On Linnik's theorem concerning exceptional L-zeros

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1. Let $p(l, k)$ denote the least prime number in the arithmetical progression

$$l, l+k, l+2k, \dots$$

where $0<l<k$, $(l, k)=1$, $k \geqq 3$.

LINNIK estimated [1], [2] $p(l, k)$ from above as follows

$$p(l, k)<k^C, \quad C \text{ a constant.} \tag{1.1}$$

LINNIK's proof of (1. 1) rests on two deep theorems concerning the distribution of zeros of L-functions. Later RODOSSKII [5] simplified the proof but its essential idea remained unchanged: the above mentioned theorems steadily stand at the bottom of it. These are (see [5], also [4]):

Theorem I. *Suppose $k \geqq 3$ and real λ to satisfy*

$$0 \leqq \lambda \leqq \log k.$$

Let $N_0=N(\lambda, k)$ be the number of zeros of $\prod_{\chi \bmod k} \mathfrak{L}(s, \chi)$ in the rectangle

$$\sigma \geqq 1-\frac{\lambda}{\log k}, \quad |t| \leqq \frac{e^\lambda}{\log k}$$

Then[1]

$$N_0 \leqq e^{c_1 \lambda}.$$

Theorem II.[2]) *There exist constants c_2, c_3 such that if β_1 is a zero of $\mathfrak{L}(s, \chi_1)$* mod $k(\geqq 3)$, *satisfying*

$$1-\frac{c_2}{\log k} \leqq \beta_1<1, \tag{1.2}$$

then writing

$$\delta_1=1-\beta_1$$

we have

$$\prod_{\chi \bmod k} \mathfrak{L}(s, \chi) \neq 0$$

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[1]) $c_1, c_2, c_3, \dots$ denote positive numerical constants throughout.

[2]) What I quote is a somewhat stronger form of the theorem, due to K. A. RODOSSKII.

for $s \neq \beta_1$ *in the domain*

$$(1.3) \qquad \sigma \geqq 1 - \frac{c_3}{\log k(|t|+1)} \cdot \log \frac{c_2 e}{\delta_1 \log k(|t|+1)}, \qquad \delta_1 \log k(|t|+1) \leqq c_2.$$

Recently Professor TURÁN obtained, using his method, an alternative (and much simpler) proof of Theorem I. It was just about that time when I was in Budapest so that, thanks to Professor TURÁN's kindness, had an opportunity to read over the manuscript of his paper (which in the meantime appeared as [8]). Professor TURÁN encouraged me also to try to prove Theorem II along similar lines. Such proof will be the subject of this paper.

For our proof of Theorem II we shall use the old inequality of PAGE (see [3])

$$(1.4) \qquad \delta_1 > \frac{c_4}{k^{1/2} \log k}$$

(and even only

$$(1.4)' \qquad \delta_1 > \frac{c_5}{k}),$$

and the following improved form of a theorem of TURÁN (see [6]: the original form was theorem X in [7])

If $z_1, z_2, \ldots, z_N$ *with*

$$|z_1| \geqq |z_2| \geqq \ldots \geqq |z_N|$$

are arbitrary complex, $g > 0$ *arbitrary and* $N \leqq M$, *then there is an integer* $\varkappa$ *with*

$$(1.5) \qquad g \leqq \varkappa \leqq g + M$$

such that

$$(1.6) \qquad |z_1^{\varkappa} + z_2^{\varkappa} + \ldots + z_N^{\varkappa}| \geqq \left(\frac{M}{23(g+M)}\right)^M \cdot |z_1|^{\varkappa}.$$

It may be noted that in all probability the present proof of Theorem II will provide better (i. e. greater) constants c_2, c_3 which would be of importance for numerical estimations of C in (1. 1).

