Publ. Math. Debrecen
48 / 3-4 (1996), 193–199

Generalized X-ray pictures

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Abstract. We generalize the definition of X-ray pictures by introducing weight functions. Then the corresponding Hammer-type problem is solved and applications are given.

In this short article we introduce weight functions into the definition of the X-ray picture and solve the corresponding Hammer-type problem by generalizing the idea of McMullen and Gardner [2]. Similarly generalized X-ray transforms, that involve weights along the straight lines, are widely considered in the theory of the Radon transform, as they occur in practice [1], [5].

Under some consistency conditions on the weight functions, our result is the analogue of the result due to Gardner and McMullen. This gives a solution to Hammer's problem on the constant curvature spaces and proves the uniqueness for the exponential X-ray picture. These results may prove useful in practice, as the hyperbolic case plays a role in radar technique [1] and the exponential case plays a role in single photon emission tomography [5].

The author thanks the Soros Foundation for supporting his visit at MIT's Department of Mathematics, where the idea of this paper was born. Thanks are also due to the Mathematische Institut of the Universität Erlangen-Nürnberg, where the article was written.

Mathematics Subject Classification: 0052, 0054.
Key words and phrases: convex geometry, X-ray picture, weight function.
Supported by the Hungarian NSF, OTKA Nr. T4427, W015425 and F016226.

Preliminaries

We call a compact set with nonempty interior in $\mathbb{R}^n$ a *body*. The *parallel beam* X-ray picture of a body $\mathcal{B}$ corresponding to a direction, i.e. to a unit vector $\omega \in \mathrm{S}^{n-1}$, is a function on $\mathbb{R}^n$ defined by

$$X^{\mathcal{B}}(x,\omega) = \int_{-\infty}^{\infty} \chi_{\mathcal{B}}(x+\lambda\omega)\,\mathrm{d}\lambda \qquad (x \in \mathbb{R}^n),$$

where $\chi_{\mathcal{B}}$ is the indicator function of $\mathcal{B}$. Note that $X^{\mathcal{B}}(x,\omega) = X^{\mathcal{B}}(x + r\omega,\omega)$ for any real r.

Hammer's problem can now be formulated in the following way: *For how many directions $\omega \in \mathrm{S}^{n-1}$ the functions $X^{\mathcal{B}}(.,\omega)$ should be known to permit the determination of the convex body $\mathcal{B}$.*

Our considerations will be concentrated to the planar case, because the definition of $X^{\mathcal{B}}$ shows that a solution to Hammer's problem on the plane gives an upperbound for the higher dimensional cases.

Let $\mu(\lambda,\omega,r)$ be a locally integrable strictly positive function given for every straight line L, where r is the signed distance of L from the origin, $\omega \in \mathrm{S}^1$ is a normal vector of L and λ is the arclength parameter, so that the point with $\lambda = 0$ is at distance $|r|$ from the origin. We suppose further, that the parameterization is anticlockwise. By this we mean the following. If $p(\lambda)$ is the point of L with parameter λ then the direction vector $\omega^{\perp} = dp(\lambda)/d\lambda$ of L is ω rotated $\pi/2$ anticlockwise. Of course, $\mu(\lambda,\omega,r) = \mu(-\lambda,-\omega,-r)$. We call μ a *weight function.*

We define the *generalized parallel beam* X-ray picture for a direction ω by

$$X^{\mathcal{B}}_{\mu}(x,\omega) = \int_{-\infty}^{\infty} \chi_{\mathcal{B}}(x+\lambda\omega^{\perp})\mu(\lambda,\omega,\langle x,\omega\rangle)\,\mathrm{d}\lambda \qquad (x \in \mathbb{R}^2).$$

where $\langle .,. \rangle$ is the usual inner product of $\mathbb{R}^2$. Obviously, $X^{\mathcal{B}}_{\mu}(x,\omega) = X^{\mathcal{B}}_{\mu}(x + r\omega^{\perp},\omega)$ for any real r.

The result

For convenience we reparameterize the weight functions. We shall write $\mu(\lambda,\alpha,r)$ instead of $\mu(\lambda,\omega_{\alpha},r)$, where ω_{α} denotes the unit vector of S^1 making the angle α with the x-axis.

