On the dependence relation over abelian groups

By V. DLAB (Khartoum)

1. It is a familiar fact that the maximal torsion subgroup of an abelian group G contains with any non-zero element g also all the solutions of the equation $nx=g$ (with a natural n) in G. In other words, the subset of all non-zero elements of the maximal torsion subgroup has the following property (P):

A subset A of non-zero elements of an abelian group G is said to have property (P) if every element of G depending on an element of A belongs to A.

It is easy to see that the property (P) does not give a characterization of the maximal torsion group. Therefore, let us consider the following „stronger" properties of a subset A of G possessed by the maximal torsion subgroup:

A subset A of non-zero elements of an abelian group G is said to have

(i) *property (P') if every element of G depending on an independent subset of A belongs to A;*

(ii) *property (P'') if every element of G depending on A belongs to A;*

(iii) *property (P^*) if every element of A depending on an independent subset I of G depends also on the intersection $I \cap A$.* [1])

Thus, property (P) is (P^*) restricted to the case when I is a single element. Further, clearly (P'') implies (P') and (P') implies (P). We shall see that (P^*) implies also (P'). In fact, we shall prove that only property (P^*) yields a characterization of the maximal torsion subgroup. Simultaneously, we shall describe the subsets of G "closed" with respect to the dependence in G, and so-called g-coverings of G.

Throughout this paper, G denotes an abelian group, T its maximal torsion subgroup (which may be (0) or G itself) and W the complement of T in G. By G_0 or T_0 we denote the subset of all non-zero elements of G or T, respectively, by $O(g)$ the order of an element g and by $G[n]$ the subgroup of elements g with $O(g) \leqq n$. For familiar concepts, as well as for some basic facts on abelian groups, in particular, those concerning the dependence relation, we refer to L. FUCHS [3], or to the original papers of T. SZELE [4] or the author [2].

2. The following lemmas can easily be verified.

Lemma 1. *If A has property (P), then it contains with every element a of infinite order the whole coset $a+T$ of G modulo T.*

Lemma 2. *If T is not a primary group,*[2]) *than every subset A with property (P) containing an element of finite order includes the whole T_0.*

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[1]) For the sake of easier formulations, we exclude the zero-element out of consideration.

Lemma 3. *If T is a p-primary group, then every subset A with property (P) contains with an element g of order $p^n (n \geqq 2)$ the coset $g+T[p^{n-1}]$ of T modulo $T[p^{n-1}]$.*

Lemma 4. *If A_α have property (P), then*
(a) *the intersection $\bigcap_\alpha A_\alpha$ and*
(b) *the union $\bigcup_\alpha A_\alpha$*
have property (P), as well.

On the basis of the preceding lemmas, we can prove then the following

Theorem 1. *Let an abelian group G be given. Then there exists a partition $(A_\lambda)_{\lambda\in\Lambda}$ of the set G_0 such that each of A_λ is a minimal non-void subset of G_0 with property (P). This partition is uniquely determined. If T is not a primary group, then T_0 is one of A_λ. If T is a primary group, then any two elements of the same subset A_λ are dependent. If G is a torsion free group, then a subset is a member of the partition $(A_\lambda)_{\lambda\in\Lambda}$ if, and only if, it forms, together with the zero-element, a minimal pure subgroup of G. — The union $\bigcup_{\lambda\in\Lambda'} A_\lambda$, $\Lambda' \subseteq \Lambda$, of some members of $(A_\lambda)_{\lambda\in\Lambda}$ has property (P) and all the subsets with property (P) can be obtained in this way.*

Proof. Consider the intersection A_g of all the subsets with property (P) containing an element $g\in G_0$. In view of Lemma 4. (a), A_g is a non-void subset with property (P), which is obviously minimal. Since all such different subsets are mutually disjoint and cover G_0, they form a partition of G_0. If T is not a primary group, then, by Lemma 2, T_0 is a member of this partition.

If T is a primary group, then G_0 contains only elements of infinite and/or prime power order and thus, if g_1 and g_2 are two independent elements of G_0, there is no element in G_0 depending on each of them. Hence, in this case, the subset A_g corresponding to an element $g\in G$ is just formed of all elements depending on g, i. e. any two elements of the same minimal subset with property (P) are dependent.

