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Generalized power means for matrix functions II

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Abstract. In a recent paper, the authors obtained matrix versions of a number of inequalities involving generalized power means. Here these results are extended to corresponding inequalities for power means of several matrices.

1. Introduction

Generalized power means are defined by [1]:

$$M_{n,a}(x;w)_p = \left\{ \frac{\sum_{i=1}^{n} w_i x_i^{a+p}}{\sum_{i=1}^{n} w_i x_i^{p}} \right\}^{1/a}, \quad a \neq 0$$

(1)

$$= \exp \left\{ \frac{\sum_{i=1}^{n} w_i x_i^{p} \log x_i}{\sum_{i=1}^{n} w_i x_i^{p}} \right\}, \quad a = 0$$

where $a, p \in \mathbb{R}$, $x, w \in \mathbb{R}_+^n$, $n \in \mathbb{N}$. Concerning inequalities for these means see [2–5].

Matrix version of such results are obtained in [6]. Here, we shall give analogous results for several matrices.

2. Preliminaries

Let $A \in C^{n\times n}$ be a normal matrix, i.e., $A^*A = AA^*$. Here A^* means $\bar{A}^t$, the transpose conjugate of A. There exists [7] a unitary matrix U such that

$$A = U^*[\lambda_1, \lambda_2, \ldots, \lambda_n]U \tag{2}$$

where $[\lambda_1, \ldots, \lambda_n]$ is the diagonal matrix $(\lambda_j \delta_{ij})$, and where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are eigenvalues of A, each appearing as often as its multiplicity. A is Hermitian if and only if $\lambda_i, i \in I_n = \{1, 2, \ldots, n\}$ are real. If A is Hermitian and all λ_i are strictly positive, then A is said to be positive definite. Assume now that $f(\lambda_i) \in C$, $i \in I_n$ is well defined. Then $f(A)$ may be defined by (see e.g. [7, p. 71] or [8, p. 90])

$$f(A) = U^*[f(\lambda_1), f(\lambda_2), \ldots, f(\lambda_n)]U. \tag{3}$$

As before, if $f(\lambda_i)$, $i \in I_n$ are all real, then $f(A)$ is Hermitian. If, also, $f(\lambda_i) > 0$, $i \in I_n$, then $f(A)$ is positive definite.

We note that for the inner product

$$(f(A)x, x) = \sum_{i=1}^{n} |y_i|^2 f(\lambda_i) \tag{4}$$

where $y \in C^n$, $y = Ux$ and so $\sum_{i=1}^{n} |y_i|^2 = \sum_{i=1}^{n} |x_i|^2$.

If A is positive definite, so that $\lambda_i > 0$, $i \in I_n$ and $f(t) = t^r$ where $t > 0$ and $r \in \mathbb{R}$, we have $f(A) = A^r$.

3. Generalized power means for several matrices

Definition 1. Let A_j, $j = 1, \ldots, k$ be positive definite Hermitian matrices; $x_j \in C^n$, $j = 1, \ldots, k$; $a, p \in \mathbb{R}$, then the generalized power mean of A_j is defined by

$$\begin{aligned} M_k^p(A; x)_a &= \left\{ \frac{\sum_{j=1}^{k} (A_j^{a+p} x_j, x_j)}{\sum_{j=1}^{k} (A_j^p x_j, x_j)} \right\}^{1/a}, \quad a \neq 0, \\ &= \exp \left\{ \frac{\sum_{j=1}^{k} ((A_j^p \log A_j) x_j, x_j)}{\sum_{j=1}^{k} (A_j^p x_j, x_j)} \right\}, \quad a = 0. \end{aligned} \tag{5}$$

First, we shall prove the following result:

Theorem 1. *Let a, b, p, q satisfy*

$$\Big||a| - |b|\Big| + a + 2p \le b + 2q. \tag{6}$$

Then for every positive definite Hermitian matrix $A_j, x_j \in C^n$, $j = 1, \dots, k$

$$M_k^p(A;x)_a \le M_k^q(A;x)_b. \tag{7}$$

Proof. Using (4) we have

$$\begin{aligned} M_k^p(A;x)_a &= \left\{ \frac{\sum_{j=1}^{k}\sum_{i=1}^{n} |y_{ji}|^2 \lambda_{ji}^{a+p}}{\sum_{j=1}^{k}\sum_{i=1}^{n} |y_{ji}|^2 \lambda_{ji}^{p}} \right\}^{1/a}, \quad a \ne 0 \\ &= \exp\left\{ \frac{\sum_{j=1}^{k}\sum_{i=1}^{n} |y_{ji}|^2 \lambda_{ji}^{p} \log \lambda_{ji}}{\sum_{j=1}^{k}\sum_{i=1}^{n} |y_{ji}|^2 \lambda_{ji}^{p}} \right\}, \quad a = 0. \end{aligned} \tag{8}$$

Now, Theorem 1 is a simple consequence of the following lemma ([2]).

