Cn an entire function of order less than one

By S. K. SINGH (Dharwar, India)

1. R. P. Boas [1] has proved the following

Theorem. *If $f(z)$ is an entire function of order* $\varrho (0 \leqq \varrho < 1)$ *then*

$$\limsup_{r\to\infty} \frac{N(r)}{Q(r)} \geqq \frac{1-\varrho}{\varrho} \tag{1}$$

where $N(r) = \int\limits_0^r \frac{n(t)}{t}\,dt$; $Q(r) = \int\limits_r^\infty \frac{n(t)}{t^2}\,dt.$

(We are assuming that $n(r)=0$ if $r<1$, which can always be done without any loss of generality.)

The purpose of this note is to give an alternative proof of the above theorem. The proof is on lines similar to Littlewood see [2, p. 136].

2. Let $\lim\sup \dfrac{N(r)}{Q(r)} = \alpha$.

If $\alpha = \infty$ then (1) is trivially true. Suppose α is finite. Then for $\beta > \alpha$ we have

$$N(r) < \beta r \int\limits_r^\infty \frac{n(t)}{t^2}\,dt \quad \text{for} \quad r \geqq r_0$$

$$= \beta r \int\limits_r^\infty \frac{N'(t)}{t}\,dt = \beta r \left[\frac{N(t)}{t}\right]_r^\infty + \beta r \int\limits_r^\infty \frac{N(t)}{t^2}\,dt = -\beta N(r) + \beta r \int\limits_r^\infty \frac{N(t)}{t^2}\,dt$$

because $\dfrac{N(t)}{t} \to 0$ as $t\to\infty$ since $\varrho < 1$. Hence $\dfrac{N(r)}{r^2}\,(1+\beta) < \dfrac{\beta}{r} \int\limits_r^\infty \dfrac{N(t)}{t^2}\,dt$. So

$$\frac{\beta}{(1+\beta)r} + \left(\frac{d}{dr}\left\{\int\limits_r^\infty \frac{N(t)}{t^2}\,dt\right\}\right) \Big/ \left(\int\limits_r^\infty \frac{N(t)}{t^2}\,dt\right) > 0. \tag{2}$$

Now

$$\int_{r_0}^{\infty} \frac{N(t)}{t^2}\,dt > N(r_0)\int_{r_0}^{\infty}\frac{dt}{t^2} = \frac{N(r_0)}{r_0} > K$$

where K is a positive constant because $N(r_0) > 0$ since $0 \leqq \varrho < 1$. Hence integrating (2) from r_0 to r we have

$$\int_{r}^{\infty} \frac{N(t)}{t^2}\,dt > K_1 r^{\frac{-\beta}{1+\beta}}. \tag{3}$$

Hence from (3) it follows that

$$\limsup_{r\to\infty} N(r) r^{\frac{-1}{1+\beta}} > 0$$

because if $N(r) r^{\frac{-1}{1+\beta}} \to 0$ as $r \to \infty$, then

$$\frac{N(t)}{t^2} < \varepsilon t^{\frac{1}{1+\beta}-2} \quad \text{for } t \geqq t_0.$$

So

$$\int_{r}^{\infty} \frac{N(t)}{t^2}\,dt < \varepsilon \int_{r}^{\infty} t^{\frac{1}{1+\beta}-2}\,dt < \varepsilon_1 r^{\frac{-\beta}{1+\beta}}$$

which contradicts (3).

Hence $N(r) > K_2 r^{\frac{1}{1+\beta}}$ for a sequence of values of r. Thus

$$\limsup_{r\to\infty} \frac{\log N(r)}{\log r} \geqq \frac{1}{1+\beta}.$$

But $\limsup\limits_{r\to\infty} \dfrac{\log N(r)}{\log r} = \limsup\limits_{r\to\infty} \dfrac{\log n(r)}{\log r} = \varrho_1$ and since $0 \leqq \varrho < 1$ so $\varrho_1 = \varrho$. Finally since β can be taken arbitrarily close to α the result follows.

Bibliography

[1] R. P. Boas, Some elementary theorems on entire functions, *Rend. Circ. Mat. Palermo* Ser. II., **1** (1952), 323–331.

[2] G. Valiron, Lectures on the general theory of integral functions, *Toulouse*, 1923.

(Received September 20, 1961.)