If G is a torsion free group, then each A_λ, $\lambda\in\Lambda$, forms (together with the zero-element) clearly a minimal pure subgroup of G. On the contrary, the set of all the non-zero elements of a minimal pure subgroup of G containing a non-zero element g is just the subset A_g of our partition.

The second part of the theorem is then a consequence of Lemma 4.

Remark 1. If T is not a primary group, then two elements of the same minimal subset having property (P) need not be dependent. Moreover, it may well happen that there is even no element depending on each of them. In order to show an example, consider the direct sum of two cyclic groups of order 2 and 6, respectively:

$$C(2)\oplus C(6,) \quad \text{where} \quad C(2)=\{g_1\},\ C(6)=\{g_2\}.$$

By our theorem, only the subset of all non-zero elements has property (P). But it is easy to check that there is no element depending on g_1 and on $3g_2$ at the same time.

On the other hand, it can be proved that, for any two independent elements g_1 and g_2 of the same minimal subset $A\subseteq G_0$ with property (P) such that there

[2]) I. e., in particular $T\neq(0)$.

is no element depending on each of them, there exist two dependent elements h_1 and h_2 in A depending on g_1 and g_2, respectively. For, first of all, according to Theorem 1, the maximal torsion subgroup T of G is not a primary group and it is $A=T_0$. Now, for $i=1, 2$, let us denote by g_i^* a suitable multiple of g_i, such that $O(g_i^*)=p_i$, where p_i is a prime. If p_1 were different from p_2, then the element $g_1^*+g_2^*$ would obviously depend on both g_1^* and g_2^*, i. e. on both g_1 and g_2, as well, the latter contradicting our assumption. Thus, $p_1=p_2$. Since T is not a primary group, there is an element h of prime order $q\neq p_1$. In order to complete the proof it suffices to put $h_1=g_1^*+h$ and $h_2=g_2^*+h$.

E. g., in the above example, the elements g_1+2g_2 and $5g_2$ can be taken for h_1 and h_2.

Now, we are going to describe the subsets having properties (P'), (P'') and (P^*), respectively. First of all, let us consider the analogies to Lemmas 1, 2, 3 and 4 if (P') or (P'') or (P^*) is read instead of (P). We shall refer to these modified lemmas as to Lemma 1′, Lemma 1″, Lemma 1* etc. One can readily verify the validity of all of them with the exception of Lemma 4′(b) and Lemma 4″(b) which do not hold. Only Lemma 4*(a) requires perhaps a brief remark concerning the proof; applying the fact that the dependence is a property of finite character it can be proved by induction.

Theorem 1′. *Let G be an abelian group. Then the sets of the partition $(A_\lambda)_{\lambda\in\Lambda}$ of G_0 described in Theorem 1 are just all the minimal non-void subsets with property (P'). The following three statements are equivalent:*

(i) *A has property (P');*

(ii) *$A\cap T$ and $A\cap W$ have property (P');*

(iii) *A is a union of some A_λ such that with any two independent elements $a(\in A)$, $b(\in A)$ also $a+b\in A$.* [3])

If T is not a primary group, then T_0 is the only subset of T with property (P'). If T is a p-primary group, then $A\subseteq T$ has property (P') if, and only if, it is a union of some A_λ such that $A\cap T[p]$ together with the zero-element is a subgroup. If G is a torsion free group, then A has property (P') if, and only if, A forms, together with the zero-element, a pure subgroup of G.

PROOF. The validity of the first assertion is due to the fact that the independent subsets of A_λ consist of a single element only.

It is easy to see that, together with T_0, also W has property (P^*). Hence, a subset A has property (P') if, and only if, both $A\cap T=A\cap T_0$ and $A\cap W$ have it.

