The estimation of the mean value of a matrix-valued discrete stationary stochastic process

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1. Let $(\Omega, \mathfrak{F}, P)$ be a probability field. Consider the totality $L_2(\Omega)$ of the quadratic functional matrices of order r, defined on the set Ω, measurable with respect the probability measure P and having square integrable elements. On $L_2(\Omega)$, which is a linear space with respect to ordinary matrix addition and left multiplication by $r\times r$ matrices of constant elements, we define the inner product as follows:

$$(\mathbf{f}, \mathbf{g}) = \int_{\Omega} \mathbf{f}(\omega)\mathbf{g}^*(\omega)\,dP(\omega) \qquad \mathbf{f}, \mathbf{g} \in L_2(\Omega).$$

For every pair $\mathbf{f}, \mathbf{g}$ the inner product $(\mathbf{f}, \mathbf{g})$ is an $r\times r$ matrix with constant elements. By the norm of $\mathbf{f}\in L_2(\Omega)$ we understand the square root of the positive semidefinite Hermitian matrix $(\mathbf{f}, \mathbf{f})$ and we denote it by $\|\mathbf{f}\|$. Accordingly, the convergence in the mean of the matrix sequence $\mathbf{f}_n \in L_2(\Omega)(n=1, 2, \dots)$ to $\mathbf{f}\in L_2(\Omega)$ means that the relation $\|\mathbf{f}_n - \mathbf{f}\| \to (0)_r$, $n\to\infty$ is fulfilled, and this holds if and only if $Sp\|\mathbf{f}_n-\mathbf{f}\|\to 0$ $(n\to\infty)$ ([3], Theorem 7). In § 3. of [1] we have shown that $L_2(\Omega)$ is, in the terminology of [3], § 2., a *quasi Hilbert space.*

Let us now consider a one parametric family $\mathbf{X}_t(\omega)$ ($t\in T$, where T is some index set) of elements of $L_2(\Omega)$, This will be called a stochastic process. By the mean value of the process we mean the integral

$$E\mathbf{X}_t = \mathbf{m}_t = \int_{\Omega} \mathbf{X}_t(\omega)\,dP(\omega)$$

and by its covariance functional matrix the functional matrix

$$(\mathbf{X}_t - \mathbf{m}_t, \mathbf{X}_s - \mathbf{m}_s) = \mathbf{R}(t, s), \qquad t, s\in T$$

of two variables. If T is an abelian group, then we say the process $\mathbf{X}_t(\omega)$ to be stationary in the wide sense, if for any $t\in T, s\in T, h\in T$,

$$\mathbf{R}(t+h, s+h) = \mathbf{R}(t, s) = \mathbf{R}(t-s)$$

holds.

If T is the set of the integers, then the process will be called discrete. By Theorem 7. of [1] the covariance functional matrix of a discrete stationary stochastic process can be represented in the form

$$\mathbf{R}(k) = (\mathbf{X}_{t+k}, \mathbf{X}_t) = \int_{-\pi}^{\pi} e^{ikx}\,d\mathbf{F}(x) \qquad (k=0, \pm 1, \pm 2, \dots),$$

where $\mathbf{F}(x)$ is a positive semidefinite Hermitian functional matrix of order r. This matrix will be called the spectral distribution functional matrix of the process.

If the elements of $\mathbf{F}(x)$ are absolutely continuous functions, then we call the spectrum of the process absolutely continuous. In this case there exists a positive semidefinite (L) integrable Hermitian functional matrix $\mathbf{f}(x)$ ([1], Theorem 9), such that

$$\mathbf{F}(x)=\frac{1}{2\pi}\int_{-\infty}^{x}\mathbf{f}(t)\,dt \quad \text{and} \quad \mathbf{f}(x)=2\pi\frac{d}{dx}\mathbf{F}(x). \text{ a. e.}$$

$\mathbf{f}(x)$ is the spectral density matrix of the process.