Lemma 1. *Let $a, b, p, q \in \mathbb{R}$. Then*

$$M_{n,a}(x;w)_p \le M_{n,b}(x;w)_q \tag{9}$$

holds for every $x \in \mathbb{R}^n$ $(x \ne 0)$, if and only if (6) *holds.*

Similarly, using other results for generalized power means we can obtain (see also [6]).

Theorem 2. *Let $a, b_1, b_2, \dots, b_s, p, q_1, \dots, q_s \in \mathbb{R}$, $s \ge 2$. Further, let*

$$\begin{aligned} Q_0 &= a^- - p, \quad Q_i = b_i^+ + q_i, \quad i = 1, \dots, s \\ Q_0^* &= a^+ + p, \quad Q_i^* = b_i^- - q_i, \quad i = 1, \dots, s \end{aligned} \tag{10}$$

where $a^+ = (|a| + a)/2$ and $a^- = (|a| - a)/2$, and for $i = 0, 1, \dots, s$, let

$$H_i = \begin{cases} \left(\sum_{\substack{j=0 \\ j \ne i}}^{s} Q_j^{-1} \right)^{-1}, & \text{when } \prod_{\substack{j=0 \\ j \ne i}}^{s} Q_j \ne 0 \\ 0, & \text{when } \prod_{\substack{j=0 \\ j \ne i}}^{s} Q_j = 0. \end{cases} \tag{11}$$

Let A_j, $j = 1, \ldots, k$, be normal matrices with eigenvalues in I $(I \subset C)$; $f_i : I \to \mathbb{R}_+$ be strictly positive functions for $i = 1, \ldots, s$; $x_j \in C^n$, $j = 1, \ldots, k$. If $Q_i \geq 0$ and $H_i \geq Q_i^*$ $(i = 0, \ldots, s)$ then

$$M_k^p((f_1 \ldots f_s)(A); x)_a \leq M_k^{q_1}(f_1(A); x)_{b_1} \ldots M_k^{q_s}(f_s(A); x)_{b_s}. \tag{12}$$

Theorem 3. *Let $a, b_1, \ldots, b_s, p, q_1, \ldots, q_s, A_j, x_j$ $(j = 1, \ldots, k)$, f_i $(i = 1, \ldots, s)$ be as in the previous theorem. If*

$$\max\{p + a^+, 1\} \leq q_i + b_i^+; \quad \max\{p - a^-, 0\} \leq \min\{q_i - b_i^-, 1\} \tag{13}$$

hold for every $i = 1, \ldots, s$, then

$$M_k^p((f_1 + \cdots + f_s)(A); x)_a \leq M_k^{q_1}(f_1(A); x)_{b_1} + \cdots + M_k^{q_s}(f_s(A); x)_{b_s}. \tag{14}$$

The reverse inequality in (14) holds if

$$\min\{p + a^+, 1\} \geq \max\{q_i + b_i^+, 0\}; \quad \min\{p - a^{-1}, 0\} \geq q_i - b_i^- \tag{15}$$

is valid for $i = 1, \ldots, s$.

Theorem 4. *Let A_j, $j = 1, \ldots, k$ be positive definite Hermitian matrices with eigenvalues λ_{ji} $(j = 1, \ldots, k;\ i = 1, \ldots, n)$ such that $0 < m \leq \lambda_{ji} \leq M$. Then*

$$M_k^q(A; x)_b \leq K(m, M) M^p(A; x)_a \tag{16}$$

where a, b, p, q are fixed numbers such that (6) holds and where $K(m, M)$ is defined by

$$K(m, M) = \Gamma_{b,q}(t_0, \gamma) / \Gamma_{a,p}(t_0, \gamma),$$

$\gamma = M/m$ and t_0 is the unique positive root of the equation

$$\lambda_{a,p}(\gamma)(\gamma^q + t)(\gamma^{b+q} + t) = \lambda_{b,q}(\gamma)(\gamma^p + t)(\gamma^{a+p} + t)$$

where, for $t > 0$,

$$\lambda_{a,p}(t) = \begin{cases} t^p \dfrac{t^{a-1}}{a}, & a \neq 0 \\ t^p \log t, & a = 0; \end{cases} \tag{17}$$

and

$$\Gamma_{a,p}(t, \gamma) = \begin{cases} ((\gamma^{a+p} + t)/(\gamma^p + t))^{1/a}, & a \neq 0 \\ \exp((\gamma^p \log \gamma)/(\gamma^p + t)), & a = 0. \end{cases} \tag{18}$$