2. Write

$$f(s, \chi) = \mathfrak{L}(s, \chi) \cdot \mathfrak{L}(s + \delta_1, \chi\chi_1), \qquad \chi \bmod k$$

and suppose $\varrho_0 = \beta_0 + i\gamma_0$, $\beta_0 \geqq \frac{1}{2}$, $\varrho_0 \neq \beta_1$ to be a zero of $f(s, \chi)$. Writing

$$\delta = 1 - \beta_0$$
$$U = 1 + |\gamma_0|$$
$$\lambda = \delta \log kU,$$

and supposing $\chi \neq \chi_0$ (the main character) we shall prove

$$(2.1) \qquad \delta_1 \log kU > c_6 e^{-c_7 \lambda}$$

The inequality (2. 1) gives clearly the statement of Theorem II. It follows in fact that $\prod_{\chi \neq \chi_0} f(s, \chi) \neq 0$ in the domain (1. 3) (with some suitable c_2, c_3). In particular

$f(s, \chi_1) \neq 0$ either, which means that if $\varrho^* = \beta^* + i\gamma^*$, $\beta^* \geqq \frac{1}{2} + \delta_1$, $\delta_1 \log k(1 + |\gamma^*|) \leqq c_2$ is a zero of $\mathfrak{L}(s, \chi_0)$ then $(\varrho^* - \delta_1$ being a zero of $f(s, \chi_1))$

$$\beta^* - \delta_1 < 1 - \frac{c_3}{\log k\,(|\gamma^*| + 1)} \log \frac{c_2 e}{\delta_1 \log k\,(|\gamma^*| + 1)}. \tag{2.2}$$

Since the statement of Theorem II is not touched by diminishing of c_2, c_3, it may obviously be assumed that $c_3 = 2c_2$. (2. 2) then yields

$$\beta^* < 1 - \frac{2c_2}{\log k\,(|\gamma^*| + 1)} \log \frac{c_2 e}{\delta_1 \log k\,(|\gamma^*| + 1)} + \delta_1 \leqq 1 -$$

$$- \frac{c_2}{\log k\,(|\gamma^*| + 1)} \cdot \log \frac{c_2 e}{\delta_1 \log k(|\gamma^*| + 1)}.$$

This shows that $\prod\limits_{\chi \bmod k} \mathfrak{L}(s, \chi) \neq 0$, apart from $s = \beta_1$, in the domain (1. 3) with some (eventually smaller than those above) constants c_2, c_3.

It will be enough to prove Theorem II for k sufficiently large. In fact, for "small" values of k c_2 may be chosen in such a way as to have no zeros of $\prod\limits_{\chi \bmod k} \mathfrak{L}(s, \chi)$ in the interval (1. 2).

We introduce a certain number $\alpha > 1$, to be numerically determined later, write for brevity $f(s, \chi) = f(s)$, and supply a number of following simple lemmas.

(i) *There exists a zero* $\varrho' = \beta' + i\gamma'$ *of* $f(s)$ *such that*

$$\beta' \geqq \beta_0, \ |\gamma' - \gamma_0| \leqq e^{5\alpha}\lambda \tag{2.3}$$

and $f(s) \neq 0$ *for*

$$\beta' + \frac{1}{\alpha \log kU} \leqq \sigma \leqq 2, \ |t - \gamma'| \leqq e^{4\alpha} \tag{2.4}$$

(ϱ' *may happen to be just* ϱ_0).

Let us consider first the rectangle $\beta_0 + \dfrac{1}{\alpha \log kU} \leqq \sigma \leqq 2$, $|t - \gamma_0| \leqq e^{4\alpha}$. If $f(s)$ did not vanish there, lemma (i) would be true with $\varrho' = \varrho_0$. If not, there would exist some f-zero $\varrho^{(1)} = \beta^{(1)} + i\gamma^{(1)}$ with $\beta^{(1)} > \beta_0 + \dfrac{1}{\alpha \log kU}$, $|\gamma^{(1)} - \gamma_0| \leqq e^{4\alpha}$. In this case we should consider the rectangle $\beta^{(1)} + \dfrac{1}{\alpha \log kU} \leqq \sigma \leqq 2$, $|t - \gamma^{(1)}| \leqq e^{4\alpha}$ and repeat the former argument. Proceeding on that way we must arrive in q steps $\left(\text{where } \dfrac{q}{\alpha \log kU} < \delta\right)$ at a $\varrho' = \beta' + i\gamma'$ satisfying (2. 4). It follows that $q < \alpha\lambda$ and

$$|\gamma' - \gamma_0| \leqq e^{4\alpha}\alpha\lambda < \lambda e^{5\alpha}, \qquad \text{Q. e. d.}$$