2. In the present paper we consider the special case of matrix valued discrete stationary processes, in which the mean value of the process has the form

$$\mathbf{m}_t=\mathbf{M}e^{it\lambda_0},$$

where λ_0 is a known numerical value, and $\mathbf{M}$ a regular matrix of order r with constant elements. We propose as our task to give an estimate of $\mathbf{M}$, if the process is being observed in the time points $t=-n, -n+1, \dots, n-1, n$. We seek for $\mathbf{M}$ an unbiased linear estimate

$$\tilde{\mathbf{M}}=\sum_{\nu=-n}^{n}\mathbf{C}_\nu\mathbf{X}_\nu,$$

for which $Sp\,\|\tilde{\mathbf{M}}-\mathbf{M}\|^2$ is minimal. Thus our problem is to determine the rxr matrices $\mathbf{C}_\nu$ of constant elements, occurring in the representation of $\tilde{\mathbf{M}}_{\min}$. The first part of our paper is devoted to the solution of this problem. The formula (5) gives an explicit expression for the matrices $\mathbf{C}_\nu$. Theorems 1, 2 and 5 tell us the conditions under which $\tilde{\mathbf{M}}_{\min}$ is a consistent estimate of $\mathbf{M}$.

Making use of the results of [2], we determine the error of the most efficient linear unbiased estimate of $\mathbf{M}$. More detailed information about this formula for errors is given by Theorems 6 and 7. In Theorem 8 we deal with the estimate $\tilde{\mathbf{M}}_L$ of $\mathbf{M}$, which is also linear and unbiased, but not efficient. We prove that the quotient of the errors of $\tilde{\mathbf{M}}_L$ and $\tilde{\mathbf{M}}_{\min}$ is assymptotically equal, if $n\to\infty$.

Theorems 3 and 4 concern the more general case, when

$$\mathbf{m}_t=\mathbf{M}\int_{-\pi}^{\pi}e^{it\lambda}\,dm(\lambda),$$

where $m(\lambda)$ is a known complex-valued function of bounded variation.

We remark that the results of the present paper are generalizations of the results in 11. 1 – 11. 3 of the book [4].

The concepts and notations from matrix theory, of which we make use in this paper, are summarized in [2], 2.

3. Let $\mathbf{X}_t(\omega)$ be a discrete stationary stochastic process. Suppose that the spectrum of the process is absolutely continuous and let the positive definite Hermitian functional matrix $\mathbf{f}(x)$ be the spectral density. Let furthermore

$$\mathbf{m}_t=\mathbf{M}e^{it\lambda_0},$$

where λ_0 is a known numerical value, and $\mathbf{M}$ a regular matrix of order r with constant elements. Suppose that we have observed the process in the time points $t=-n, -n+1, \ldots, n-1, n$ and let the observed values be given by the matrices $\mathbf{X}_{-n}, \mathbf{X}_{-n+1}, \ldots, \mathbf{X}_{n-1}, \mathbf{X}_n$. We propose to give an estimate of $\mathbf{M}$, i. e. to find a functional matrix

$$\tilde{\mathbf{M}}=\tilde{\mathbf{M}}(\mathbf{X}_{-n}, \mathbf{X}_{-n+1}, \ldots, \mathbf{X}_{n-1}, \mathbf{X}_n),$$

for which

$$\mathrm{Sp}\,\|\tilde{\mathbf{M}}-\mathbf{M}\|^2 \tag{1}$$

is minimal. We restrict ourselves to linear unbiased estimations

$$\tilde{\mathbf{M}}=\sum_{\nu=-n}^{n}\mathbf{C}_\nu\mathbf{X}_\nu,$$

where the quantities $\mathbf{C}_\nu$ are the matrices to be determined, subject to the condition (1) and also to the following $E\tilde{\mathbf{M}}=\mathbf{M}$.

The condition that $\tilde{\mathbf{M}}$ should be unbiased means that the relation

$$E\tilde{\mathbf{M}}=\sum_{\nu=-n}^{n}\mathbf{C}_\nu E\mathbf{X}_\nu=\sum_{\nu=-n}^{n}\mathbf{C}_\nu\mathbf{M}e^{i\nu\lambda_0}=\mathbf{M}$$

must necessarily hold, and thus in view of the regularity of the condition

$$\sum_{\nu=-n}^{n}\mathbf{C}_\nu e^{i\nu\lambda_0}=\mathbf{E}_r \tag{2}$$

must be fulfilled.

