By the distance of two matrices $\mathbf{f}, \mathbf{g} \in L_2(\Omega)$ we understand the matrix $\|\mathbf{f}-\mathbf{g}\|$. The matrix sequence $\mathbf{f}_n \in L_2(\Omega)$ $(n = 1, 2, \ldots)$ converges in the mean to the matrix $\mathbf{f} \in L_2(\Omega)$, if for $n \to \infty$ the relation $\|\mathbf{f}_n - \mathbf{f}\| \to (\mathbf{O})_r$ holds. By Theorem 6 from [3] a necessary and sufficient condition for this convergence is the validity of $\mathrm{Sp}\,\|\mathbf{f}_n - \mathbf{f}\| \to 0$ $(n \to \infty)$. It is easy to see that $L_2(\Omega)$ is complete with respect to this concept of convergence.

For any natural number n, there exist in this space n linearly independent elements.

On the basis of the properties enumerated, we call $L_2(\Omega)$ *quasi Hilbert space*, in accordance with the terminology of [3] § 2.

4. Let us now consider a certain set of elements of $L_2(\Omega)$ depending on a single parameter. An element of this set will be denoted by $\mathbf{X}_t(\omega)$, where $t \in T$ and T is an index set. The elements of $\mathbf{X}_t(\omega)$ are clearly random variables, and so the totality of the matrices $\mathbf{X}_t(\omega)$, $t \in T$ can be called stochastical process. For a fixed value of t $\mathbf{X}_t(\omega)$ is a point of the quasi Hilbert space, and $\mathbf{X}_t(\omega)$, $t \in T$ is a curve in this space.

If a topology is defined on the set T, then we call the process in the usual manner continuous in the mean, if for $\tau \to t$ the relation

$$\|\mathbf{X}_\tau - \mathbf{X}_t\| \to (\mathbf{O})_r$$

also holds. From this we infer that for the continuity in the mean of $\mathbf{X}_t(\omega)$ it is necessary and sufficient that $\operatorname{Sp}\|\mathbf{X}_\tau - \mathbf{X}_t\| \to 0$ for $\tau \to t$.

The matrix

$$E\mathbf{X}_t(\omega) = \mathbf{m}_t = \int_\Omega \mathbf{X}_t(\omega)\, dP(\omega)$$

consisting of the expected values of the elements of $\mathbf{X}_t(\omega)$, will be called the mean value of the matrix $\mathbf{X}_t(\omega)$.

Definition 1. *The functional matrix of two variables*

$$(\mathbf{X}_t - \mathbf{m}_t, \mathbf{X}_s - \mathbf{m}_s) = \mathbf{R}(t, s), \quad t, s \in T$$

is called the covariant functional matrix of the process $\mathbf{X}_t(\omega)$.

Without restriction to generality, we can suppose in the sequel that $\mathbf{m}_t \equiv (\mathbf{O})_r$.

Definition 2. *If T is an abelian group, then the process $\mathbf{X}_t(\omega)$ is called stationary if for any $t \in T$, $s \in T$, $h \in T$*

$$\mathbf{R}(t+h, s+h) = \mathbf{R}(t, s) = \mathbf{R}(t-s).$$

Theorem 1. *If the stationary stochastical process $\mathbf{X}_t(\omega)$ is continuous in the mean, then the covariant functional matrix $\mathbf{R}(t)$ is continuous for all values of t.*

PROOF. Suppose $\mathbf{R}(t)$ is not continuous in some point t. Then there exists a positive number $\varepsilon' > 0$, such that for any value of h the matrix $\mathbf{R}(t+h) - \mathbf{R}(t)$ has at least one element with an absolute value greater than ε'. We show that this leads to a contradiction.