A weight function μ is called a *plane weight*, if there is a function F so that

$$F(\alpha,r)\mu(\lambda,\alpha,r)=\mu(r\cos\alpha-\lambda\sin\alpha,0,r\sin\alpha+\lambda\cos\alpha)$$

for all the possible parameters. Note that F is strictly positive by the definition of μ. The following lemma holds for plane weight functions.

Lemma 1. *Let μ be a plane weight. If the bodies $\mathcal{B}_1$ and $\mathcal{B}_2$ have the same X-ray pictures in a direction α, i.e. $X_\mu^{\mathcal{B}_1}(.,\alpha)=X_\mu^{\mathcal{B}_2}(.,\alpha)$, then they have the same mass with respect to the density $\mu(y,0,x)\,\mathrm{d}y\,\mathrm{d}x$.*

Pproof. We have for $i\in\{1,2\}$

$$\begin{aligned}M_i(\alpha)&=\int_{-\infty}^{\infty}X_\mu^{\mathcal{B}_i}(r\omega_\alpha,\alpha)F(\alpha,r)\,\mathrm{d}r\\&=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\chi_{\mathcal{B}_i}(r\omega_\alpha+\lambda\omega_\alpha^\perp)\mu(\lambda,\alpha,r)F(\alpha,r)\,\mathrm{d}\lambda\,\mathrm{d}r.\end{aligned}$$

Substituting $y=r\cos\alpha-\lambda\sin\alpha$ and $x=r\sin\alpha+\lambda\cos\alpha$ we obtain

$$M_i(\alpha)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\chi_{\mathcal{B}_i}(y\bar\omega^\perp+x\bar\omega)\mu(y,0,x)\,\mathrm{d}y\,\mathrm{d}x,$$

where $\bar\omega^\perp=\omega_\alpha\cos\alpha-\omega_\alpha^\perp\sin\alpha$ and $\bar\omega=\omega_\alpha\sin\alpha+\omega_\alpha^\perp\cos\alpha$. Obviously, $\bar\omega$ and $\bar\omega^\perp$ make a fixed orthonormal basis and so $M_i(\alpha)$ is the mass of $\mathcal{B}_i$ with respect to the density $\mu(y,0,x)\,\mathrm{d}y\,\mathrm{d}x$. Since $X_\mu^{\mathcal{B}_1}(.,\alpha)=X_\mu^{\mathcal{B}_2}(.,\alpha)$, the first equation implies $M_1(\alpha)=M_2(\alpha)$. $\square$

In what follows the mass, volume and the likes are meant with respect to the density $\mu(y,0,x)\,\mathrm{d}y\,\mathrm{d}x$. A simple general observation appeared in [2], and used by Volčič and by Gardner [3] later, gives

Lemma 2. *Let μ be a plane weight and $\mathcal{B}$ be a body. If*

$$\int_{-\infty}^{\varrho(\alpha)}X_\mu^{\mathcal{B}_i}(r\omega_\alpha,\alpha)F(\alpha,r)\,\mathrm{d}r=\int_{\varrho(\alpha)}^{+\infty}X_\mu^{\mathcal{B}_i}(r\omega_\alpha,\alpha)F(\alpha,r)\,\mathrm{d}r$$

then the centroid of $\mathcal{B}$ is on the straightline $L(\varrho(\alpha),\alpha)$, where $\varrho(\alpha)$ is the signed distance of L from the origin, and $\omega_\alpha\in\mathrm{S}^1$ is a normal vector of L.

Proof. By the condition and Lemma 1 the masses of the two parts of $\mathcal{B}$ as cut by L are equal, therefore L should go through the centroid of $\mathcal{B}$. $\square$

The following theorem is the analogue of Theorem 2 in [2] except that the weight function is involved. Our proof follows the proof of Theorem 2 in [2]; therefore we go into details only where it is necessary.

Theorem 1. *If μ is a plane weight and $\mathcal{B}_1 \not\equiv \mathcal{B}_2$ are convex bodies then the set A of the directions α for which $X_\mu^{\mathcal{B}_1}(.,\alpha) = X_\mu^{\mathcal{B}_2}(.,\alpha)$ is affinely equivalent to a subset of the directions of diagonals of some regular polygon.*

Proof. Since any three directions are affinely equivalent to a subset of the directions of diagonals of a regular polygon we may suppose that $|A| \geq 4$. A convex polygon $\mathcal{P}$ is said to be an A-polygon, if any straight line through any vertex p of $\mathcal{P}$ with direction in A either goes through another vertex of $\mathcal{P}$, or supports $\mathcal{P}$ in p alone. We prove through several steps.