4. Generalized quasi-arithmetic means for several matrices

Definition 2. Let $\phi : I \longrightarrow \mathbb{R}_+$ ($I \subset \mathbb{R}$) be a strictly positive function, $F : I \longrightarrow \mathbb{R}$ a strictly monotone function, A_j, $j = 1, \dots, k$, Hermitian matrices with eigenvalues in I, $x_j \in C^n$, $j = 1, \dots, k$. The generalized quasi-arithmetic mean of A_j is, for $x \neq 0$,

$$F_k(A; x, \phi) = F^{-1}\left\{ \frac{\sum_{j=1}^{k} ((\phi.F)(A_j)x_j, x_j)}{\sum_{j=1}^{k} (\phi(A_j)x_j, x_j)} \right\}. \tag{19}$$

The following result is a consequence of Theorem 1 from [9, p. 262]:

Theorem 5. *Let K, L, M be three differentiable strictly monotone functions from the closed interval I to $\mathbb{R}$; ϕ, ψ, χ, three functions from I to $\mathbb{R}_+$; $f : I^2 \longrightarrow I$ such that for all $u, v, s, t, \in I$ the following inequality is valid.*

$$\begin{aligned} &\left(\frac{M \circ f(u,v) - M \circ f(t,s)}{M' \circ f(t,s)} \right) \frac{\chi \circ f(u,v)}{\chi \circ f(t,s)} \\ &\leq \left(\frac{K(u) - K(t)}{K'(t)} \right) \frac{\phi(u)}{\phi(t)} f_1(t,s) + \left(\frac{L(v) - L(s)}{L'(s)} \right) \frac{\psi(v)}{\psi(s)} f_2(t,s). \end{aligned}$$

Let A_j, $j = 1, \dots, k$ be normal matrices with eigenvalues in J and let $g, h : J \longrightarrow I$ be given functions; $x_j \in C^n$, $j = 1, \dots, k$. Then

$$f(K_k(g(A); x, \phi), L_k(h(A); x, \psi)) \geq M_k(f(g(A), h(A)); x, \chi). \tag{20}$$

Theorem 6. *With the notation of the previous theorem,*

$$M_k(A; x, \chi) \leq K_k(A; x, \phi) \tag{21}$$

if, for all $u, t \in J$,

$$\frac{M(u) - M(t)}{M'(t)} \frac{\chi(u)}{\chi(t)} \leq \left(\frac{K(u) - K(t)}{K'(t)} \right) \frac{\phi(u)}{\phi(t)}. \tag{22}$$

Proof. Immediate from the previous theorem taking $f(x, y) = x$, $g(x) = x$.

An important special case of Theorem 5 is when $f(x,y) = x+y$. Then we get

$$K_k(g(A);x,\phi) + L_k(h(A);x,\psi) \geq M_k(g(A)+h(A);x,\chi) \tag{23}$$

holds if for all u, v, s, t in J, we have

$$\frac{M(u+v)-M(t+s)}{M'(t+s)}\frac{\chi(u+v)}{\chi(t+s)} \leq \frac{K(u)-K(t)}{K'(t)}\frac{\phi(u)}{\phi(t)} + \frac{L(v)-L(s)}{L'(s)}\frac{\psi(v)}{\psi(s)}.$$

Generalized quasi-arithmetic means are not only extensions of generalized power means but are also generalizations of the quasi-arithmetic means. Therefore, in the next section, we give some other results for the quasi-arithmetic means for matrix functions.

