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Oscillation criteria for second order half-linear differential equations with functional arguments

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Abstract. Oscillation criteria for the second order half-linear differential equations with functional arguments of the form

$$[r(t)|y'(t)|^{\alpha-1}y'(t)]' + p(t)f(y(t)), y(g(t))) = 0, \tag{$*$}$$

are established,where $\alpha > 0$ is a constant and $g(t) \to \infty$ as $t \to \infty$. These results exhibit a surprising similarity in the oscillatory behavior existing between $(*)$ and the corresponding differential equation

$$y''(t) + p(t)f(y(t), y(g(t))) = 0.$$

1. Introduction

Consider the following three second order differential equations

$$[r(t)|y'(t)|^{\alpha-1}y'(t)]' + p(t)f(y(t)), y(g(t))) = 0, \tag{E}$$

$$[r(t)|y'(t)|^{\alpha-1}y'(t)]' + p(t)|y(g(t))|^{\beta-1}y(g(t)) = 0, \tag{E$_1$}$$

$$[r(t)|y'(t)|^{\alpha-1}y'(t)]' + \lambda p(t)|y(t)|^{\alpha-1}y(t) = 0, \tag{E$_2$}$$

where

(a) $p, g \in C([t_0, \infty); \Re)$ for some $t_0 \geq 0$ and $\lim\limits_{t\to\infty} g(t) = \infty$;
(b) $r \in C^1([t_0, \infty), (0, \infty))$;

(c) $f \in C(\Re \times \Re, \Re)$, $f(x,y)$ has the same sign of x and y when they have the same sign, that is,

$$f(x,y)\begin{cases} >0 & \text{if } x>0, y>0, \\ <0 & \text{if } x<0, y<0; \end{cases}$$

(d) α and β are positive constants.

Throughout this paper, we define

$$\pi(t) := \int_t^\infty (r(s))^{-\frac{1}{\alpha}} ds,\ t \geq t_0.$$

In [1], Elbert established the existence and uniqueness of solutions to the initial value problem for equation (E_2) on $[t_0, \infty)$. Note that any constant multiple of a solution of (E_2) is also a solution of (E_2). A nontrivial solution is called oscillatory if it has arbitrarily large zeros; otherwise it is said to be nonoscillatory. The equation (E_2) is nonoscillatory [resp. oscillatory] if all of its solutions are nonoscillatory [resp. oscillatory].

Surprisingly, some similar properties between equation (E_2) and the linear equation

$$(E_0) \qquad (p(t)y')' + q(t)y = 0,\ t \geq 0$$

have been observed by Elbert [1, 2], Mirzov [9, 10, 11], Kusano and Naito [4], Kusano and Yoshida [5], Kusano et al [6], Li and Yeh [7, 8]. For example, Sturmian theory for (E_0) has been extended in a natural way to (E_2) by Elbert [1], Li and Yeh [7]. They showed that the zeros of two linearly independent solutions of (E_2) separate each other. If $\alpha = 1$ and $r(t) = 1$, then Travis [12] and Yeh [13] established some sufficient conditions on the oscillation of (E). Travis [12] also gave a sufficient condition on the oscillation of $y'(t)$ for any solution of (E_2). For the other result, we refer to Kusano and Lalli [3].

The purpose of this paper is to extend the results of Travis [12] and Yeh [13] to the equation (E) by using a result (lemma 3 below) of Li and Yeh [7].

2. Oscillations of Equation (E)

Theorem 1. *Let $\pi(t_0) = \infty$; and*

(C_1) *there exist a constant k and a function $h(t) \in C([t_0, \infty), \Re)$ such that $h(t) \le g(t)$ and $0 < k \le h'(t) \le 1$*

(C_2) *there exist a constant $M > 0$ such that $|y| \ge M$ implies*

$$\liminf_{|w|\to\infty} |\frac{f(y,w)}{|w|^{\alpha-1}w}| \ge \varepsilon > 0$$

for some $\varepsilon > 0$.

(C_3) $p(t) \ge 0$ *and* $\limsup_{t\to\infty} A(t,t_0)^{-\lambda} \int_{t_0}^t A(t,s)^{\lambda} p(s)ds = \infty$,

where $A(t,s) = \int_s^t (r(u))^{\frac{-1}{\alpha}} du; \lambda > 1$.