It is also possible to formulate the problem in a different way. Indeed,

$$\|\tilde{\mathbf{M}}-\mathbf{M}\|^2=(\tilde{\mathbf{M}}-\mathbf{M}, \tilde{\mathbf{M}}-\mathbf{M})=\left(\sum_{k=-n}^{n}\mathbf{C}_k(\mathbf{X}_k-\mathbf{M}e^{ik\lambda_0}),\ \sum_{l=-n}^{n}\mathbf{C}_l(\mathbf{X}_l-\mathbf{M}e^{il\lambda_0})\right)=$$

$$=\sum_{k=-n}^{n}\sum_{l=-n}^{n}\mathbf{C}_k\mathbf{R}(k-l)\mathbf{C}_l^*=\mathbf{CRC}^*,$$

where

$$\mathbf{C}=(\mathbf{C}_{-n}, \mathbf{C}_{-n+1}, \ldots, \mathbf{C}_n)$$

and $\mathbf{R}$ is a covariance matrix of order $(2n+1)r$.

Let us introduce the notation

$$\mathbf{A}^*=(e^{-in\lambda_0}\mathbf{E}_r, e^{-i(n-1)\lambda_0}\mathbf{E}_r, \ldots, e^{in\lambda_0}\mathbf{E}_r).$$

This enables us to write condition (2) in the form

$$\mathbf{CA}=\mathbf{E}_r. \tag{3}$$

Thus our task is to determine the matrix $\mathbf{C}$ which minimizes

$$\mathrm{Sp}\,\mathbf{CRC}^*$$

under the condition (3). We can solve this task as follows:

An arbitrary matrix $\mathbf{C}$ satisfying condition (3) can be written in the form

$\mathbf{C}=\mathbf{K}\mathbf{A}^*\mathbf{R}^{-1}+\mathbf{D}$, where $\mathbf{K}=(\mathbf{A}^*\mathbf{R}^{-1}\mathbf{A})^{-1}$ and $\mathbf{D}\mathbf{A}=(\mathbf{O})_r$. The matrix $\mathbf{R}$ occurring here, and thus also the matrix $\mathbf{K}$, has an inverse by Theorem 1 in [1].

Then

$$\mathbf{C}\mathbf{R}\mathbf{C}^*=(\mathbf{K}\mathbf{A}^*\mathbf{R}^{-1}+\mathbf{D})\mathbf{R}(\mathbf{K}\mathbf{A}^*\mathbf{R}^{-1}+\mathbf{D})^*=$$
$$=\mathbf{K}\mathbf{A}^*\mathbf{R}^{-1}\mathbf{A}\mathbf{K}+\mathbf{D}\mathbf{A}\mathbf{K}+\mathbf{K}(\mathbf{D}\mathbf{A})^*+\mathbf{D}\mathbf{R}\mathbf{D}^*.$$

Since $\mathbf{D}\mathbf{A}\mathbf{K}=\mathbf{K}(\mathbf{D}\mathbf{A})^*=(\mathbf{O})_r$ and $\mathbf{D}\mathbf{R}\mathbf{D}^*$ is a positive definite matrix, $\mathrm{Sp}\,\mathbf{C}\mathbf{R}\mathbf{C}^*$ will be minimal for $\mathbf{D}=(\mathbf{O})_r$. In this case the value of the minimum is

$$\mathrm{Sp}\,\mathbf{Q}_n=\mathrm{Sp}\,(\mathbf{A}^*\mathbf{R}^{-1}\mathbf{A})^{-1} \tag{4}$$

and we have for the minimizing matrix

$$\mathbf{C}=(\mathbf{A}^*\mathbf{R}^{-1}\mathbf{A})^{-1}\mathbf{A}^*\mathbf{R}^{-1}. \tag{5}$$

In accordance with the terminology generally accepted, we say that the sequence $\tilde{\mathbf{M}}_n$ is a strongly consistent sequence of estimates if

$$\lim_{n\to\infty}\|\tilde{\mathbf{M}}_n-\mathbf{M}\|^2=(\mathbf{O})_r.$$

It is our purpose to answer the question, whether there exists a sequence consistent estimates in the above case?

