# On monotonic sequences defined by a recurrence relation

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**Introduction.** The subject of the present paper is the recurrence relation

$$x_{n+1}=\lambda x_n+F_n, \qquad n=0,1,2,\ldots \tag{1}$$

where $\{x_n\}$ is the sequence to be defined, while $F_n$ is a given sequence and $\lambda$ is a given real number.

There are infinitely many sequences $x_n$ satisfying relation (1). They are all contained in the formula

$$x_n=\lambda^n x_0+\sum_{i=0}^{n-1}\lambda^i F_{n-i-1}, \qquad n=1,2,\ldots \tag{2}$$

where $x_0$ may be taken as an arbitrary real number. We are omitting the easy proof of the equivalence of (1) and (2). Thus formula (2) gives an uncountable family of solutions of (1). But sometimes a particular solution can be chosen by some additional requirements.

In the present paper we shall establish some conditions of the uniqueness and existence of monotonic sequences satisfying relation (1). In the case $\lambda=-1$ this problem has been investigated in [1], [4]. (Regarding the case $\lambda=1$, compare also [2], [3]). We shall also give some applications of the obtained results to the theory of functional equations.

**1. Uniqueness.** We shall prove the following

**Theorem 1.** *If* $\lambda<0$ *and*

$$\lim_{n\to\infty}\frac{F_n}{\lambda^n}=0, \tag{3}$$

*then there may exist at most one monotonic sequence* $x_n$, *satisfying relation* (1). *If it actually does exist, then the series*

$$\sum_{k=0}^{\infty}\frac{F_{n+k}}{\lambda_{k+1}} \tag{4}$$

*converges for every n and the sequence* $x_n$ *is given by the formula*

$$x_n=-\sum_{k=0}^{\infty}\frac{F_{n+k}}{\lambda^{k+1}}. \tag{5}$$

PROOF. From relation (1) the formula

$$x_{n+m}=\lambda^m x_n+\sum_{i=0}^{m-1}\lambda^i F_{n+m-i-1}, \qquad \begin{matrix} n=0, 1, 2, \dots \\ m=1, 2, \dots \end{matrix} \tag{6}$$

follows easily (e. g. by induction). Let us suppose that there exists an increasing sequence $x_n$, satisfying (1). Thus we have for arbitrary $n\geqq 0\ p\geqq 1$

$$x_{n+2p+1}=\lambda x_{n+2p}+F_{n+2p}\geqq x_{n+2p},$$

whence, since $\lambda<0$,

$$x_{n+2p}\leqq -\frac{F_{n+2p}}{\lambda-1}.$$

Now we make use of (6), setting $m=2p$. Thus

$$\lambda^{2p}x_n+\sum_{i=0}^{2p-1}\lambda^i F_{n+2p-i-1}\leqq -\frac{F_{n+2p}}{\lambda-1},$$

whence

$$x_n\leqq -\sum_{i=0}^{2p-1}\frac{F_{n+2p-i-1}}{\lambda^{2p-i}}-\frac{F_{n+2p}}{(\lambda-1)\lambda^{2p}},$$

and after a change of the index of summation

$$x_n\leqq -\sum_{k=0}^{2p-1}\frac{F_{n+k}}{\lambda^{k+1}}-\frac{F_{n+2p}}{(\lambda-1)\lambda^{2p}}. \tag{7}$$

Similary, starting from the relation

$$x_{n+2p+2}=\lambda x_{n+2p+1}+F_{n+2p+1}\geqq x_{n+2p+1},$$

we can get the inequality

$$x_n\geqq -\sum_{k=0}^{2p}\frac{F_{n+k}}{\lambda^{k+1}}-\frac{F_{n+2p+1}}{(\lambda-1)\lambda^{2p+1}}. \tag{8}$$

It follows from (7) and (8) that

$$-\frac{F_{n+2p}}{\lambda^{2p+1}}-\frac{F_{n+2p+1}}{\lambda^{2p+1}(\lambda-1)}\leqq x_n+\sum_{k=0}^{2p-1}\frac{F_{n+k}}{\lambda^{k+1}}\leqq -\frac{F_{n+2p}}{(\lambda-1)\lambda^{2p}}. \tag{9}$$

Hence, as $p\to\infty$, according to (3) we obtain formula (5), and consequently also the uniqueness of the sequence $x_n$. Also the convergence of series (4) follows immediately from (9).