Then (E) *is oscillatory.*

Proof. Assume the contrary. Then (E) has a nonoscillatory solution y(t). Without loss of generality, we may assume that $y(t) > 0$ on $[T, \infty)$ for some $T \ge t_0$. It is easily to verify that $y'(t) > 0$ for large t. Let

$$w(t) = \frac{r(t)|y'(t)|^{\alpha-1}y'(t)}{|y(h(t))|^{\alpha-1}y(h(t))}.$$

Then $w(t)$ satisfies

$$w'(t) = -p(t)\frac{f(y(t), y(g(t)))}{|y(h(t))|^{\alpha-1}y(h(t))} - \alpha\frac{y'(h(t))}{y(h(t))}h'(t)w(t), \tag{1}$$

for $t \ge T$.

Since $y'(t) > 0$ for large t, $\lim_{t\to\infty} y(t)$ exists either as a finite or infinite limit. If $\lim_{t\to\infty} y(t) = b$ is finite, then

$$\lim_{t\to\infty} \frac{f(y(t), y(g(t)))}{|y(g(t))|^{\alpha-1}y(g(t))} = \frac{f(b,b)}{b^{\alpha}} > 0.$$

If $\lim_{t\to\infty} y(t) = \infty$, then, by ($C_2$), we have that

$$\frac{f(y(t), y(g(t)))}{|y(g(t))|^{\alpha-1}y(g(t))} \ge \varepsilon > 0$$

for all large t. Let $\varepsilon_1 = \min\{\varepsilon, \frac{f(b,b)}{2b^\alpha}\}$. Since $y(t)$ is increasing for large t, we have that

$$p(t)\frac{f(y(t), y(g(t)))}{|y(h(t))|^{\alpha-1}y(h(t))} \geq p(t)\frac{f(y(t), y(g(t)))}{|y(g(t))|^{\alpha-1}y(g(t))} \geq \varepsilon_1 p(t), \tag{2}$$

and

$$\alpha\frac{y'(h(t))}{y(h(t))}h'(t)w(t) \geq \alpha k r^{\frac{-1}{\alpha}}(t) w^{\frac{1+\alpha}{\alpha}}(t), \tag{3}$$

for all large t. It follows from (1), (2) and (3) that

$$\begin{aligned} w'(t) &\leq -\varepsilon_1 p(t) - \alpha k r^{\frac{-1}{\alpha}}(t) w^{\frac{1+\alpha}{\alpha}}(t) \\ &\leq -\varepsilon_1 p(t) \leq 0 \end{aligned} \tag{4}$$

for $t \geq T$,where T is large enough. This implies

$$\int_T^t A^\lambda(t,s)w'(s)ds \leq -\varepsilon_1 \int_T^t A^\lambda(t,s)p(s)ds. \tag{5}$$

Since

$$\int_T^t A^\lambda(t,s)w'(s)ds = \lambda \int_T^t A^{\lambda-1}(t,s)r^{\frac{-1}{\alpha}}(s)w(s)ds - w(T)A^\lambda(t,T),$$

we get, by (5)

$$\varepsilon_1 A^{-\lambda}(t,t_0)\int_T^t A^\lambda(t,s)(s)p(s)ds \leq w(T)\left\{\frac{A(t,T)}{A(t,t_0)}\right\}^\lambda \leq w(T).$$

Hence

$$\limsup_{t\to\infty} \varepsilon_1 A^{-\lambda}(t,t_0)\int_T^t A^\lambda(t,s)p(s)ds \leq w(T),$$

which contratticts condition (C_3). Thus, our proof is complete.

We say that (E_2) is strongly oscillatory if (E_2) is oscillatory for every $\lambda > 0$. In order to discuss the next two theorems, we need the following three lemmas:

Lemma 2 (Kusano et all. [6]).