Moreover, a subset A which has property (P') is clearly a union of some A_λ and contains with any two independent elements also their sum. In particular, if the maximal torsion subset T is a p-primary group and $A\subset T$, then $A\cap T[p]$, together with the zero-element, is a subgroup. On the contrary, a subset A with property (P) (i. e., by Theorem 1, a union of some A_λ) such that the sum of any two its independent elements belongs to A again, contains even every linear combination $k_1g_1+k_2g_2+\dots+k_ng_n$ of independent elements $g_1, g_2, \dots, g_n$ of A. Hence, such a subset has property (P').

[3]) Thus, A has property (P') if, and only if, the subset $\bar{A}$ corresponding to A in the natural homomorphism $G\to\bar{G}=G/T$ forms, together with the zero-element, a pure subgroup of $\bar{G}$.

It remains to consider a subset A of a p-primary maximal torsion subgroup T such that it has property (P) and $A \cap T[p]$ forms (together with the zero-element) a subgroup. If g_1, g_2 are two independent elements of A, then g_1+g_2 is also in A. For, in the case that $O(g_1) \neq O(g_2)$, this follows immediately from Lemma 3′. In the other case, there is a natural k such that $kg_1 \neq 0$ and $kg_2 \neq 0$ belong to $A \cap T[p]$. Hence, $k(g_1+g_2) \neq 0$ is an element of A, i. e. $g_1+g_2 \in A$, as well.

The previous conclusions then complete the proof.

REMARK 2. Let us point out that as another consequence of the previous theorem the following equivalence is established:

A subset $A(\subseteq G_0)$ has property (P') if, and only if, every element of G_0 which depends on an independent subset of A containing one or two elements belongs to A.

Theorem 1″. *Let G be an abelian group. If G is not a torsion group, then a non-void subset A has property (P'') if, and only if, it forms, together with the zero-element, a pure subgroup of G icluding T. If G is a torsion group, then a non-void subset $A \neq G_0$ has property (P'') if, and only if, G is a primary group and A forms, together with the zero-element, an elementary pure proper subgroup (i. e. a non-trivial elementary direct summand) of G.* [4])

PROOF. First of all, we deduce immediately that a subset A has property (P'') if, and only if, it is a union of some A_λ (of Theorem 1) such that $A \cup (0)$ is a subgroup. Then, with the aid of Lemma 1″ and Lemma 3″ together with Theorem 1 and Theorem 1′, we can readily derive both parts of Theorem 1″.

Theorem 1*. *Let G be an abelian group. If there is a non-void proper subset A of G_0 having property (P^*), then G is a mixed group and A is either T_0 or W.*

PROOF. Denote by B the complement of A in G_0; it is $B \neq \emptyset$. Both A and B are unions of some A_λ (of Theorem 1). Thus, both A and B contain with an element also its multiples. In order to prove the theorem it suffices therefore to show that there are no elements $a \in A$ and $b \in B$ such that $O(a)=O(b)=\infty$ or $O(a)=O(b)=p$ with a prime p (for $O(a)=p$ and $O(b)=q$ with different primes p, q implies that both a and b are in the same A_λ, i. e. either both in A or in B).

Suppose, on the contrary, that there are $a \in A$ and $b \in B$ of the same order. Then, not only a and b, but also a and $a+b$ and further, b and $a+b$ are independent. Since a depends obviously on the subset $I=(b, a+b)$, $a+b$ can not belong neither to B $(A \cap I = \emptyset)$ nor to A $(A \cap I = (a+b))$, a contradiction. The theorem follows.

REMARK 3. Again, we have proved simultaneously that *a subset $A(\subseteq G_0)$ has property (P^*) if, and only if, every element of A depending on an independent subset I which consists of one or two elements of G depends on $I \cap A$.*

Corollary. *If G is a mixed group, then the maximal torsion subgroup is the only non-zero proper subgroup H of G such that $H \cap G_0$ has property (P^*).*

3. Let us conclude this paper with the following two remarks concerning „closed" subsets of an abelian group and coverings of an abelian group by subgroups.

For the purpose of the first question, let us introduce the following notation.

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[4]) In particular, if G is a primary elementary group, then A has property (P'') if, and only if, A is, together with the zero-element, a subgroup.