If we assume that $x_n$ is decreasing, we obtain the inequalities

$$-\sum_{k=0}^{p-1}\frac{F_{n+k}}{\lambda^{k+1}}-\frac{F_{n+2p}}{(\lambda-1)\lambda^{2p}}\leqq x_n\leqq -\sum_{k=0}^{2p}\frac{F_{n+k}}{\lambda^{k+1}}-\frac{F_{n+2p+1}}{(\lambda-1)\lambda^{2p+1}}.$$

and we proceed further analogically.

This completes the proof.

From Theorem 1 immediately follows

**Corollary 1.** *If* $\lambda < -1$ *and*

$$\limsup_{n\to\infty} |F_{n+1} - F_n| < \infty \tag{10}$$

*then there may exist at most one monotonic sequence* $x_n$, *satisfying* (1).

In fact, it is obvious that for $\lambda < -1$ condition (10) implies the fulfilment of (3).

Condition (10) is for $\lambda < -1$ stronger than condition (3). But for $\lambda = -1$ even the condition

$$\lim_{n\to\infty} |F_{n+1} - F_n| = 0 \tag{11}$$

is weaker than (3). As has been proved in [1], condition (11) guarantees the uniqueness of the monotonic solution of (1). This together with our theorem 1 gives the following

**Corollary 2.** *If* $\lambda \leqq -1$ *and condition* (11) *is fulfilled, then there may exist at most one monotonic sequence, satisfying* (1).

Thus Theorem 1 and Corollaries 1 and 2 give some conditions of the uniqueness of monotonic solutions of (1) in the case $\lambda < 0$. On the other hand, if $\lambda > 0$ the uniqueness of monotonic solutions does not occur. To see this it is enough to show that the homogeneous equation

$$x_{n+1} = \lambda x_n, \qquad n = 0, 1, 2, \ldots \tag{12}$$

has always infinitely many monotonic solutions. But it is so, in fact, for, according to (2), the general solution of (12) has the form

$$x_n = \lambda^n x_0, \qquad n = 0, 1, 2, \ldots \tag{13}$$

The sequence (13) is increasing if $(\lambda - 1)x_0 > 0$, and decreasing if $(\lambda - 1)x_0 < 0$. Thus, since the sum of two increasing, resp. decreasing sequences is itself increasing, resp. decreasing, in the case $\lambda > 0$ there are either infinitely many monotonic sequences fulfilling (1), or none.

But we can obtain the uniqueness of the solution of (1) also for $\lambda > 0$, replacing the requirement of the monotonity of $x_n$ by another condition, somewhat similar to condition (10). Namely, we shall prove the following

**Theorem 2.** *If* $|\lambda| > 1$, *than there may exist at most one sequence* $x_n$, *satisfying relation* (1) *and fulfilling the condition*

$$\limsup_{n\to\infty} |x_{n+1} - x_n| < \infty. \tag{14}$$

Proof. Since the difference of two sequences fulfilling condition (14) also fulfils condition (14), it is enough to prove that the only sequence satisfying the homogeneous relation (12) and fulfilling (14) is (for $|\lambda| > 1$)

$$x_n = 0, \qquad n = 0, 1, 2, \ldots\ . \tag{15}$$

Formula (6) has now the form

$$x_{n+m} = \lambda^m x_n.$$

If there were an index $N$ such that $x_N \neq 0$, we should have

$$\limsup_{n\to\infty} |x_{n+1}-x_n| = \limsup_{n\to\infty} |\lambda-1|\,|x_n| = \limsup_{n\to\infty} |\lambda-1|\,|\lambda|^{n-N}|x_N| = \infty,$$

in contradiction with (14). Thus (15) holds, which was to be proved.

In the case $|\lambda|<1$ every sequence of the form (13) fulfils condition (14). Consequently the latter does not guarantee the uniqueness of $x_n$ for $|\lambda|<1$.