Let us write now

$$\delta' = 1 - \beta'$$
$$U' = 1 + |\gamma'|$$
$$\lambda' = \delta' \log kU'.$$

(ii) *We have* $\lambda, \lambda' \geqq c_8$ $(0 < c_8 < 1)$ (see [4], 130, Satz 6. 9).

(iii) *Let* $Q = Q(t_0, \omega)$ *be the square*

$$t_0 - \omega \leqq t \leqq t_0 + \omega$$
$$1 - 2\omega \leqq \sigma \leqq 1,$$

where $0 < \omega \leqq 1$. *Then the number of zeros of* $f(s)$ *in* Q:

$$V(\omega, t_0, \chi) < c_9 \omega \log k\,(|t_0| + 1) \tag{2. 5}$$

(2. 5) is trivial for $\omega < c_{10}(\log k(|t_0|+1))^{-1}$ with a suitable c_{10} since then $V(\omega, t_0, \chi) = 0$ (see [4], 130, Satz 6. 9). Hence it may be assumed

$$\omega \geqq c_{10}(\log k\,(|t_0|+1))^{-1} \tag{2. 6}$$

Noting the approximate formula (see [4], 350)

$$\frac{f'}{f}(s_0) = \sum_{|\gamma - t_0| \leqq 1} \frac{1}{s_0 - \varrho} + 0(\log k\,(|t_0|+1)),$$

where $s_0 = 1 + \omega + it_0$ and $\varrho = \beta + i\gamma$ runs through the zeros of $f(s)$, using further (2. 6) and the (simple) inequalities

$$R\frac{f'}{f}(s_0) \leqq \frac{c_{11}}{\omega}$$

and

$$R\sum_{|\gamma - t_0| \leqq 1} \frac{1}{s_0 - \varrho} \geqq R\sum_{\varrho \in Q} \frac{1}{s_0 - \varrho} \geqq \frac{V(\omega, t_0, \chi)}{10\omega}$$

we obtain the result.

3. Let integer r, to be fixed later, satisfy the inequalities

$$c_{12}\lambda' \leqq r \leqq c_{13}\lambda', \tag{3. 1}$$

where $c_{12} = \frac{12}{c_8}$, $c_{13} = c_{12} + c_9$. We have in particular

$$r \geqq 12, \ c_{12} \geqq 12. \tag{3. 2}$$

Let us put further

$$A = \frac{2\alpha}{\delta'}, \ B = \frac{\alpha}{\delta'}.$$

We have

$$\frac{f'}{f}(s) = -\sum_{n=1}^{\infty} b_n \Lambda(n) n^{-s}, \ \sigma > 1,$$

where

$$b_n = \begin{cases} \chi(p^m)\{1 + \chi_1(p^m)p^{-m\delta_1}\} & \text{if} \quad n = p^m, \ m \geqq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Starting from the integral

$$J_r(\varrho')=-\frac{1}{2\pi i}\int\limits_{(2)}\left\{e^{Aw}\frac{e^{Bw}-e^{-Bw}}{2Bw}\right\}^r\cdot\frac{f'}{f}(\varrho'+w)\,dw$$

we have by term-by-term integration

$$J_r(\varrho')=\sum_{n=1}^{\infty}\frac{b_n\Lambda(n)}{n^{\varrho'}}\cdot\frac{1}{2\pi i}\int\limits_{(2)}\left\{e^{Aw}\frac{e^{Bw}-e^{-Bw}}{2Bw}\right\}^r\frac{dw}{n^w}. \tag{3. 3}$$