Consider indeed the Gramian matrix

$$\mathbf{G}=\mathbf{G}(\mathbf{X}_{t+h}-\mathbf{X}_t,\mathbf{X}_0)=\begin{pmatrix}(\mathbf{X}_{t+h}-\mathbf{X}_t,\mathbf{X}_{t+h}-\mathbf{X}_t) & (\mathbf{X}_{t+h}-\mathbf{X}_t,\mathbf{X}_0)\\ (\mathbf{X}_0,\mathbf{X}_{t+h}-\mathbf{X}_t) & (\mathbf{X}_0,\mathbf{X}_0)\end{pmatrix},$$

which is positive semidefinite. We show that the existence of a value t for which $\mathbf{R}(t)$ is not continuous would imply the existence of a $2r$-dimensional row-vector $\mathbf{z}$ such that

$$\mathbf{z}\mathbf{G}\mathbf{z}^*<0$$

which is impossible.

Clearly

$$\mathbf{R}(t+h)-\mathbf{R}(t)=(\mathbf{X}_{t+h}-\mathbf{X}_t,\mathbf{X}_0).$$

Now put

$$\|\mathbf{X}_{t+h}-\mathbf{X}_t\|^2=(a_{ik}),\quad (\mathbf{X}_{t+h}-\mathbf{X}_t,\mathbf{X}_0)=(b_{ik})$$

$$(\mathbf{X}_0,\mathbf{X}_{t+h}-\mathbf{X}_t)=(\bar{b}_{ki}),\quad (\mathbf{X}_0,\mathbf{X}_0)=(c_{ik})$$

$$\mathbf{z}=(\mathbf{z}_1,\mathbf{z}_2),$$

where

$$\mathbf{z}_1=(0,\ldots,0,\overset{j}{\bar{x}},0,\ldots,\overset{r}{\breve{0}}),\quad \mathbf{z}_2=(0,\ldots,0,\overset{k}{\breve{1}},0,\ldots,\overset{r}{\breve{0}}).$$

With these notations

$$\mathbf{z}\mathbf{G}\mathbf{z}^*=xa_{jj}\bar{x}+2V(xb_{jk})+c_{kk},$$

where $a_{jj}\geqq 0$, $c_{kk}\geqq 0$ and $V(xb_{jk})$ stands for the real part of the complex number xb_{jk}. Now chose x so that the inequality $V(xb_{jk})>\max\,(1,c_{kk})$ should hold. Let moreover $\varepsilon>0$ be such that $\varepsilon'>\varepsilon$ and $\max(1,c_{kk})>\varepsilon$ hold, and choose for this ε a h so small, that $xa_{jj}\bar{x}<\varepsilon$. Then

$$(-x)a_{jj}(-\bar{x})+c_{kk}+2V(-xb_{jk})<0.$$

Theorem 1 implies in particular that if the stationary stochastical process $\mathbf{X}_t(\omega)$ is continuous in the mean, then $\operatorname{Sp}\mathbf{R}(t)$ is a function continuous at the point $t=0$. Conversely:

Theorem 2. *If* $\operatorname{Sp}\mathbf{R}(t)$ *is continuous at the point* $t=0$, *then the stationary stochastical process* $\mathbf{X}_t(\omega)$ *is continuous in the mean.*

PROOF. Clearly

$$\|\mathbf{X}_{t+h}-\mathbf{X}_t\|^2=2\mathbf{R}(0)-\mathbf{R}(h)-\mathbf{R}^*(h).$$

Now, if $\operatorname{Sp}\mathbf{R}(t)$ is a function continuous at the point $t=0$, then

$$\lim_{h\to 0}\operatorname{Sp}\|\mathbf{X}_{t+h}-\mathbf{X}_t\|^2=\lim_{h\to 0}\operatorname{Sp}\big(2\mathbf{R}(0)-\mathbf{R}(h)-\mathbf{R}^*(h)\big)=0,$$

and this implies

$$\|\mathbf{X}_{t+h}-\mathbf{X}_t\|^2\to(\mathbf{O})_r,\quad \text{for}\quad h\to 0.$$

Theorems 1 and 2 together imply the following

Theorem 3. *In order that the stationary stochastical process $\mathbf{X}_t(\omega)$ be continuous in the mean, it is necessary and sufficient that the spur of the covariant functional matrix $\mathbf{R}(t)$ is continuous at the point $t=0$.*

In what follows we shall restrict ourselves to the case when t runs through the points of the real axis.