**2. Existence.** As is easily apparent from the form

$$F_n = x_{n+1} - \lambda x_n$$

of relation (1), if $\lambda<0$ a necessary condition for the existence of an increasing (decreasing) sequence $x_n$ satisfying (1) is that the sequence $F_n$ be increasing (decreasing), but it is not sufficient. If the series (5) converges, its sum always satisfies the relation (1), but is not always monotonic. Some conditions for the existence of the monotonic solution of (1) are established, however, by the following

**Theorem 3.** *If*

$$|\lambda| > \limsup_{n\to\infty} \sqrt[n]{|F_n|} \tag{16}$$

*and either*

a) $\lambda>0$ *and the sequence* $F_n$ *is increasing (decreasing); or*

b) $\lambda<0$ *and the sequence*

$$u_n \overset{\text{def}}{=} F_{n+1} + \lambda F_n \tag{17}$$

*is increasing (decreasing),*

*then formula* (5) *actually defines a decreasing (increasing) sequence, satisfying relation* (1). *In case* b) *it is the unique monotonic sequence satisfying* (1).

Proof. The convergence of (4) follows immediately from (16) in view of the Cauchy—Hadamard theorem on power series. Thus (5) really defines a sequence satisfying (1). This sequence is evidently decreasing (increasing), if $\lambda>0$ and $F_n$ is increasing (decreasing), as sum of decreasing (increasing) sequences. When $\lambda<0$ we have

$$x_n = -\sum_{k=0}^{\infty} \frac{F_{n+k}}{\lambda^{k+1}} = -\sum_{i=0}^{\infty} \frac{F_{n+2i}}{\lambda^{2i+1}} - \sum_{i=0}^{\infty} \frac{F_{n+2i+1}}{\lambda^{2i+2}}$$

$$= -\sum_{i=0}^{\infty} \frac{F_{n+2i+1}+\lambda F_{n+2i}}{\lambda^{2i+2}} = -\sum_{i=0}^{\infty} \frac{u_{n+2i}}{(\lambda^2)^{i+1}}$$

whence it follows (since $\lambda^2>0$) that the sequence $x_n$ is decreasing (increasing) if $u_n$ is increasing (decreasing).

As the convergence of series (4) implies condition (3), it follows from Theorem 1 that in case b) the sequence $x_n$ given by (5) is the unique monotonic sequence satisfying (1). This completes the proof.

The convergence of the series (4) is not necessary for the existence of a monotonic solution of (1). This is clear from the example of the recurrence relation

$$x_{n+1} + 2x_n = 2^{n+2}, \tag{18}$$

which is satisfied by the increasing sequence[1]) $x_n = 2^n$, while

$$\sum_{k=0}^{\infty} \frac{F_{n+k}}{\lambda^{k+1}} = \sum_{k=0}^{\infty} 2^{n+1} = \infty.$$

From Theorem 3 results, however, the following

**Corollary 3.** *If $\lambda < 0$ and the series* (4) *diverges, then there may exist monotonic sequences satisfying* (1) *only if* $\frac{F_n}{\lambda^n} \nrightarrow 0$.

Now we shall find a necessary and sufficient condition of the existence of a monotonic sequence satisfying (1) in the case $\lambda < 0$. To this purpose we write

$$S_p \overset{\text{def}}{=} -\sum_{k=0}^{p} \frac{F_k}{\lambda^{k+1}} - \frac{F_{p+1}}{(\lambda-1)\lambda^{p+1}}$$

and we put further

$$\overline{\sigma_e} \overset{\text{def}}{=} \sup_{p=1,2,\dots} S_{2p}, \quad \underline{\sigma_e} \overset{\text{def}}{=} \inf_{p=1,2,\dots} S_{2p}, \quad \overline{\sigma_0} \overset{\text{def}}{=} \sup_{p=1,2,\dots} S_{2p-1}, \quad \underline{\sigma_0} \overset{\text{def}}{=} \inf_{p=1,2,\dots} S_{2p-1}.$$

We shall prove the following

**Theorem 4.** *Let $\lambda$ be negative. In order that a sequence $x_n$, defined by the recurrence relation* (1) *(or, what ammounts to the same, by formula* (2)*) be increasing, it is necessary and sufficient that*

$$\overline{\sigma_e} \leqq x_0 \leqq \underline{\sigma_0}. \tag{19}$$

*Similary, in order that a sequence $x_n$, defined by relation* (1), *be decreasing, it is necessary and sufficient that*

$$\overline{\sigma_0} \leqq x_0 \leqq \underline{\sigma_e}. \tag{20}$$

PROOF. We shall prove only the first part of the Theorem, the proof for decreasing sequences is quite analogical.