Noting that the integrals at the right-hand side of (3. 3) vanish for $n\geqq e^{(A+B)r}$ and also, if we move the line of integration to-say-$\sigma=-1$, for $n\leqq e^{(A-B)r}$, we obtain

$$J_r(\varrho')=\sum_{e^{(A-B)r}\leqq n\leqq e^{(A+B)r}}\frac{b_n\Lambda(n)}{n^{\varrho'}}R_r(n,A,B), \tag{3. 4}$$

where

$$R_r(n,A,B)\overset{\text{def}}{=}\frac{1}{2\pi i}\int\limits_{(0)}\left\{e^{Aw}\frac{e^{Bw}-e^{-Bw}}{2Bw}\right\}^r\frac{dw}{n^w}=\frac{1}{2\pi}\int\limits_{-\infty}^{+\infty}e^{(Ar-\log n)it}\left(\frac{\sin Bt}{Bt}\right)^r dt.$$

Hence it is easily seen that

$$|R_r(n,A,B)|\leqq\frac{2}{B\pi}<\frac{\delta'}{\alpha}. \tag{3. 5}$$

By (3. 4), (3. 5) we get, putting $h_1=\alpha c_{12}$, $h_2=3\alpha c_{13}$,

$$|J_r(\varrho')|\leqq\frac{\delta'}{\alpha}\sum_{(kU')^{h_1}\leqq p^m\leqq(kU')^{h_2}}\frac{1+\chi_1(p^m)p^{-m\delta_1}}{p^{(1-\delta')m}}\log p\leqq$$

$$\leqq 3c_{13}\lambda'(kU')^{3\alpha c_{13}\delta'}\sum_{(kU')^{h_1}\leqq p^m\leqq(kU')^{h_2}}\frac{1+\chi_1(p^m)p^{-m\delta_1}}{p^m}$$

and finally

$$|J_r(\varrho')|\leqq 3c_{13}e^{(3\alpha c_{13}+1)\lambda'}\sum_{(kU')^{h_1}\leqq p^m\leqq(kU')^{h_2}}\frac{1+\chi_1(p^m)p^{-m\delta_1}}{p^m}, \tag{3. 6}$$

where $h_1=\alpha c_{12}$, $h_2=3\alpha c_{13}$.

4. It follows by the theorem of residues

(4. 1)

$$J_r(\varrho')=-\sum_{\varrho}\left(e^{A(\varrho-\varrho')}\frac{e^{B(\varrho-\varrho')}-e^{-B(\varrho-\varrho')}}{2B(\varrho-\varrho')}\right)^r-\frac{1}{2\pi i}\int\limits_{(-\frac{1}{2}-\beta')}\left\{e^{Aw}\frac{e^{Bw}-e^{-Bw}}{2Bw}\right\}^r\cdot\frac{f'}{f}(\varrho'+w)\,dw,$$

where ϱ denotes the zeros of $f(s)$. Further we have for $w=-\frac{1}{2}-\beta'+it$ (see [4], 227)

$$\left|\frac{f'}{f}(\varrho'+w)\right|\leqq c_{14}\log k\,(|t|+|\gamma'|+1)\leqq c_{14}\log k\,(|t|+1)(|\gamma'|+1)=$$

$$=c_{14}(\log k\,U'+\log(|t|+1)),$$

whence

$$\left| \int_{(-1/2-\beta')} \left\{ e^{Aw} \frac{e^{Bw}-e^{-Bw}}{2Bw} \right\}^r \cdot \frac{f'}{f}(\varrho'+w)\,dw \right| \leqq$$

$$\leqq c_{14} \frac{e^{-\frac{\alpha}{\delta'} r(1/2+\beta')}}{B^r} \left(\int_{-\infty}^{+\infty} \frac{\log kU'}{(1+t^2)^{r/2}} dt + \int_{-\infty}^{+\infty} \frac{\log(|t|+1)}{(1+t^2)^{r/2}} dt \right) \leqq c_{15} \frac{1}{B^r} e^{-\frac{\alpha}{\delta'} c_{12}\lambda'} \log k\, U' \leqq$$

$$\leqq \frac{c_{15} \log kU'}{\alpha^r (kU')^{\alpha c_{12}}} < \frac{c_{15} \log kU'}{\alpha^r (kU')^{12}} < \frac{1}{\alpha^r},$$

if k is sufficiently large.