Theorem 4. *If $\mathbf{X}_t(\omega)$ is a stationary stochastical process and $t_k \in T$ $(k=1, 2, \ldots, n)$, then the covariant matrix of order nr*

$$\mathbf{R}_n = (\mathbf{R}(t_k - t_l)) \qquad (k, l = 1, 2, \ldots, n)$$

is positive semidefinite and Hermitian.

PROOF. Put $\mathbf{z} = (\mathbf{z}_1, \ldots, \mathbf{z}_n)$ where $\mathbf{z}_j = (z_j^1, \ldots, z_j^r)$ $(j=1, \ldots, n)$ is an arbitrary nr-dimensional row-vector, and consider the quadratic form

$$\mathbf{z}\mathbf{R}_n\mathbf{z}^* = \sum_{k=1}^{n}\sum_{l=1}^{n} \mathbf{z}_k \mathbf{R}(t_k - t_l)\mathbf{z}_l^*.$$

Taking into account the definition of the matrix $\mathbf{R}(t_k - t_l)$ we obtain that

$$\mathbf{z}\mathbf{R}_n\mathbf{z}^* = \sum_{k=1}^{n}\sum_{l=1}^{n} \int_{\Omega} \mathbf{z}_k \mathbf{X}_{t_k} \mathbf{X}_{t_l}^* \mathbf{z}_l^* \, dP(\omega) = \int_{\Omega} \left(\sum_{k=1}^{n} \mathbf{z}_k \mathbf{X}_{t_k}\right)\left(\sum_{k=1}^{n} \mathbf{z}_k \mathbf{X}_{t_k}\right)^* dP(\omega),$$

where the integrand is nonnegative, and thus $\mathbf{R}_n$ is positive semidefinite.

Consider now the matrix

$$\mathbf{R}_n^* = (\mathbf{R}^*(t_l - t_k)) \qquad (k, l = 1, 2, \ldots, n).$$

Since $\mathbf{R}(t)$ is a quasi-Hermitian matrix, i. e.

$$\mathbf{R}^*(t) = (\mathbf{X}_{t+h}, \mathbf{X}_h)^* = (\mathbf{X}_h, \mathbf{X}_{t+h}) = \mathbf{R}(-t), \tag{2}$$

we have $\mathbf{R}_n^* = \mathbf{R}_n$, i. e. $\mathbf{R}_n$ is also Hermitian.

Let us introduce the following notations:

$$\xi_k = (0, \ldots, 0, \overset{k}{\breve{1}}, 0, \ldots, \overset{r}{\breve{0}}) \qquad (k = 1, \ldots, r)$$

$$\left.\begin{aligned} \xi_{kl} &= (0, \ldots, 0, \overset{k}{\breve{1}}, 0, \ldots, 0, \overset{l}{\breve{1}}, 0, \ldots, \overset{r}{\breve{0}}) \\ \eta_{kl} &= (0, \ldots, 0, \overset{k}{\breve{i}}, 0, \ldots, 0, \overset{l}{\breve{1}}, 0, \ldots, \overset{r}{\breve{0}}) \end{aligned}\right\} \qquad (k, l = 1, \ldots, r;\ k < l),$$

where i is the imaginary unit.

Theorem 5. *The elements of the matrix* $\mathbf{R}(t)$ *are uniquely determined by the expressions*

$$\xi_k \mathbf{R}(t)\xi_k^* \qquad (k=1,\ldots,r),$$
$$\xi_{kl}\mathbf{R}(t)\xi_{kl}^*,\ \ \boldsymbol{\eta}_{kl}\mathbf{R}(t)\boldsymbol{\eta}_{kl}^* \qquad (k,l=1,\ldots,r;\ k<l).$$

PROOF. If the elements of $\mathbf{R}(t)$ are denoted by $r_{kl}(t)$ $(k,l=1,\ldots,r)$, then

(3) $$\xi_k \mathbf{R}(t)\xi_k^* = r_{kk}(t) \qquad (k=1,\ldots,r).$$

Moreover, in case $k,l=1,\ldots,r$ one has

(4) $$\xi_{kl}\mathbf{R}(t)\xi_{kl}^* = r_{kk}(t)+r_{ll}(t)+r_{kl}(t)+r_{lk}(t)$$

and

(5) $$\boldsymbol{\eta}_{kl}\mathbf{R}(t)\boldsymbol{\eta}_{kl}^* = r_{kk}(t)+r_{ll}(t)+i[r_{kl}(t)-r_{lk}(t)].$$

From (4) and (5)

(6) $$r_{kl}(t)+r_{lk}(t)=\xi_{kl}\mathbf{R}(t)\xi_{kl}^*-r_{kk}(t)-r_{ll}(t),$$

resp.