If $\int^\infty (r(s))^{\frac{-1}{\alpha}} ds = \infty$, *then* ($E_2$) *is strongly oscillatory if and only if* (C_4) $p \geq 0$ *is integrable on* $[t_0, \infty)$ *and* $\limsup_{t\to\infty} A^\alpha(t,t_0)\int_t^\infty p(s)ds = \infty$, *where* $A(t,t_0)$ *is defined as in Theorem 1.*

Lemma 3 (Kusano and Natio [4]). *If $\int^{\infty}(r(s))^{\frac{-1}{\alpha}}ds < \infty$, then* ($E_2$) *is strongly oscillatory if and only if*
(C_5) *$p \geq 0$ is integrable on $[t_0, \infty)$, $\int_{t_0}^{\infty}(\pi(t))^{\alpha+1}p(t)dt < \infty$, and*

$$\limsup_{t\to\infty} \pi^{-1}(t)\int_t^{\infty}(\pi(s))^{\alpha+1}p(s)ds = \infty.$$

Lemma 4 (Li and Yeh [7]). *Equation* (E) *is nonoscillatory if and only if there is a function $\omega \in C^1[T, \infty)$ for some $T \geq t_0$, satisfying*

$$\omega'(t) + p(t) + \alpha r^{-\frac{1}{\alpha}}(t)|\omega(t)|^{\frac{\alpha+1}{\alpha}} \leq 0.$$

Theorem 5. *Let* (C_1) *and* (C_2) *hold and $\pi(t_0) = \infty$. If condition* (C_4) *holds, then* (E) *is oscillatory.*

Proof. Assume the contrary. Then (E) has a nonoscillatory solution y(t). With loss of generality, we may assume that $y(t) > 0$ on $[T, \infty)$ for some $T \geq t_0$. Let

$$w(t) = \frac{r(t)|y'(t)|^{\alpha-1}y'(t)}{|y(h(t))|^{\alpha-1}y(h(t))}.$$

As in the proof of theorem 1, we have

$$w'(t) + \varepsilon_1 p(t) + \alpha k r^{\frac{-1}{\alpha}}(t) w^{\frac{1+\alpha}{\alpha}}(t) \leq 0.$$

If $u(t) = k^{\alpha}w(t)$, then

$$u'(t) + \varepsilon_1 k^{\alpha}p(t) + \alpha r^{\frac{-1}{\alpha}}(t) u^{\frac{1+\alpha}{\alpha}}(t) \leq 0.$$

It follows from lemma 4 that

$$[r(t)|u'(t)|^{\alpha-1}u'(t)]' + \varepsilon_1 k^{\alpha}p(t)|u(t)|^{\alpha-1}u(t) = 0$$

is nonoscillatory. However, this contradicts the fact that (E_2) is strongly oscillatory (by Lemma 2).

Similiarly, using Lemma 3 and Lemma 4, we can prove the following

Theorem 6. *Let* (C_1) *and* (C_2) *hold and $\pi(t_0) < \infty$. If condition* (C_5) *holds, then* (E) *is oscillatory.*

3. Oscillation of the derivative of a solution of (E_1)

Theorem 7. *Assume $g(t)$ is differentiable, $g'(t) \geq 0$, and $\int^{\infty} p(t)dt = \infty$. If $\int^{\infty} r^{\frac{-1}{\alpha}}(t)dt = \infty$, then $y'(t)$ is oscillatory for any solution $y(t)$ of* (E_1).

Proof. If $y(t)$ oscillates, then there is nothing to prove. If $y(t)$ is ulimately positive, then so is $y(g(t))$. Suppose $y'(t) > 0$ for all large t. Then

$$w(t) = \frac{r(t)|y'(t)|^{\alpha-1}y'(t)}{|y(g(t))|^{\beta-1}y(g(t))}$$

satisfies the equation

$$w'(t) = -p(t) - w(t)\frac{\beta[y(g(t))]^{\beta-1}y'(g(t))g'(t)}{|y(g(t))|^{\beta-1}y(g(t))} \leq -p(t).$$