This and (4. 1) give

$$|J_r(\varrho')| \geqq |\sum_{\varrho} a_\varrho^r| - \frac{1}{\alpha^r} \qquad (\text{if } k \geqq c_{16}), \tag{4. 2}$$

where

$$a_\varrho = e^{A(\varrho-\varrho')} \frac{e^{B(\varrho-\varrho')} - e^{-B(\varrho-\varrho')}}{2B(\varrho-\varrho')} \qquad \text{if } \varrho \neq \varrho'$$

$$a_{\varrho'} = 1.$$

5. In this section we shall estimate from above the sum

$$I = \sum_{\varrho \in \overline{Q}(\gamma', \delta')} |a_\varrho|^r.$$

Let $K_{\mu\nu}$ $\left(\mu = 1, 2, \ldots, \left[\frac{2}{\delta'}\right]; \nu = 1, 2, \ldots\right)$ denote the square

$$\beta' - \mu\delta' \leqq \sigma \leqq \beta' - (\mu - 1)\delta',$$

$$\gamma' + \nu\delta' \leqq t \leqq \gamma' + (\nu+1)\delta'.$$

The number of f-zeros in $K_{\mu\nu}$ is by (iii)

$$\leqq V\left(\gamma' + \left(\nu + \frac{1}{2}\right)\delta', \ \frac{\mu+1}{2}\delta', \chi\right) \leqq \frac{c_9(\mu+1)}{2} \delta' \log k \left(U' + \left(\nu + \frac{1}{2}\right)\delta' \right),$$

whence

$$I_{\mu\nu} \overset{\text{def}}{=} \sum_{\varrho \in K_{\mu\nu}} |a_\varrho|^r \leqq c_9 \frac{(\mu+1)}{2} \delta' \log k \left(U' + \left(\nu + \frac{1}{2}\right)\delta' \right) e^{-\alpha(\mu-1)r} \cdot \frac{1}{(\alpha\nu)^r}.$$

Noting that $\frac{1}{\nu} \log k \left(U' + \left(\nu + \frac{1}{2}\right)\delta' \right)$ decreases for $\nu \geqq 1$ we have

$$I_{\mu\nu} \leqq \frac{c_9}{2} (\mu+1)\delta' \log k \left(U' + \frac{3}{2}\delta' \right) \cdot \frac{e^{-\alpha(\mu-1)r}}{\alpha^r \nu^{r-1}} < c_9(\mu+1) \frac{\lambda' e^{-\alpha(\mu-1)r}}{\alpha^r \nu^{r-1}}.$$

Further

$$I_\mu \overset{\text{def}}{=} \sum_{\nu=1}^{\infty} I_{\mu\nu} \leqq c_9(\mu+1)\lambda' \frac{e^{-\alpha(\mu+1)r}}{\alpha^r} \sum_{\nu=1}^{\infty} \frac{1}{\nu^{11}} = c_{17}(\mu+1)\frac{\lambda' e^{-\alpha(\mu-1)r}}{\alpha^r}$$

and

$$\sum_{\mu \leqq 2/\delta'} I_\mu \leqq c_{17}\frac{\lambda'}{\alpha^r} \sum_{\mu=1}^{\infty} \frac{(\mu+1)}{e^{\alpha(\mu-1)r}} < c_{17}\frac{\lambda'}{\alpha^r}\sum_{\mu=1}^{\infty}\frac{\mu+1}{e^{\mu-1}} = \frac{c_{18}\lambda'}{\alpha^r}.$$

Next we estimate the number of zeros in the squares

$$K_\mu: \quad \begin{gathered} \beta' - \mu\delta' \leqq \sigma \leqq \beta' - (\mu-1)\delta', \\ \gamma' \leqq t \leqq \gamma' + \delta', \\ \mu = 2, 3, \ldots, \left[\frac{2}{\delta'}\right]. \end{gathered}$$