(7) $$r_{kl}(t)-r_{lk}(t)=-i[\boldsymbol{\eta}_{kl}\mathbf{R}(t)\boldsymbol{\eta}_{kl}^*-r_{kk}(t)-r_{ll}(t)]$$

follows, where on the right hand side there occur functions already known. Taking the sum and the difference respectively of (6) and (7), we get the assertion of the theorem.

Theorem 6. *If* $\mathbf{X}_t(\omega)$ *is a stationary process, continuous in the mean, then*

$$\mathbf{R}(t)=(\mathbf{X}_{t+h},\mathbf{X}_h)=\int_{-\infty}^{\infty} e^{itx}\,d\mathbf{F}(x) \qquad (-\infty<t<\infty),$$

where $\mathbf{F}(x)$ *is a positive semidefinite Hermitian matrix of order r, the elements of which are functions of limited variation, and for any row-vector* $\mathbf{z}=(z_1,\ldots,z_r)$

$$\mathbf{z}\mathbf{F}(x)\mathbf{z}^*$$

is a monotone nondecreasing function. This functional matrix is uniquely determined in any point x, which is a point of continuity for all of its elements.

If T denotes the set of integers on the real axis, then the totality of matrices $\mathbf{X}_t(\omega)$ is called a *discrete stochastical process.* The following theorem is about such processes:

Theorem 7. *If* $\mathbf{X}_t(\omega)$ *is a discrete stationary stochastical process, then*

(8) $$\mathbf{R}(k)=(\mathbf{X}_{k+l},\mathbf{X}_l)=\int_{-\pi}^{\pi} e^{ikx}\,d\mathbf{F}(x) \qquad (k=0,\pm1,\pm2,\ldots),$$

where the elements of the positive semidefinite Hermitian matrix $\mathbf{F}(x)$ *of order* r *are functions of limited variation, and for any row-vector* $\mathbf{z}=(z_1,\ldots,z_r)$

$$\mathbf{z}\mathbf{F}(x)\mathbf{z}^*$$

is a monotone nondecreasing function. This functional matrix is uniquely determined in any point x, which is a point of continuity of all its elements.

The proofs of the two Theorems 6 and 7 being completely analogous, we restrict ourselves to proving Theorem 7.

Let $\mathbf{z}=(z_1,\ldots,z_r)$ be an arbitrary row-vector, and let us denote by $\varphi_{\mathbf{z}}(k)$ the function $\mathbf{z}\mathbf{R}(k)\mathbf{z}^*$. First we show that for any positive integer n the matrix

$$\mathbf{\Phi}_{\mathbf{z}}=\begin{pmatrix}\varphi_{\mathbf{z}}(0) & \varphi_{\mathbf{z}}(1) & \cdots\ \varphi_{\mathbf{z}}(n-1)\\ \varphi_{\mathbf{z}}(-1) & \varphi_{\mathbf{z}}(0) & \cdots\ \varphi_{\mathbf{z}}(n-2)\\ \vdots & \vdots & \vdots\\ \varphi_{\mathbf{z}}(-n+1) & \varphi_{\mathbf{z}}(-n+2) & \cdots\ \varphi_{\mathbf{z}}(0)\end{pmatrix}$$

is positive semidefinite and Hermitian. Let indeed $\mathbf{v}=(v_1,\ldots,v_n)$ an arbitrary n-dimensional row-vector. Then

$$\mathbf{v}\mathbf{\Phi}_{\mathbf{z}}\mathbf{v}^*=\sum_{k=1}^{n}\sum_{l=1}^{n}v_k\varphi_{\mathbf{z}}(k-l)\overline{v_l}=$$