If $x_n$ is an increasing sequence satisfying (1), then we have by (7) and (8) ($n=0$)

$$S_{2p} \leqq x_0 \leqq S_{2p-1}, \qquad p = 1, 2, \dots, \tag{21}$$

whence (19) follows immediately. On the other hand, let $x_n$ be a sequence satisfying (1) (and thus of the form (2)), and let us assume that inequalities (19) hold. Then inequalities (21) hold too, and we have by (2)

$$x_{2p+1} - x_{2p} = \lambda^{2p+1} x_0 + \sum_{i=0}^{2p} \lambda^i F_{2p-i} - \lambda^{2p} x_0 - \sum_{i=0}^{2p-1} \lambda^i F_{2p-i-1} =$$

$$= \lambda^{2p}(\lambda - 1) x_0 + \sum_{i=0}^{2p} \lambda^i F_{2p-i} - \sum_{i=1}^{2p} \lambda^{i-1} F_{2p-i} =$$

$$= \lambda^{2p}(\lambda - 1) x_0 + (\lambda - 1) \sum_{i=1}^{2p} \lambda^{i-1} F_{2p-i} + F_{2p} =$$

$$= \lambda^{2p}(\lambda - 1)\left[x_0 + \sum_{k=0}^{2p-1} \frac{F_k}{\lambda^{k+1}} + \frac{F_{2p}}{(\lambda-1)\lambda^{2p}}\right] = \lambda^{2p}(\lambda - 1)(x_0 - S_{2p-1}) \geqq 0,$$

[1]) There are also other monotonic sequences satisfying relation (18). We shall find all of them later.

and similarly

$$x_{2p+2}-x_{2p+1}=\lambda^{2p+1}(\lambda-1)(x_0-S_{2p})\geqq 0.$$

This means that the sequence $x_n$ is increasing, which was to be proved.

Now we are able to determine all monotonic sequences satisfying relation (18). We have

$$S_p=-\sum_{k=0}^{p}\frac{2^{k+2}}{(-2)^{k+1}}-\frac{2^{p+3}}{(-3)(-2)^{p+1}}=-2\sum_{k=0}^{p}(-1)^{k+1}+\frac{4}{3}(-1)^p,$$

whence

$$\overline{\sigma_e}=\underline{\sigma_e}=\frac{2}{3},\ \ \overline{\sigma_0}=\sigma_0=\frac{4}{3}.$$

Consequently a sequence $x_n$ satisfying relation (18) is monotonic (increasing) if and only if

$$\frac{2}{3}\leqq x_0\leqq\frac{4}{3}.$$

For $x_0=1$ we obtain the formerly found solution $x_n=2^n$.

**3. Consequences for functional equations.** The results obtained can be applied to establish some conditions of uniqueness and existence of solutions of the functional equation

$$\varphi[f(t)]=\lambda\varphi(t)+F(t). \tag{22}$$

Here $\varphi(t)$ is the unknown function, $f(t)$ and $F(t)$ are given functions, and $\lambda\neq 0$ is a given real number. Throughout this section we shall assume that the function $F(t)$ is defined in an interval (a, b) (finite or not) and the function $f(t)$ is strictly increasing in (a, $b$), $f(t)>t$ in $(a, b)$, $f(t)\in(a, b)$, for $t\in(a, b)$. By a solution of equation (22) will be understood any function $\varphi(t)$ defined in $(a, b)$ and satisfying equation (22) for $t\in(a, b)$.