If we take into account the representation given in [1] (formula (8)) of the covariance functional matrix, then we get

$$\|\tilde{\mathbf{M}}-\mathbf{M}\|^2=\sum_{k=-n}^{n}\sum_{l=-n}^{n}\mathbf{C}_k\mathbf{R}(k-l)\mathbf{C}_l^*=$$
$$=\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sum_{k=-n}^{n}\mathbf{C}_k e^{ikx}\right)\mathbf{f}(x)\left(\sum_{k=-n}^{n}\mathbf{C}_k e^{ikx}\right)^* dx=$$
$$=\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathbf{P}_n(z)\mathbf{f}(x)\mathbf{P}_n^*(z)\,dx \qquad (z=e^{ix}),$$

where $\mathbf{P}_n(z)=\sum_{k=-n}^{n}\mathbf{C}_k z^k$. By theorem 4 in [2] this implies that

$$\mathrm{Sp}\,\mathbf{Q}_{n+1}\leqq\mathrm{Sp}\,\mathbf{Q}_n$$

i. e. the limit $\lim_{n\to\infty}\mathrm{Sp}\,\mathbf{Q}_n$ exits. Thus the question is, when will this limit be equal to zero?

Theorem 1. *For a bounded functional matrix* $\mathbf{f}(x)$ *there exists a consistent estimate, and*

$$\mathrm{Sp}\,\mathbf{Q}_n=0(n^{-1}).$$

This theorem follows from Theorem 6 in [2], the conditions of that theorem being satisfied by $\mathbf{f}(x)$.

It is easy to see that there exists a consistent estimate also in a more general case. Indeed, we have the following

Theorem 2. *If the spectrum of the process is absolutely continuous, then there exists a consistent estimate.*

PROOF. Consider the estimate

$$\tilde{\mathbf{M}}_L = \frac{1}{2n+1} \sum_{k=-n}^{n} \mathbf{X}_k e^{-ik\lambda_0},$$

i. e. let $\mathbf{C}_k = \frac{1}{2n+1} e^{-ik\lambda_0} \mathbf{E}_r$ $(k=-n, -n+1, \dots, n)$. This is an unbiased estimate of $\mathbf{M}$, for

$$E\tilde{\mathbf{M}}_L = \frac{1}{2n+1} \sum_{k=-n}^{n} \mathbf{M} e^{-ik\lambda_0} e^{ik\lambda_0} = \mathbf{M}.$$

On the other hand

$$\|\tilde{\mathbf{M}}_L - \mathbf{M}\|^2 = \frac{1}{2\pi} \frac{1}{(2n+1)^2} \int_{-\pi}^{\pi} \left| \sum_{k=-n}^{n} e^{i(x-\lambda_0)k} \right|^2 \mathbf{f}(x)\,dx =$$

$$= \frac{1}{2\pi} \frac{1}{(2n+1)^2} \int_{-\pi}^{\pi} \left(\frac{\sin (n+\frac{1}{2})(x-\lambda_0)}{\sin \frac{1}{2}(x-\lambda_0)} \right)^2 \mathbf{f}(x)\,dx,$$

hence

$$Sp\, \|\tilde{\mathbf{M}}_L - \mathbf{M}\|^2 = \frac{1}{2\pi} \frac{1}{(2n+1)^2} \int_{-\pi}^{\pi} \left(\frac{\sin (n+\frac{1}{2})(x-\lambda_0)}{\sin \frac{1}{2}(x-\lambda_0)} \right)^2 Sp\, \mathbf{f}(x)\,dx. \tag{6}$$