Similarly as before

$$\sum_{2 \leqq \mu \leqq 2/\delta'} \sum_{\varrho \in K_\mu} |a_\varrho|^r \leqq \sum_{2 \leqq \mu \leqq 2/\delta'} e^{-(\mu-1)\alpha r} \cdot \frac{1}{\alpha^r} \cdot \frac{c_9}{2}(\mu+1)\lambda' < c_9\frac{\lambda'}{\alpha^r}\sum_{\mu=2}^{\infty}\frac{\mu+1}{e^{(\mu-1)\alpha r}} < \frac{c_{19}\lambda'}{\alpha^r}.$$

Finally, bearing in mind (i), we have to cope with the zeros in the domains

$$D_1: \quad \begin{gathered} \beta' \leqq \sigma \leqq \beta' + \frac{1}{\alpha \log kU} \\ \gamma' + \delta' \leqq t \leqq \gamma' + e^{4\alpha} \end{gathered}$$

and

$$D_2: \quad \begin{gathered} \beta' \leqq \sigma < 1 \\ t > \gamma' + e^{4\alpha}. \end{gathered}$$

We have

$$I_{D_1} \overset{\text{def}}{=} \sum_{\varrho \in D_1} |a_\varrho|^r \leqq \frac{c_9}{2}\sum_{\nu=1}^{\infty} \frac{e^{(A+B)\frac{r}{\alpha \log kU}}}{(B\nu\delta')^r}\delta' \log k\left(U' + \left(\nu+\frac{1}{2}\right)\delta'\right) \leqq$$

$$\leqq c_9 \frac{\exp\left(\frac{3}{\delta'}\cdot\frac{r}{\log kU}\right)}{\alpha^r}\lambda' \sum_{\nu=1}^{\infty}\frac{1}{\nu^{11}} = c_{20}\frac{\lambda'}{\alpha^r}\exp\left(\frac{3}{\delta'}\frac{r}{\log kU}\right).$$

But

$$\frac{3}{\delta'}\frac{r}{\log kU} \leqq \frac{3c_{13}\log kU'}{\log kU} \leqq \frac{3c_{13}\log\{k(U+\lambda e^{5\alpha})\}}{\log kU} < \frac{3c_{13}\log\{k(U+e^{5\alpha}kU)\}}{\log_k U}$$

$$< \frac{3c_{13}(2\log kU + 6\alpha)}{\log kU} < 7c_{13}, \quad \text{if} \quad k \geqq k_0(\alpha).$$

Hence

$$I_{D_1} \leqq c_{21}\frac{\lambda'}{\alpha^r} \quad \text{for} \quad k \geqq k_0(\alpha)$$

Further

$$\sum_{\varrho\in D_2}|a_\varrho|^r \leqq c_9 e^{3\alpha r}\sum_{\nu=\left[\frac{e^{4\alpha}}{\delta'}\right]}^{\infty}\frac{\delta'\log\{k(U+(\nu+{}^1/_2)\delta')\}}{(B\nu\delta')^r} < c_9 e^{3\alpha r}\frac{\lambda'}{\alpha^r}\sum_{\nu=\left[\frac{e^{4\alpha}}{\delta'}\right]}^{\infty}\frac{1}{\nu^{r-1}}$$

$$< c_9 e^{3\alpha r}\frac{\lambda'}{\alpha^r}e^{-4\alpha(r-3)}\sum_{\nu=1}^{\infty}\frac{1}{\nu^2} < c_{22}\frac{\lambda'}{\alpha^r}$$

(the latter by (3. 2)).