Integrating the above inequality, we obtain

$$w(x) \leq w(\alpha) - \int_{\alpha}^{x} p(t)dt.$$

It follows from $\int^{\infty} p(t)dt = \infty$ that $y'(t) < 0$ for all large t, which is a contradiction. Suppose now $y'(t) < 0$ for all large t. It is easy to see that $\int^{\infty} p(t)dt = \infty$ implies that there exists a positive constant T such that

$$\int_{T}^{t} p(t)dt \geq 0$$

for $t \geq T$. Hence, we have

$$\begin{aligned} &\int_{T}^{t} p(s)(|y(g(s))|^{\beta-1}y(g(s)))ds \\ &= |y(g(t))|^{\beta-1}y(g(t)) \int_{T}^{t} p(s)ds \\ &-\beta \int_{T}^{t} (y(g(s)))^{\beta-1}y'(g(s))g'(s) \int_{T}^{s} p(r)drds \geq 0, \qquad t \geq T. \end{aligned} \tag{6}$$

Now integrating equation (E_1) and using (6), we have

$$r(t)|y'(t)|^{\alpha-1}y'(t) \leq r(T)|y'(T)|^{\alpha-1}y'(T) := |c|^{\alpha-1}c < 0,$$

i.e.,

$$|y'(t)|^{\alpha-1}y'(t) \leq \frac{r(T)|y'(T)|^{\alpha-1}y'(T)}{r(t)} < 0,$$

thus

$$y'(t) \le c r^{\frac{-1}{\alpha}}(t) < 0$$

for some $c < 0$. Integrating it from T to $t\,(\ge T)$, we obtain

$$y(t) - y(T) \le c \int_T^t r^{\frac{-1}{\alpha}}(s)ds.$$

Thus $y(t) < 0$ for t large enough, which contradicts the fact that $y(t)$ is positive for large t. This completes the proof.

Example 8. Let $y(t)$ be a solution of

$$\frac{d}{dt}\phi(y') + \frac{\sin t}{2 - \sin t}\phi(y) = 0$$

for $t \ge 0$, where $\phi(u) = |u|^{\alpha-1}u$, then, by theorem 7, $y'(t)$ is oscillatory because

$$\int^\infty \frac{\sin t}{2 - \sin t}dt = \infty.$$

References

[1] Á. Elbert, A half-linear second order differential equation, *Colloquia Math. Soc. János Bolyai* **30**: Qualitative Theory of Differential Equations (1979), pp. 153–180, *Szeged*.

[2] Á. Elbert, Oscillation and nonoscillation theorems for some nonlinear ordiary differential equations, *Lecture Notes in Mathematics* **964**: Ordianry and Partial Differential Equations (1982), pp. 187–212.

[3] T. Kusano and B. S. Lalli, On oscillation of half-linear functional differential equations with deviating arguments, *Hiroshima Math. J.* **24** (1994), 549–563.

[4] T. Kusano and Y. Naito, Oscillation and nonoscillation criteria for second order quasilinear differential equations, (*preprint*).

[5] T. Kusano and N. Yoshida, Nonoscillation theorems for a class of quasilinear differential equations of second order, *J. Math. Anal. Appl.* **189** (1995), 115–127.

[6] T. Kusano, Y. Naito and A. Ogata, Strong oscillation and nonoscillation of quasilinear differential equations of second order, *Differential Equations and Dynamical Systems* **2** (1994), 1–10.

[7] H. J. Li and C. C. Yeh, Sturmian comparison theorem for half-linear second order differential equation, *Proc. of the Royal Soc. of Edinburgh* (*to appear*).

[8] H. J. Li and C. C. Yeh, An integral criterion for oscillation of nonlinear differential equations, *Japonica Math.* **41** (1995), 185–188.

[9] D. D. Mirzov, On the oscillation of a system of nonlinear differential equations, *Diffferencial'nye Uravnenija* **9** (1973), 581–583.

[10] D. D. Mirzov, On some analogs of Sturm's and Kneser's theorems for nonlinear systems, *J. Math. Anal. Appl.* **53** (1976), 418–425.

[11] D. D. Mirzov, On the oscillation of solutions of a system of differential equations, *Mat. Zametki* **23** (1978), 401–404.
[12] C. Travis, Oscillation theorems for second-order differential equations with functional arguments, *Proc. Amer. Math. Soc.* **31** (1972), 199–202.
[13] C. C. Yeh, An oscillation criterion for second order nonlinear differential equations with functional arguments, *J. Math. Anal. Appl.* **76** (1980), 72–76.

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