$\mathrm{Sp}\, \mathbf{f}(x)$ is (L) integrable, and so, by the Riemann-Lebesgue lemma, its Fourier coefficients tend to zero. If the n-th Fejér mean of the Fourier series of $\mathrm{Sp}\, \mathbf{f}(x)$ is denoted by $\sigma_n(x)$, then the right hand side of (6) equals $\frac{\sigma_n(\lambda_0)}{2n+1}$. As is known, $\lim_{n\to\infty} a_n = 0$ implies

$$\lim_{n\to\infty} \frac{(n+1)a_0 + na_1 + \dots + 2a_{n-1} + a_n}{n+1} = 0,$$

hence $\frac{\sigma_n(\lambda_0)}{2n+1} \to 0$ $(n\to\infty)$ and thus the right hand side of (6) converges to zero, if $n\to\infty$.

Let us now consider the more general case, when

$$\mathbf{m}_t = \mathbf{M} \int_{-\pi}^{\pi} e^{itx} dm(x),$$

where $m(x)$ is a complex-valued function of bounded variation. In this case the spectral distribution functional matrix is not necessarily absolutely continuous. Then we have to minimize

$$\text{Sp}\,\|\tilde{\mathbf{M}}-\mathbf{M}\|^2=\text{Sp}\,\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathbf{P}_n^*(z)\mathbf{P}_n(z)\,d\mathbf{F}(x)\quad (z=e^{ix}) \tag{7}$$

with the auxiliary condition

$$\int_{-\pi}^{\pi}\mathbf{P}_n(z)\,dm(x)=\mathbf{E}_r\quad (z=e^{ix}). \tag{8}$$

For this case we prove the following

Theorem 3.

$$\lim_{n\to\infty}\text{Min}\,Sp\,\|\tilde{\mathbf{M}}-\mathbf{M}\|^2=\lim_{n\to\infty}Sp\,\mathbf{Q}_n\leqq\frac{1}{2\pi}\sum_{j=1}^{r}H_j^{-1}, \tag{9}$$

where $H_j=\int_{-\pi}^{\pi}\frac{|dm(x)|^2}{dF_{jj}(x)}$ *is the so called Hellinger integral, and* $F_{jj}(x)$ $(j=1,2,\dots,r)$ *is the j-th diagonal element of the matrix* $\mathbf{F}(x)$

Proof. Let

$$\mathbf{P}_n(z)=\begin{pmatrix} p_{11}(z) & & (0)\\ & p_{22}(z) & \\ & & \ddots \\ (0) & & p_{rr}(z)\end{pmatrix}$$

and suppose that

$$\int_{-\pi}^{\pi}\mathbf{P}_n(z)\,dm(x)=\mathbf{E}_r\quad (z=e^{ix}).$$

In this case

$$Sp\,\mathbf{Q}_n\leqq\text{Min}\,\frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{j=1}^{r}|p_{jj}(z)|^2dF_{jj}(x)= \tag{10}$$

$$=\sum_{j=1}^{r}\text{Min}\,\frac{1}{2\pi}\int_{-\pi}^{\pi}|p_{jj}(z)|^2dF_{jj}(x)\quad (z=e^{ix}).$$

In [5], p. 569 it is proved that

$$\lim_{n\to\infty}\text{Min}\,\frac{1}{2\pi}\int_{-\pi}^{\pi}|p_{jj}(z)|^2dF_{jj}(x)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{|dm(x)|^2}{dF_{jj}(x)},$$

if only $\int_{-\pi}^{\pi} p_{jj}(z)\,dm(x)=1$. This, together with (10), already implies the assertion of our theorem.

A consideration of (9) immediately yields the following

Theorem 4. *If the Hellinger integrals H_j ($j=(1, 2, \dots, r)$ are all divergent, then the estimate $\tilde{\mathbf{M}}=\sum_{\nu=-n}^{n} \mathbf{C}_\nu \mathbf{X}_\nu$ satisfying condition* (8) *is consistent.*

In Theorems 1 and 2 the function $m(x)$ is constant, for $x\leqq\lambda_0$ and also for $x>\lambda_0$, while at $x=\lambda_0$ it has a jump of magnitude one. Clearly, in this case the Hellinger integral H_j is divergent or convergent depending on whether the function $F_{jj}(x)$ is continuous or not at the point $x=\lambda_0$. This fact is expressed by the following

Theorem 5. *If $\mathbf{m}_t=\mathbf{M}e^{it\lambda_0}$ and the elements $F_{jj}(x)$ ($j=1, 2, \dots, r$) in the main diagonal of the spectral distribution functional matrix are continuous at the point $x=\lambda_0$, then there exists a consistent estimate of* $\mathbf{M}$.