Estimating similarly in domains symmetrical to $K_{\mu\nu}$, K_μ, D_1, D_2 with respect to $t=\gamma'$, we obtain all in all

$$I < c_{23}\frac{\lambda'}{\alpha^r} \quad \text{for} \quad k \geqq k_0(\alpha),$$

whence and by (4. 2)

$$|J_r(\varrho')| \geqq \Big|\sum_{\varrho\in Q(\gamma',\delta')} a_\varrho^r\Big| - c_{24}\frac{\lambda'}{\alpha^r} \quad \text{for} \quad k\geqq k_0(\alpha) \tag{5. 1}$$

6. Now we shall estimate from below the sum

$$\Big|\sum_{\varrho\in Q(\gamma',\delta')} a_\varrho^r\Big| \overset{\text{def}}{=} S_r.$$

We use (1. 6) putting $g=c_{12}\lambda'$, $M=c_9\lambda'$, whence ($N\leqq M$ and (1. 5) being obviously satisfied owing to (iii) and (3. 1)) with a suitable r

$$S_r > \left(\frac{c_9}{23c_{13}}\right)^{c_9\lambda'} = e^{-c_{25}\lambda'}. \tag{6. 1}$$

Fixing now α numerically so as to have

$$e^{-c_{25}\lambda'} - c_{24}e^{-\lambda'(c_{12}\log\alpha-1)} > e^{-2c_{25}\lambda'},$$

and by (5. 1), (6. 1), we get

$$|J_r(\varrho')| > e^{-2c_{25}\lambda'} \tag{6. 2}$$

with (already!) $k\geqq c_{26}$.

(6. 2) and (3. 6) give with some $c_{27}\geqq 12$

$$\sum_{(kU')^{c_{27}}\leqq p^m\leqq(kU')^{c_{28}}}\frac{1+\chi_1(p^m)p^{-m\delta_1}}{p^m} > e^{-c_{29}\lambda'}. \tag{6. 3}$$

Let us note that owing to $|U'-U|\leqq c_{30}\lambda\leqq c_{30}\log kU$ we have $\log kU'\leqq 2\log kU$ and similarly $\log kU\leqq 2\log kU'$. Noting further that $\delta'\leqq\delta$ we get from (6. 3), with some $c_{31}\geqq 6$,

$$\sum_{(kU)^{c_{31}}\leqq p^m\leqq(kU)^{c_{32}}}\frac{1+\chi_1(p^m)p^{-m\delta_1}}{p^m} > e^{-c_{33}\lambda}. \tag{6. 4}$$

7. It may be assumed that

$$(7.1)\qquad 1-\frac{1}{p^{m\delta_1}}\leqq e^{-c_{34}\lambda}\quad\text{for}\quad p^m\leqq (kU)^{c_{32}}$$

with an arbitrarily fixed c_{34}.

For, in the opposite case we should have

$$e^{-c_{34}\lambda}<1-\frac{1}{(kU)^{\delta_1 c_{32}}},$$

whence

$$e^{c_{34}\lambda}>\frac{1}{1-\frac{1}{(kU)^{\delta_1 c_{32}}}}>\frac{1}{c_{32}\delta_1\log kU},$$

i. e. (2. 1).

(7. 1) gives then

$$\sum_{\substack{(kU)^{c_{31}}\leqq p^m\leqq (kU)^{c_{32}}\\ \chi_1(p^m)=-1}}\frac{1+\chi_1(p^m)p^{-\delta_1 m}}{p^m}<e^{-c_{34}\lambda}\left(\sum_{(kU)^{c_{31}}\leqq p\leqq (kU)^{c_{32}}}\frac{1}{p}+\sum_p\sum_{m=2}^{\infty}\frac{1}{p^m}\right)<c_{35}e^{-c_{34}\lambda}$$

whence, by (6. 4) and with c_{34} sufficiently large,

$$(7.2)\qquad \sum_{\substack{(kU)^{c_{31}}\leqq p^m\leqq (kU)^{c_{32}}\\ \chi_1(p^m)=1}}\frac{1}{p^m}>e^{-c_{36}\lambda},\ (c_{31}\geqq 6),$$

k being supposed to be $\geqq c_{26}$.