Let $f^n(t)$ denote the $n$-th iterate of the function $f(t)$:

$$f^0(t)=t,\ f^n(t)=f[f^{n-1}(t)]\qquad n=1, 2, \ldots;\ t\in(a, b).$$

Putting $x_n\overset{\text{def}}{=}\varphi[f^n(t)]$, $F_n\overset{\text{def}}{=}F[f^n(t)]$, we obtain as an immediate consequence of theorem 1 the following

**Theorem 1′.** *If* $\lambda<0$ *and*

$$\lim_{n\to\infty}\frac{F[f^n(t)]}{\lambda^n}=0,\quad \textit{for}\quad t\in(a, b),$$

*then there may exist at most one solution of equation* (22), *semimontonic* $\{f\}$ *in* $(a, b)$ [2]). *If such a solution actually does exist, then it is given by the formula*

$$\varphi(t)=-\sum_{k=0}^{\infty}\frac{F[f^k(t)]}{\lambda^{k+1}},\qquad t\in(a, b), \tag{23}$$

*and the series on the right-hand side of* (23) *converges.*

[2]) The definition of functions semimonotonic $\{f\}$ has been given in [1]. $\varphi(t)$ is semi-increasing $\{f\}$ if $\varphi[f(t)]\geqq\varphi(t)$, semi-decreasing $\{f\}$ if $\varphi[f(t)]\leqq\varphi(t)$.

**Corollary 1′.** *If* $\lambda < -1$ *and*

$$\limsup_{n\to\infty} |F[f^{n+1}(t)] - F[f^n(t)]| < \infty \quad \textit{for} \quad t \in (a, b),$$

*then the equation* (22) *may have at most one solution semimonotonic* $\{f\}$ *in* $(a, b)$.

An analogue of Corollary 2 is also true (cf. [1]):

**Corollary 2′.** *If* $\lambda \leqq -1$ *and*

$$\lim_{n\to\infty} |F[f^{n+1}(t)] - F[f^n(t)]| = 0 \quad \textit{for} \quad t \in (a, b),$$

*then equation* (22) *may have at most one solution semimonotonic* $\{f\}$ *in* $(a, b)$.

If $\lambda > 0$, the equation (22) possesses either infinitely many solutions which are semimonotonic $\{f\}$ in $(a, b)$, or none. If moreover $\lambda \neq 1$, we may replace the word "semimonotonic $\{f\}$" in the preceding sentence by the word "monotonic" (for $\lambda = 1$ the situation is somewhat different; cf. [2], [3] under suitable conditions there may exist at most a one parameter family of monotonic solutions of (22)). To prove this we need only to show that if $\lambda > 0$, $\lambda \neq 1$, then the homogeneous equation

$$\varphi[f(t)] = \lambda\varphi(t)$$

has infinitely many monotonic solutions. But its general solution is given by

$$\varphi(t) = \lambda^p \tilde{\varphi}[f^{-p}(t)] \quad \text{for} \quad t \in \langle f^p(t_0), f^{p+1}(t_0) \rangle,$$

where $t_0 \in (a, b)$ and $\tilde{\varphi}(t)$ is an arbitrary function defined in $\langle t_0, f(t_0) \rangle$.

This solution is evidently

a) increasing, if $(\lambda - 1)\tilde{\varphi}(t) > 0$, $\tilde{\varphi}(t)$ is increasing in $\langle t_0, f(t_0) \rangle$ and $\lim_{t\to f(t_0)-0} \tilde{\varphi}(t) \leqq \lambda\tilde{\varphi}(t_0)$;

b) decreasing, if $(\lambda - 1)\tilde{\varphi}(t) < 0$, $\tilde{\varphi}(t)$ is decreasing in $\langle t_0, f(t_0) \rangle$ and $\lim_{t\to f(t_0)-0} \tilde{\varphi}(t) \geqq \lambda\tilde{\varphi}(t_0)$.