Let us now return to the case when $\mathbf{m}_t=\mathbf{M}e^{it\lambda_0}$ and the spectrum is absolutely continuous. By Theorem 2 there exists a consistent estimate. Of course, our aim is to find the „*best one*" of these consistent estimates. Let the matrices $\mathbf{X}_0, \mathbf{X}_1, \dots, \mathbf{X}_n$ stand for the observations. If the "*best*" consistent estimate of $\mathbf{M}$ is denoted by $\tilde{\mathbf{M}}_{\min}$, i. e. if $\tilde{\mathbf{M}}_{\min}$ is the linear combination of the matrices $\mathbf{X}_0, \mathbf{X}_1, \dots$ $\dots \mathbf{X}_n$, for which $\mathrm{Sp}\,\|\tilde{\mathbf{M}}_{\min}-\mathbf{M}\|^2$ is minimal, then in view of (5) one can write

$$\tilde{\mathbf{M}}_{\min}=(\mathbf{A}^*\mathbf{R}^{-1}\mathbf{A})^{-1}\mathbf{A}^*\mathbf{R}^{-1}\mathbf{X},$$

where $\mathbf{X}=(\mathbf{X}_0, \mathbf{X}_1, \dots, \mathbf{X}_n)$. Moreover, the mean square deviation of the error will be

$$Sp\,\|\tilde{\mathbf{M}}_{\min}-\mathbf{M}\|^2=Sp\,(\mathbf{A}^*\mathbf{R}^{-1}\mathbf{A})^{-1}.$$

Although these are explicit expressions, the effective determination of the values required is rather cumbersome, since it involves the uneasy task of determining the inverse of the matrix $\mathbf{R}$, possibly of high order. Therefore it is desirable to give approximations at least for the most frequently occuring cases.

In order to be able to do this, first we remark that the polynomial matrix minimizing the expression $\mathrm{Sp}\,\|\tilde{\mathbf{M}}_{\min}-\mathbf{M}\|^2$ under the condition $\mathbf{P}_n(e^{i\lambda_0})=\mathbf{E}_r$, has, by Theorem 3 from [2], the form

$$\mathbf{P}_n(z)=\mathbf{S}_n^{-1}(e^{i\lambda_0}, e^{i\lambda_0})\mathbf{S}_n(e^{i\lambda_0}, z),$$

and the minimum itself is equal to $\mathrm{Sp}\,\mathbf{S}_n^{-1}(e^{i\lambda_0}, e^{i\lambda_0})$. On the basis of this we can prove the following

Theorem 6. *If for the positive definite* (L) *integrable Hermitian functional matrix $\mathbf{f}(x)$ the relation*

(11) $$\mathbf{f}(x)=\big(\mathbf{A}(x)\mathbf{A}^*(x)\big)^{-1} \quad (z=e^{ix})$$

holds, where $\mathbf{A}(z)$ *is a trigonometrical polynomial matrix of degree* p, *then*

$$\lim_{n\to\infty} \frac{n\, Sp\, \|\tilde{\mathbf{M}}_{\min} - \mathbf{M}\|^2}{Sp\, \mathbf{f}(\lambda_0)} = 1. \tag{12}$$

PROOF. If $\mathbf{f}(x)$ has the form (11), then the polynomial matrix occuring in the representation of $\mathbf{S}_j(\alpha, \alpha)$ is

$$\boldsymbol{\phi}_{m+p}(z) = z^m \mathbf{A}(z) \qquad (m = 0, 1, 2, \dots),$$

since

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} z^k \mathbf{A}(z) \big(\mathbf{A}(z) \mathbf{A}^*(z)\big)^{-1} \mathbf{A}^*(z) \bar{z}^l dx = \begin{cases} (\mathbf{O})_r, & \text{for } k \neq l \\ \mathbf{E}_r & \text{for } k = l. \end{cases} \quad (z = e^{ix})$$