If A is a subset of G_0, denote by $cl(A)$ or $cl_i(A)$ or $cl_{ii}(A)$ the subset of G_0 of all the elements g such that there is an element in A or there is an independent subset of A or there is a subset of A such that g depends on it, respectively. Clearly, the operations $A \to cl(A)$, $A \to cl_i(A)$ and $A \to cl_{ii}(A)$ are extensive and isotone. The example of $C(2) \oplus C(6)$ in Remark 1 shows that they are not idempotent. On the other hand, the operations $Cl(A)$, $Cl_i(A)$ and $Cl_{ii}(A)$ (defined again on the system of all subsets A of G_0) mapping A onto the least subset of G_0 with property (P), (P') and (P''), respectively, are besides being extensive and isotone also idempotent, i. e. they are closure operations in the sense of E. H. MOORE (see e. g. G. BIRKHOFF [1]). Thus, all the „closed" subsets of the first type (i. e. all the subsets with property (P)) or of the second type (i. e. all the subsets with property (P')) or of the third type (i. e. all the subsets with property (P''))form a complete lattice, in which infimum means intersection, respectively.

Now, using the symbol $cl^{(n)}(A)$ for $cl(cl^{(n-1)}(A))$, $n \geqq 2$, [5]) and similarly for $cl_i(A)$ and $cl_{ii}(A)$, we can formulate

Theorem 2. *Let G be an abelian group and $A \subseteq G_0$. Then, in general, $cl(A) \neq cl^{(2)}(A) \neq cl^{(3)}(A) = cl^{(n)}(A) = Cl(A)$ for $n \geqq 3$, and similarly for $cl_i(A)$ and $cl_{ii}(A)$. If the maximal torsion subgroup of G is a primary group (i. e. especially, if G is a primary or a torion free group), then $cl(A) = Cl(A)$, $cl_i(A) = Cl_i(A)$ and, in general, $cl_{ii}(A) \neq cl_{ii}^{(2)}(A) = cl_{ii}^{(n)}(A) = Cl_{ii}(A)$ for $n \geqq 2$ unless G is a torsion free or an elementary primary group, in which case also $cl_{ii}(A) = Cl_{ii}(A)$.* [6])

PROOF. First of all, for a subset $A \subseteq G_0$, evidently $cl(A) \subseteq cl_i(A) \subseteq cl_{ii}(A)$ and $Cl(A) \subseteq Cl_i(A) \subseteq Cl_{ii}(A)$; further, $Cl(A)$ is a union of those A_λ of Theorem 1 which are not disjoint with A.

Let us remind that if an element g of infinite order depends on a set B and each element of B depends on a set A, then g depends on A. And moreover, if an arbitrary element g depends on an independent set I consisting of elements of infinite and/or prime power order and each element of I depends on A, then g depends on A.

Now, let G be a non-primary torsion group and A a non-void subset of G_0. Then, in view of Theorem 1, $Cl(A) = G_0$ and, according to Remark 1, $cl^{(3)}(A) = Cl(A)$. Hence, also $cl_i^{(3)}(A) = Cl_i(A) = G_0$ and $cl_{ii}^{(3)}(A) = Cl_{ii}(A) = G_0$ and the example of Remark 1 shows that, in general, $cl_{ii}^{(2)}(A) \neq cl_{ii}^{(3)}(A)$.

If G is a primary group, then, for a non-void subset $A \subseteq G_0$, clearly $cl(A) = Cl(A)$ and $cl_i(A) = Cl_i(A)$. The corresponding equality does not hold, in general, for $cl_{ii}(A)$: If $G = C_1(4) \oplus C_2(4)$, where $C_1(4) = \{g_1\}$ and $C_2(4) = \{g_2\}$, then, for $A = (g_1)$, we have $cl_{ii}(A) = cl_i(A) = cl(A) = (g_1, 2g_1, 3g_1, g_1 + 2g_2, 3g_1 + 2g_2)$, but $Cl_{ii}(A) = G_0$. Nevertheless, we are going to prove that, for every $A \subseteq G_0$, $cl_{ii}^{(2)}(A) = Cl_{ii}(A)$. If G is, in particular, an elementary group, then, evidently, $cl_{ii}(A) = Cl_{ii}(A)$; the same app'ies, by virtue of Theorem 1'', if G is a primary group and $Cl_{ii}(A) \neq G_0$. Thus, let G be a non-elementary p-primary group and $A \subseteq G_0$ such that $Cl_{ii}(A) = G_0$. Then, there is an element $g_0 \in cl_i(A)$ such that $0(g_0) = p^2$; for, otherwise, $cl_i(A) = Cl_i(A) = Cl_{ii}(A) \neq G_0$. Now, let $g \in G_0$ with $0(g) = p^N$. Then,