$$=\sum_{k=1}^{n}\sum_{l=1}^{n}v_k\mathbf{z}\mathbf{R}(k-l)\mathbf{z}^*\overline{v_l}=\sum_{k=1}^{n}\sum_{l=1}^{n}(v_k\mathbf{z})\mathbf{R}(k-l)(v_l\mathbf{z})^*$$

and by Theorem 4 the right hand side is nonnegative. Moreover, by (2) we have

$$\varphi_{\mathbf{z}}(-k)=\mathbf{z}\mathbf{R}(-k)\mathbf{z}^*=\mathbf{z}\mathbf{R}^*(k)\mathbf{z}^*=\big(\mathbf{z}\mathbf{R}(k)\mathbf{z}^*\big)^*=\overline{\varphi_{\mathbf{z}}(k)},$$

and from this it follows that $\mathbf{\Phi}_{\mathbf{z}}$ is also Hermitian.

The two above mentioned properties of the matrix $\mathbf{\Phi}_{\mathbf{z}}$ are by a well-known theorem ([4], p. 186) necessary and sufficient for the validity of the representation

$$\varphi_{\mathbf{z}}(k)=\int_{-\pi}^{\pi}e^{ikx}dH_{\mathbf{z}}(x)\qquad(k=0,\pm1,\pm2,\ldots),\tag{9}$$

where $H_{\mathbf{z}}(x)$ is a nondecreasing function of limited variation, uniquely determined in all its points of continuity.

By Theorem 5 the elements $r_{kl}(j)$ can uniquely be determined by linear combinations of the functions $\varphi_{\xi_k}(j)$, $\varphi_{\xi_{kl}}(j)$ and $\varphi_{\eta_{kl}}(j)$. Taking into account the representation (9) for $\varphi_{\mathbf{z}}(j)$, we get

$$r_{jl}(k)=\int_{-\pi}^{\pi}e^{ikx}dF_{jl}(x)\qquad(j,l=1,\ldots,r;\ k=0,\pm1,\pm2,\ldots),\tag{10}$$

where the functions $F_{jl}(x)$ are of limited variation and are uniquely determined in all their points of continuity.

From the theorem mentioned there follows also that if $\mathbf{F}(x)=(F_{jl}(x))$ $(j, l=1, \ldots, r)$, then for any row-vector $\mathbf{z}=(z_1, \ldots, z_r)$

$$\mathbf{z}\mathbf{F}(x)\mathbf{z}^*$$

is a monotone nondecreasing function. Indeed, by (9) and (10) for any row-vector $\mathbf{z}=(z_1, \ldots, z_r)$ the relation

$$\varphi_{\mathbf{z}}(k)=\mathbf{z}\mathbf{R}(k)\mathbf{z}^*=\int_{-\pi}^{\pi} e^{ikx}\,d(\mathbf{z}\mathbf{F}(x)\mathbf{z}^*)$$

holds, where $\mathbf{z}\mathbf{F}(x)\mathbf{z}^*$ is a monotone nondecreasing function.

Finally we show that $\mathbf{F}(x)$ is Hermitian and positive semidefinite. From (2) there follows

$$\mathbf{R}^*(k)=\int_{-\pi}^{\pi} e^{-ikx}\,d\mathbf{F}^*(x)=\int_{-\pi}^{\pi} e^{-ikx}\,d\mathbf{F}(x)=\mathbf{R}(-k),$$

i. e. in case

$$\int_{-\pi}^{\pi} e^{-ikx}\,d[\mathbf{F}(x)-\mathbf{F}^*(x)]=(\mathbf{O})_r.$$

The completeness of the trigonometrical system implies $\mathbf{F}(x)=\mathbf{F}^*(x)$ almost everywhere.

Since $\mathbf{z}\mathbf{F}(x)\mathbf{z}^*$ is a monotone nondecreasing function,

$$\mathbf{z}\mathbf{F}(x)\mathbf{z}^*=\int_{-\pi}^{x}\mathbf{z}\,d\mathbf{F}(t)\mathbf{z}^*=\int_{-\pi}^{x} d(\mathbf{z}\mathbf{F}(t)\mathbf{z}^*)\geqq 0$$

also holds. This completes the proof of the theorem.