Step 1. $\operatorname{Int}\mathcal{B}_1 \cap \operatorname{Int}\mathcal{B}_2 \neq 0$. Taking two directions α_1 and α_2 from A, we see by Lemma 2 that $\mathcal{B}_1$ and $\mathcal{B}_2$ have common centroids.

Step 2. *If R is a component of either* $\operatorname{Int}\mathcal{B}_1 - \mathcal{B}_2$ *or* $\operatorname{Int}\mathcal{B}_2 - \mathcal{B}_1$ *then it is an open set bounded by an arc of $\partial\mathcal{B}_1$ and an arc of $\partial\mathcal{B}_2$ with common distinct endpoints.*

Step 3. *If R is a component of* $\operatorname{Int}\mathcal{B}_1 - \mathcal{B}_2$ *then for any $\alpha \in A$*

$$\bar{R} = \bigcup_{L\|\omega_\alpha, L\cap R\neq 0} ((\operatorname{Int}\mathcal{B}_2 - \mathcal{B}_1) \cap L)$$

is a component of $\operatorname{Int}\mathcal{B}_2 - \mathcal{B}_1$ *with the mass of R.* Since $X_\mu^{\mathcal{B}_1}(.,\alpha) = X_\mu^{\mathcal{B}_2}(.,\alpha)$ we have

$$\int_{-\infty}^{\infty} \chi_{\operatorname{Int}\mathcal{B}_2-\mathcal{B}_1}(r\omega_\alpha + \lambda\omega_\alpha^\perp)\mu(\lambda,\alpha,r)\,\mathrm{d}\lambda = \int_{-\infty}^{\infty} \chi_{\operatorname{Int}\mathcal{B}_1-\mathcal{B}_2}(r\omega_\alpha^\perp + \lambda\omega_\alpha)\mu(\lambda,\alpha,r)\,\mathrm{d}\lambda$$

for each r. Since μ is strictly positive this means that, a straight line L going through a point of $\partial\mathcal{B}_1 \cap \partial\mathcal{B}_2$ must go through exactly one further point of $\partial\mathcal{B}_1 \cap \partial\mathcal{B}_2$. Therefore $\bar{R}$ must be a component and its mass is the same as of R by Lemma 1 (since the above formula says $X_\mu^R(.,\alpha) = X_\mu^{\bar{R}}(.,\alpha)$).

Step 4. *Iterating Step* 3 *using any sequence in A generates only a finite set S of components.* Clearly $\mathcal{B}_1 \cup \mathcal{B}_2$ has finite measure.

Step 5. *The polygon $\mathcal{P}$ of the endpoints of the components in S is an A-polygon.*

Step 6. *If $\mathcal{P}$ is an A-polygon, then so is its midpoint polygon $M(\mathcal{P})$.* The midpoint polygon's vertices are the midpoints of the edges of the original polygon $\mathcal{P}$.

Step 7. *If $\mathcal{P}_0 = \mathcal{P}$ and $\mathcal{P}_{k+1} = M(\mathcal{P}_k)/\mathrm{Vol}(M(\mathcal{P}_k))$ then $\{\mathcal{P}_{2k}\}_{k=0}^{\infty}$ converges to an affinely regular polygon.* For this result of M. G. Darboux see [6]. □

Theorem 2. *Either any infinite set of parallel beam X-ray pictures, or certain sets of four prescribed parallel beam X-ray pictures are enough to distinguish any two convex bodies.*

Proof. For the first part we just have to observe that no infinite set of directions can be affinely equivalent to a subset of the directions of diagonals of any regular polygon. Further, the coordinates of any set of directions of diagonals of any affinely regular polygon constitutes an algebraic system of numbers, because the numbers $\cos(2\pi/k)$ $(k \in \mathbb{N})$ are algebraic. Therefore, the four directions only have to be chosen so that their coordinates constitute a transcendental system of numbers. □

In the case of $\mu \equiv 1$ for any three directions a number of easy examples show the same parallel beam X-ray pictures. For general plane weight μ the construction of such examples seems very hard, but we conjecture

For any three directions and any plane weight there exist convex bodies with the same generalized parallel beam X-ray pictures.