5. The quasi-arithmetic means for several matrices

Definition 3. Let $A_j,\ j = 1,\ldots,k$ be Hermitan matrices with eigenvalues $\lambda_{ji} \in J$ ($j = 1,\ldots,k;\ i = 1,\ldots,n$). Suppose that $F : J \to \mathbb{R}$ is a continuous and strictly monotone function. The quasi-arithmetic F-means is defined by

$$F_k(A;x) = F^{-1}\left\{\sum_{j=1}^{k}(F(A_j)x_j, x_j)\right\} \tag{24}$$

where $x_j \in C^n,\ j = 1,\ldots,k$ with $\sum_{j=1}^{k}(x_j,x_j) = 1$.

Theorem 7. *Let F, G be two continuous functions with domain J, G increasing (decreasing). Then*

$$F_k(A;x) \leq G_k(A;x) \tag{25}$$

holds if G is convex (concave) with respect to F, i.e., if the function $\phi(t) = (G \circ F^{-1})(t)$ is convex (concave). If G is decreasing (increasing) and G is convex (concave) with respect to F, inequality (25) is reversed.

This is a consequence of Theorem 4 from [9, pp. 226]. Similarly, we can use results for three means [9, pp. 246–253].

Let $K : [k_1,k_2] \to \mathbb{R},\ L : [\ell_1,\ell_2] \to \mathbb{R},\ M : [m_1,m_2] \to \mathbb{R},\ f : [k_1,k_2] \times [\ell_1,\ell_2] \to [m_1,m_2];\ g : J \to [k_1,k_2],\ h : J \to [\ell_1,\ell_2]$ be given functions and let A_j be a Hermitian matrix with eigenvalues $\lambda_{ji} \in J$ ($j =$

$1, \dots, k;\ i = 1, \dots, n$). Here K, L and M are twice differentiable and strictly monotone functions, M is increasing. Consider the inequality

$$f(K_k(g(A);x), L_k(h(A);x)) \geq M_k(f(g(A),\ h(A));x) \tag{26}$$

or its reverse.

Theorem 8. *Inequality* (26) *holds if the function* $H(s,t) = M(f(K^{-1}(s), L^{-1}(t)))$ *is concave. If H is convex, then inequality* (26) *is reversed.*

Remark. Theorem 8 is a generalization of Hörder's and Minkowski's inequalities. Thus, if $f(u,v) = u+v$, when $H(s,t) = M(K^{-1}(s)+L^{-1}(t))$, $E = K'/K''$, $T = L'/L''$, $S = M'/M''$ and all of $K', L', M', K'', L'', M''$ are positive, then (56) holds if $S(u+v) \geq E(u) + T(v)$. Moreover, if $f(u,v) = uv$ when $H(s,t) = M(K^{-1}(s)L^{-1}(t))$,

$$A(u) = \frac{K'(u)}{K'(u) + uK''(u)}, \quad B(u) = \frac{L'(u)}{L'(u) + uL''(u)},$$
$$C(u) = \frac{M'(u)}{M'(u) + uM''(u)}$$

and K', L', M', A, B, C are all positive, then (56) holds if $C(uv) \geq A(u) + B(v)$.

A special case of Theorem 8 is also [9, p. 253]

$$F_k\left(\tfrac{1}{2}(g(A) + h(A)); x\right) \leq \tfrac{1}{2}\{F_k(g(A);x) + F_k(h(A);x)\} \tag{27}$$

where the function F has continuous second derivatives and is strictly increasing and strictly convex and F'/F'' is concave.

We now give two converse inequalities. These inequalities can be obtained as consequences of results from [10]:

Theorem 9. *Let F and G be two strictly monotone continuous functions defined on J, G increasing (decreasing) and G convex (concave) with respect to F. Let $A_j,\ j = 1, \dots, k$ be Hermitian matrices with eigenvalues in $[m, M]$. Then*

$$\begin{aligned}(F(M) - F(m))G_k(A;x) - (G(M) - G(m))F_k(A;x) \\ \leq F(M)G(m) - G(M)F(m).\end{aligned} \tag{28}$$

If G is decreasing (increasing) and G is convex (concave) with respect to F, inequality (28) *is reversed.*

Theorem 10. *Let $\phi(u,v)$ be a real function defined on $J \times J$, nondecreasing in u, G increasing and convex with respect to F. Let A_j, $j = 1, \dots, k$ be Hermitian matrices with eigenvalues in $[m, M]$. Then*

$$\phi(G_k(A;x), F_k(A;x)) \tag{29}$$
$$\leq \max_{\vartheta\in[0,1]} \phi[G^{-1}(\vartheta G(m) + (1-\vartheta)G(M)),\ F^{-1}(\vartheta F(m) + (1-\vartheta)F(M))].$$

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