Theorems 6 and 7 are generalizations of the spectral representation known for the ordinary case ([5], pp. 168—170). The functional matrix $\mathbf{F}(x)$ determined by these theorems will become uniquely determined, if in its points of discontinuity we require continuity from the left. The functional matrix $\mathbf{F}(x)$, uniquely determined in this way, will be called the *spectral distribution functional matrix* of the stationary stochastical process $\mathbf{X}_t(\omega)$.

Theorem 8. *The spectral distribution functional matrix* $\mathbf{F}(x)$ *has a unique representation of the form*

$$\mathbf{F}(x)=\mathbf{F}_a(x)+\mathbf{F}_l(x)+\mathbf{F}_s(x),$$

where all three matrices occurring in the decomposition are Hermitian, the elements of $\mathbf{F}_a(x)$ *are absolutely continuous functions, those of* $\mathbf{F}_l(x)$ *"stair-*

functions", those of $\mathbf{F}_s(x)$ *continuous functions and* $\mathbf{F}'_s(x)$ *is almost everywhere the null matrix.*

PROOF. Since the elements of $\mathbf{F}(x)$ are functions of bounded variation, the unique decomposition holds. ([6], p. 47.) The fact that the matrices figuring in the decomposition are Hermitian can be seen thus:

Let $F_{jk}(x)$, $F_a^{jk}(x)$, $F_l^{jk}(x)$ and $F_s^{jk}(x)$ $(j, k = 1, \ldots, r)$ denote the elements of the matrices $\mathbf{F}(x)$, $\mathbf{F}_a(x)$, $\mathbf{F}_l(x)$ and $\mathbf{F}_s(x)$ respectively. $\mathbf{F}(x)$ is Hermitian, and consequently

$$F_{jk}(x) = \overline{F_{kj}(x)},$$

but then there follows from the unicity of the decomposition the validity also of the relations

$$F_a^{jk}(x) = \overline{F_a^{kj}(x)}, \quad F_l^{jk}(x) = \overline{F_l^{kj}(x)} \quad \text{and} \quad F_s^{jk}(x) = \overline{F_s^{kj}(x)}$$

and this is just what the theorem says.

If $\mathbf{F}(x)$ is equal almost everywhere to $\mathbf{F}_a(x)$, then we say that the *spectrum of the process is absolutely continuous.*

Theorem 9. *If the spectrum of the stationary stochastical process* $\mathbf{X}_t(\omega)$ *is absolutely continuous, then there exists a functional matrix* $\mathbf{f}(x)$, *positive, semidefinite Hermitian and (L) integrable, for which*

$$\mathbf{F}(x) = \frac{1}{2\pi}\int_{-\infty}^{x} \mathbf{f}(t)\,dt \quad \textit{and} \quad \mathbf{f}(x) = 2\pi \frac{d}{dx}\mathbf{F}(x).$$

PROOF. We must prove only that $\mathbf{f}(x)$ is positive semidefinite. By Theorems 6 and 7 for any row-vector the function

$$\mathbf{z}\mathbf{F}(x)\mathbf{z}^*$$

is monotone nondecreasing, but then

$$\frac{d}{dx}(\mathbf{z}\mathbf{F}(x)\mathbf{z}^*) = \mathbf{z}\frac{d}{dx}\mathbf{F}(x)\mathbf{z}^* = \frac{1}{2\pi}\mathbf{z}\mathbf{f}(x)\mathbf{z}^* \geqq 0.$$

Conversely, if $\mathbf{f}(x)$ is a positive semidefinite Hermitian and (L) integrable functional matrix, then for any row-vector $\mathbf{z} = (z_1, \ldots, z_r)$ the function

$$\mathbf{z}\mathbf{F}(x)\mathbf{z}^* = \int_{-\infty}^{x} \mathbf{z}\mathbf{f}(t)\mathbf{z}^*\,dt$$

is monotone nondecreasing, for the integrand is nonnegative and positive, semidefinite Hermitian at the same time.

The functional matrix $\mathbf{f}(x)$ determined here is called the *spectral density* of the process $\mathbf{X}_t(\omega)$.

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