From theorem 2 follows, however, the following

**Theorem 2′.** *If* $|\lambda| > 1$, *then equation* (22) *may have at most one solution that fulfils the condition*

$$\limsup_{n\to\infty} |\varphi[f^{n+1}(t)] - \varphi[f^n(t)]| < \infty \quad \textit{for} \quad t \in (a, b). \tag{24}$$

Making use of theorem 2′ we can prove another form of corollary 1′:

**Corollary 1″.** *If* $\lambda < -1$ *and there exists an* $t_0 \in (a, b)$ *such that*

$$\limsup_{n\to\infty} |F[f^{n+1}(t_0)] - F[f^n(t_0)]| < \infty, \tag{25}$$

*then equation* (22) *may have at most one monotonic solution in* $(a, b)$.

PROOF. We have by (22)

$$F[f^n(t)]-F[f^{n-1}(t_0)]=\varphi[f^{n+1}(t_0)]-\varphi[f^n(t_0)]-\lambda\{\varphi[f^n(t_0)]-\varphi[f^{n-1}(t_0)]\},$$

whence it follows by (25), in view of the facts that the functions $\varphi(t)$ and $f(t)$ are monotonic, the sequence $f^n(t_0)$ is increasing, and $\lambda<0$, that

$$\limsup_{n\to\infty}|\varphi[f^{n+1}(t_0)]-\varphi[f^n(t_0)]|<\infty. \tag{26}$$

We have further for $t\in<t_0,f(t_0)>$

$$f^n(t_0)\leqq f^n(t)\leqq f^{n+1}(t)\leqq f^{n+2}(t_0),$$

whence it follows on account of the monotonity of $\varphi(t)$, that for $t\in<t_0,f(t_0)>$

$$|\varphi[f^{n+1}(t)]-\varphi[f^n(t)]|\leqq|\varphi[f^{n+2}(t_0)]-\varphi[f^n(t_0)]|\leqq|\varphi[f^{n+2}(t_0)]-\varphi[f^{n+1}(t_0)]|+$$
$$+|\varphi[f^{n+1}(t_0)]-\varphi[f^n(t_0)]|.$$

Consequently, according to (26), condition (24) is fulfilled for all $t\in<t_0,f(t_0)>$. But since each number from the interval $(a,b)$ can be represented in the form $t=f^N(\bar{t})$, where $\bar{t}\in<t_0,f(t_0)>$ and $N$ is an integer (positive, negative, or zero), (24) is fulfilled for all $t\in(a,b)$ and our assertion follows from Theorem 2′.

From theorem 3 we obtain the following

**Theorem 3′.** *If* $|\lambda|>\limsup_{n\to\infty}\sqrt[n]{|F[f^n(t)]|}$ *for all* $t\in(a,b)$ *and either*

a) $\lambda>0$ *and the function* $F(t)$ *is semiincreasing* $\{f\}$ *(semidecreasing* $\{f\}$*) in* $(a,b)$*; or*

b) $\lambda<0$ *and the function* $F[f(t)]+\lambda F(t)$ *is semiincreasig* $\{f\}$ *(semidecreasing* $\{f\}$*) in* $(a,b)$*;*

*then formula* (23) *actually defines a function, which is semidecreasing* $\{f\}$ *(semiincreasing* $\{f\}$*) in* $(a,b)$ *and satisfies equation* (22). *In the case* b) *it is the unique solution of* (22), *seminotonic* $\{f\}$ *in* $(a,b)$.

**Coroliary 3′.** *If* $\lambda<0$ *and the series* (23) *diverges, then there may exist semimonotonic* $\{f\}$ *solutions of* (22) *only if*

$$\frac{F[f^n(t)]}{\lambda^n}\to 0$$

*for a certain* $t_0\in(a,b)$.

Theorem 4 has no simple analogue for equation (22).

It is evident that in Theorems 1′ and 3′ and in Corollaries 1′, 2′ and 3′ the words "semimonotonic $\{f\}$", "semiincreasing $\{f\}$" and "semidecreasing $\{f\}$" may be replaced by the words "monotonic", "increasing" and "decreasing" respectively (of course, in hypothesis and in statements). In the case of Theorem 1′ and the Corollaries it follows from the fact that (since $f(t)>t$) every monotonic function is also semimonotonic $\{f\}$. For the Theorem 3′ a proof would be necessary, but it is quite analogous to the proof of Theorem 3 and thus we omit it.

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