Note on commutable mappings

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Let us consider a system of mappings $x \to F_u x$ of a set X into itself, where u ranges over the elements of a set U. These mappings are called *commutable* if

$$(1) \qquad \forall u, v \in U, \quad x \in X, \quad F_u F_v x = F_v F_u x$$

holds. Commutable mappings play an impotant role e. g. in connection with the groups of iteratives of a given function [2]. A system of mappings $x \to F_u x$ is *transitive* if

$$\forall x \in X, \quad F_U x = X$$

holds, where the set of elements $F_u x$ $(u \in U)$ is denoted by $F_U x$.

Theorem. *Every transitive system of commutable mappings of a set X has the form*

$$x \to F_u x = x + \varphi u,$$

where $+$ is an abelian group operation on X and φ: $U \to X$ is a mapping of U onto the whole of X.

PROOF. First we prove that every transitive system of commutable mappings contains only $1-1$ and onto mappings.

In fact, we see that

$$F_u X = F_u F_U x = F_U F_u x = F_U y = X$$

is true for every $u \in U$ i. e. $x \to F_u x$ maps X onto the whole of X. Further, the suppositions

$$F_u x = F_u y = x_0, \quad y = F_v x$$

(such a v always exists by the supposition of transitivity) imply that

$$F_v x_0 = F_v F_u x = F_u F_v x = F_u y = x_0$$

and, consequently, for arbitrary $z = F_w x_0$,

$$F_v z = F_v F_w x_0 = F_w F_v x_0 = F_w x_0 = z$$

hence

$$y = F_v x = x$$

holds. Thus $x \to F_u x$ is $1-1$.

After this, choosing a fixed $x_0 \in X$, let us introduce the notation

$$\varphi u = F_u x_0 .$$

By this mapping of U onto X we can define a binary operation $x+y$ on X by

$$x+y=F_u x, \quad y=\varphi u. \tag{2}$$

This relation defines $x+y$ uniquely and independently from the special choice of u since, for $u \neq v$ with $y=\varphi u=\varphi v$, we have the same

$$x+y=F_u x=F_v x$$

as

$$F_u F_w x_0 = F_w F_u x_0 = F_w \varphi u = F_w \varphi v = F_w F_v x_0 = F_v F_w x_0$$

holds for all $x=F_w x_0$. Now, we verify that $x+y$ is an abelian group operation. In fact, if we define u, v by $x=F_u x_0$, $y=F_v x_0$, we have

$$x+y=F_v x=F_v F_u x_0=F_u F_v x_0=F_u y=y+x, \tag{3}$$

further,

$$(z+x)+y=F_v F_u z=F_u F_v z=(z+y)+x, \tag{4}$$

and applying successively (3), (4) and again (3) we have

$$(x+y)+z=(y+x)+z=(y+z)+x=x+(y+z).$$

Finally, as we have seen, $x \to x+y=F_v x$ is a $1-1$ mapping of X onto itself.

So, by (2), we have obtained the most general form of transitive system of commutable mappings. On the other hand, it is easy to verify, that these mappings $x \to F_u x = x+\varphi u$ are forming a transitive system of commutable mappings, if $x+y$ is an abelian group operation and φ maps U onto X.

Thus our theorem is proved.

Supposing that X is a topological space and $F_u x$ is a continuous function of x, the grou poperation $x+y$ defined by (2) will be a topological fuction of x, moreover, being $x+y$ commutative, it is topological also with respect to y group. Being every continuous group defined on an interval isomorphic to the real additive group, in the special case where X is an interval we have the following

Corollary. *Every transitive system of commutable and continuous mappings of a real interval X is isomorphic to the translations*

$$x \to F_u x = f^{-1}[f(x)+\varphi(u)],$$

where f is a $1-1$ mapping of X onto the real axis with inverse mapping f^{-1} and φ is a mapping of U onto the real axis.

This corollary seems to be useful in order to solve the functional equation of translation. See [1], [3].

Bibliography

[1] J. Aczél, Vorlesungen über Funktionalgleichungen und ihre Anwendungen, *Basel,* 1961.

[2] L. Berg, Unstetige Iterationsgruppen, *Publ. Math. Debrecen* **9** (1962), 47—56.

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