Writing

$$S=\sum_{n\leqq (kU)^2}\frac{a_n}{n},\qquad\text{where}\qquad a_n=\sum_{d|n}\chi_1(d)=\prod_{p^m|n,\,p^{m+1}\nmid n}\{1+\chi_1(p)+\dots+\chi_1(p^m)\},$$

we get by (7. 2)

$$Se^{-c_{36}\lambda}<\sum_{n_1\leqq (kU)^2}\frac{a_{n_1}}{n_1}\sum_{\substack{(kU)^{c_{31}}\leqq p^m\leqq (kU)^{c_{32}}\\ \chi_1(p^m)=1}}\frac{1}{p^m}\leqq\sum_{n_1\leqq (kU)^2}\frac{a_{n_1}}{n_1}\sum\frac{2}{p^{m(p)}},$$

with $m(p)$ being the minimal exponent such that $(kU)^{c_{31}}\leqq p^{m(p)}\leqq (kU)^{c_{32}}$ and $\chi_1(p^{m(p)})=1$. It may be noted that numbers $n\leqq (kU)^{c_{32}+2}$ can be represented in $\frac{c_{32}+2}{c_{31}}$ ways at most as $n_1p^{m(p)}$ whence

$$Se^{-c_{36}\lambda}<2\,\frac{c_{32}+2}{c_{31}}\sum_{(kU)^{c_{31}}\leqq n\leqq (kU)^{c_{32}+2}}\frac{a_n}{n}.$$

Hence and by the formula (see [4], 362)

$$\sum_{n\leqq x}\frac{a_n}{n}=\mathfrak{L}(1,\chi_1)(\log x+c_{37})+\mathfrak{L}'(1,\chi_1)+O\left(k\frac{\log x}{x^{1/2}}\right)$$

we obtain

$$\mathfrak{L}(1,\chi)+\frac{1}{kU}>\frac{S}{\log kU}e^{-c_{38}\lambda}. \tag{7.3}$$

8. The conclusion of the present proof runs along the previous lines (e. g. those of [4], 362—363). However, for the sake of completeness, I shall reproduce it here. Starting from the identity

$$\zeta(s)\mathfrak{L}(s,\chi_1)=\sum_{n=1}^{\infty}\frac{a_n}{n^s},\ \sigma>1,$$

we have with $X=(kU)^{-3/2}$

$$\sum_{n=1}^{\infty}a_n n^{-\beta_1}e^{-Xn}=\frac{1}{2\pi i}\int_{(2)}X^{\beta_1-s}\Gamma(s-\beta_1)\zeta(s)\mathfrak{L}(s,\chi_1)ds.$$

Moving the line of integration from $\sigma=2$ to $\sigma=-\frac{1}{2}$ and using the theorem of residues we get

$$\sum_{n=1}^{\infty}a_n n^{-\beta_1}e^{-Xn}=X^{-\delta_1}\Gamma(\delta_1)\mathfrak{L}(1,\chi_1)+O\left((kU)^{-1/2}\right) \tag{8.1}$$

and also, in view of $a_n\leqq d(n)\leqq n$ ($d(n)$ number of divisors of n)

$$\sum_{n>(kU)^2}a_n n^{-\beta_1}e^{-Xn}\leqq\sum_{n>(kU)^2}ne^{-Xn}\leqq\frac{c_{39}}{kU}. \tag{8.2}$$

Further, with sufficiently small $\delta_1\log kU$,

$$\sum_{n\leqq(kU)^2}a_n n^{-\beta_1}e^{-Xn}=\sum_{n\leqq(kU)^2}\frac{a_n}{n}n^{\delta_1}e^{-Xn}\leqq(kU)^{2\delta_1}S<c_{40}S,$$

whence, together with (8. 1) and (8. 2),

$$S>c_{41}\frac{\mathfrak{L}(1,\chi_1)}{\delta_1}-\frac{c_{42}}{(kU)^{1/2}}.$$

Using now (7. 3) we obtain

$$e^{-c_{43}\lambda}<\delta_1\log kU+\frac{\log kU}{kU}.$$

Hence and by (1. 4)′ the desired (2. 1) follows.

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