Thus for $n > p$ we have

$$\mathbf{S}_n(e^{i\lambda_0}, e^{i\lambda_0}) = \sum_{\nu=0}^{p} \boldsymbol{\phi}_\nu(e^{i\lambda_0}) \boldsymbol{\phi}_\nu^*(e^{i\lambda_0}) + \sum_{\nu=p+1}^{n} \mathbf{A}(e^{i\lambda_0}) \mathbf{A}^*(e^{i\lambda_0}) =$$

$$= \sum_{\nu=0}^{p} \boldsymbol{\phi}_\nu(e^{i\lambda_0}) \boldsymbol{\phi}_\nu^*(e^{i\lambda_0}) + (n-p+1) \mathbf{f}^{-1}(\lambda_0).$$

Since $Sp\, \|\tilde{\mathbf{M}}_{\min} - \mathbf{M}\|^2 = Sp\, \mathbf{S}_n^{-1}(e^{i\lambda_0}, e^{i\lambda_0})$ and $Sp \sum_{\nu=0}^{p} \boldsymbol{\phi}_\nu(e^{i\lambda_0}) \boldsymbol{\phi}_\nu^*({}^{i\lambda_0})$ is bounded, (12) holds.

Theorem 7. *If* $\mathbf{f}(x)$ *is a positive definite Hermitian continuous functional matrix, then*

$$\lim_{n\to\infty} \frac{n\, Sp\, \|\tilde{\mathbf{M}}_{\min} - \mathbf{M}\|^2}{Sp\, \mathbf{f}(\lambda_0)} = 1.$$

PROOF. Since $\mathbf{f}(x)$ is a positive definite Hermitian continuous functional matrix, there exist positive definite Hermitian trigonometrical polynomial matrices $\mathbf{P}_1(e^{ix})$ and $\mathbf{P}_2(e^{ix})$ so that for any row vector $\mathbf{z} = (z_1, z_2, \dots, z_r)$ we have

$$\mathbf{z}\mathbf{P}_1^{-1}(e^{ix})\mathbf{z}^* \leqq \mathbf{z}\mathbf{f}(x)\mathbf{z}^* \leqq \mathbf{z}\mathbf{P}_2^{-1}(e^{ix})\mathbf{z}^* \tag{13}$$

and the inequalities

$$\begin{gathered} 0 < \mathbf{z}\big(\mathbf{f}(x) - \mathbf{P}_1^{-1}(e^{ix})\big)\mathbf{z}^* < \frac{\varepsilon}{2r}, \\ 0 < \mathbf{z}(\mathbf{P}_2^{-1}(e^{ix}) - \mathbf{f}(x))\mathbf{z}^* < \frac{\varepsilon}{2r} \end{gathered} \tag{14}$$

hold for any $\varepsilon > 0$. By the generalization [6] of the well known theorem of FEJÉR, for any polynomial matrix with the above property there is an $\mathbf{A}(e^{ix}) = \sum_{k>0} \mathbf{A}_k e^{ikx}$ such that

$$\mathbf{P}_1(e^{ix}) = \mathbf{A}_1(e^{ix}) \mathbf{A}_1^*(e^{ix}) \quad \text{and} \quad \mathbf{P}_2(e^{ix}) = \mathbf{A}_2(e^{ix}) \mathbf{A}_2^*(e^{ix}).$$

Thus by Theorem 6

$$\mu_n\big(e^{i\lambda_0},\ \mathbf{P}_1^{-1}(e^{ix})\big) \cong \frac{1}{n}\, Sp\ \mathbf{P}_1^{-1}(e^{i\lambda_0})$$