[5]) Of course, $cl^{(1)}(A)$ here means $cl(A)$.

[6]) In fact, we shall prove that if G is a primary group and $Cl_{ii}(A) \neq G_0$, then also $cl_{ii}(A) = Cl_{ii}(A)$.

$g_0 + p^{N-1}g \in cl_i(A)$ and thus, $g \in cl_{ii}(cl_i(A))$. Hence, $cl_{ii}^{(2)}(A) \supseteqq cl_{ii}(cl_i(A)) = G_0$, i. e., in fact, $cl_{ii}^{(2)}(A) = Cl_{ii}(A)$.

Arguments of a routine nature then complete the proof.

Now, let us turn our attention to the question of the following coverings of a non-zero abelian group G by its subgroups:

A collection $(G_\gamma)_{\gamma\in\Gamma}$ of non-zero subgroups of G is said to be a *g-covering of* G if $\bigcup_{\gamma\in\Gamma} G_\gamma = G$ and $G_{\gamma_1} \cap G_{\gamma_2} = (0)$ for every $\gamma_1 \neq \gamma_2$.

The relation between this concept and our previous investigations is described in

Theorem 3. *If $(G_\gamma)_{\gamma\in\Gamma}$ is a g-covering of an abelian group G, then each $G_\gamma \cap G_0$ has property (P''). In particular, each $G_\gamma \cap G_0$ is a union of some A_λ (of Theorem 1) and G_γ is pure in G.*

Proof. Clearly, every element depending on $G_\gamma \cap G_0$ should belong to $G_\gamma \cap G_0$. The rest is then obvious.

We have seen (Theorem 1″ and Theorem 1) that torsion free groups (of rank $\geqq 2$) possess non-trivial g-coverings (i. e. g-coverings consisting of proper subgroups). On the contrary, with the exception of elementary primary groups, they are the only groups permitting non-trivial g-coverings:

Theorem 4. *If there is a non-trivial g-covering of an abelian group G, then G is either an elementary primary or a torsion free group.*

Proof. In view of Theorem 1″, G can not be a mixed group; for, in this case, every member of a g-covering would contain the (non-zero) maximal torsion subgroup. Neither can G be, by the same theorem, a torsion group containing an element of order pq or p^2 with primes p, q. The theorem follows.

The structure of subgroups and therefore also the structure of g-coverings of an elementary primary group is very simple. It is interesting that from the point of view of g-coverings the behaviour of a torsion free group is to a certain extent very similar.

For two g-coverings $(G_\gamma)_{\gamma\in\Gamma}$ and $(G_\delta)_{\delta\in\Delta}$ of a torsion free group G, define the relation $(G_\gamma)_{\gamma\in\Gamma} \leqq (G_\delta)_{\delta\in\Delta}$ if for every G_γ there exists G_δ such that $G_\gamma \subseteqq G_\delta$. Then the set of all the g-coverings of G is (partly) ordered. Moreover, we can easily prove

Theorem 6. *All the g-coverings of an abelian torsion free group G form a complete lattice with respect to the ordering given above. The g-covering $(A_\lambda)_{\lambda\in\Lambda}$ described in Theorem 1 is the least and the g-covering consisting of G alone the greatest element.*

For any family $(H_\omega)_{\omega\in\Omega}$ of pure non-zero subgroups of G such that $H_{\omega_1} \cap H_{\omega_2} = (0)$ for every $\omega_1 \neq \omega_2$, there exists a g-covering containing all the subgroups H_ω, $\omega \in \Omega$, as members.

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