Applications

We now turn to Hammer's question on the constant curvature spaces. These spaces are the hyperbolic, the Euclidean and the elliptic space corresponding to the curvatures -1, 0, $+1$. Again, we restrict our considerations to the two dimensional spaces $\mathcal{M}_\kappa^2$ for $\kappa \in \{-1, 0, 1\}$. Fix a point O in $\mathcal{M}^2$, one of the three planes of constant curvature.

A set of points in these spaces is said to be convex if it contains the geodesic segment joining any two of its points. A convex set is called convex body if its interior is not empty and it is compact. The geodesics are parallel if there is a geodesic through O perpendicular to all of them. The X-ray picture of a convex domain $\mathcal{B}$ is defined on a geodesic g through O as the function that gives the lengths of the geodesic segments the domain $\mathcal{B}$ cuts out from the geodesics perpendicular to g.

Following [4] we now present maps from each constant curvature planes to the Euclidean plane, which maps the geodesics to straight lines.

The Riemannian metric on $\mathcal{M}^2$ is completely described by the 'projector function' $\pi : \mathbb{R}_+ \to \mathbb{R}_+$. This function generates a geodesic correspondence between the constant curvature planes and the Euclidean plane via the geodesic polar coordinatization:

$$\tilde{\pi}\colon \mathcal{M}^2 \to \mathbb{R}^2 \qquad \big(\mathrm{Exp}_O\, r\omega \mapsto \pi(r)\omega\big),$$

where Exp is the exponential mapping, ω is a unit vector in $\mathrm{T}_O\mathcal{M}^2$, $r \in \mathbb{R}_+$ and $\mathbb{R}^2$ is identified with $\mathrm{T}_O\mathcal{M}^2$. We have by [4]

$$\pi(r) = \begin{cases} \tanh r & \text{if } \kappa = -1,\ \mathcal{M}^2 = \mathbb{H}^2, \\ r & \text{if } \kappa = 0,\ \mathcal{M}^2 = \mathbb{R}^2, \\ \tan r & \text{if } \kappa = +1,\ \mathcal{M}^2 = \mathbb{P}^2. \end{cases}$$

The map $\tilde{\pi}$ transfers parallel geodesics into parallel straight lines, while transforming the arclength of the geodesic into the weight

$$\mu_\kappa(\lambda, \alpha, r) = \frac{\sqrt{1 + \kappa r^2}}{1 + \kappa(\lambda^2 + r^2)}$$

on the corresponding straight line, where $\mu_{-1}(\lambda, \alpha, r) = 0$ for $\lambda^2 + r^2 \geq 1$ [4, Th. 2.1]. Since the weight μ_κ is a plane weight, the following result, that has been already proved for $\kappa = 0$ in [2] and for $\kappa = 1$ in [3], is a simple consequence of Theorem 2.

Theorem 3. *Either any infinite set of parallel beam X-ray pictures, or certain sets of four prescribed parallel beam X-ray pictures are enough to distinguish any two convex bodies on any constant curvature space.*

The exponential X-ray picture is defined by the weight [5]

$$\mu(\lambda, \omega, r) = \mu_1(\omega, r) \exp(\mu_2 \langle \bar{\omega}, \lambda\omega^\perp + r\omega \rangle),$$

where $\bar{\omega}$ is a fixed unit vector, μ_2 is a constant and $\mu_1(\omega, r)$ is a positive real function of its variables. The uniqueness result in the following theorem is easily proved by the fact that the exponential weight is a plane weight.

Theorem 4. *Either any infinite set of exponential parallel beam X-ray pictures, or certain sets of four prescribed exponential parallel beam X-ray pictures are enough to distinguish any two convex bodies.*

We call attention again, that in both of these two theorems the problem for three directions is open.

Closing the paper, we note that using a projectivity and our Theorem 2 one can easily prove that certain sets of four prescribed X-ray pictures of collinear point sources distinguish any two convex bodies. Moreover, this can also be done for the generalized X-ray pictures of point sources defined in [3].

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(Received November 2, 1993; revised June 26, 1995)