(15)

$$\mu_n\big(e^{i\lambda_0},\ \mathbf{P}_2^{-1}(e^{ix})\big) \cong \frac{1}{n}\, Sp\ \mathbf{P}_2^{-1}(e^{i\lambda_0}),$$

where $\mu_n(\alpha, \mathbf{f}) = \text{Min}\ Sp \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathbf{P}_n(z)\mathbf{f}(x)\mathbf{P}_n^*(z)\,dx$ under the condition $\mathbf{P}_n(\alpha) = \mathbf{E}_r$. The inequalities (14) imply

$$0 < Sp\,\mathbf{f}(\lambda_0) - Sp\ \mathbf{P}_1^{-1}(e^{i\lambda_0}) < \frac{\varepsilon}{2}$$

$$0 < Sp\ \mathbf{P}_2^{-1}(e^{i\lambda_0}) - \mathbf{f}(\lambda_0) < \frac{\varepsilon}{2}$$

and

$$0 < Sp\ \mathbf{P}_2^{-1}(e^{i\lambda_0}) - Sp\ \mathbf{P}_1^{-1}(e^{i\lambda_0}) < \varepsilon.$$

From these inequalities, and from (13), (15) and Theorem 5 in [2] the relation

$$\mu_n(e^{i\lambda_0}, \mathbf{f}) \cong \frac{1}{n}\, Sp\,\mathbf{f}(\lambda_0).$$

follows.

Theorem 8. *If* $\mathbf{f}(x)$ *is a positive definite Hermitian continuous functional matrix and*

$$\tilde{\mathbf{M}}_L = \frac{1}{n+1} \sum_{\nu=0}^{n} e^{-i\nu\lambda_0}\,\mathbf{X}_\nu,$$

i. e. $\mathbf{C}_\nu = \frac{1}{n+1} e^{-i\nu\lambda_0}\mathbf{E}_r$ $(\nu = 0, 1, \ldots, n)$, *then*

$$\lim_{n\to\infty} \frac{n\, Sp\, \|\tilde{\mathbf{M}}_L - \mathbf{M}\|^2}{Sp\,\mathbf{f}(\lambda_0)} = 1.$$

Proof. Consider the equality

$$Sp\,\|\tilde{\mathbf{M}}_L - \mathbf{M}\|^2 = \frac{1}{(n+1)^2}\frac{1}{2\pi}\int_{-\pi}^{\pi} \left|\sum_{\nu=0}^{n} e^{i\nu(x-\lambda_0)}\right|^2 Sp\,\mathbf{f}(x)\,dx =$$

$$= \frac{1}{(n+1)^2}\frac{1}{2\pi}\int_{-\pi}^{\pi} \left(\frac{\sin\frac{n+1}{2}(x-\lambda_0)}{\sin\frac{1}{2}(x-\lambda_0)} \right)^2 Sp\,\mathbf{f}(x)\,dx.$$

Now, if $\sigma_n(x)$ denotes the n-th Cesaro mean of the Fourier series of Sp $\mathbf{f}(x)$, then

$$Sp \, \|\tilde{\mathbf{M}}_L - \mathbf{M}\|^2 = \frac{\sigma_n(\lambda_0)}{n+1}.$$

By our condition Sp $\mathbf{f}(x)$ is continuous, and so by the limit theorem of FEJÉR for the Cesaro means of the Fourier series

$$\lim_{n\to\infty} \sigma_n(\lambda_0) = Sp \, \mathbf{f}(\lambda_0)$$

hence the theorem follows.

Theorems 7 and 8 jointly imply the following

Theorem 9. *If* $\mathbf{f}(x)$ *is a positive definite Hermitian continuous functional matrix, then the unbiased and consistent estimate* $\tilde{\mathbf{M}}_L$ *of* $\mathbf{M}$ *is assymptotically equal to the estimate* $\tilde{\mathbf{M}}_{\min}$, *i. e.*

$$Sp \, \|\tilde{\mathbf{M}}_L - \mathbf{M}\|^2 \cong Sp \, \|\tilde{\mathbf{M}}_{\min} - \mathbf{M}\|^2 \cong \frac{1}{n} \, Sp \, \mathbf{f}(\lambda_0).$$

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(Received November 25, 1961.)