General algebraic dependence relations

By V. DLAB (Khartoum)

§ 1. Introduction

Many papers have been dedicated to the problem of an axiomatization of a dependence relation in mathematics. The methods of solutions of this question — frequently given by the purpose for which the related axiomatic system is constructed — differ often already in the choice of primitive undefined terms (even if the systems are sometimes equivalent). The binary relation „an element (or a set) depends on a set"[1]) is the primary object for one of the first axiomatic systems, for the system of B. L. VAN DER WAERDEN [28] and for closely related systems of O. TEICHMÜLLER [27], G. PICKERT [22], A. KERTÉSZ [12], M. N. BLEICHER and G. B. PRESTON [2] and K. G. JOHNSON [7]; axiomatic systems based on the concept of an independent (or a dependent) set are introduced by H. WHITNEY [29], T. NAKASAWA [19], [20], [21], O. HAUPT, G. NÖBELING and CHR. PAUC [8] and R. RADO [23] and that based on the concept of a rank function by H. WHITNEY [29] and R. RADO [24]; an axiomatization of a dependence relation in terms of orderings (of lattices) was carried out by S. MACLANE [14], in terms of closure operations (of exchange structures) by N. BOURBAKI [3] and J. SCHMIDT [25] and in terms of algebraic operations (of abstract algebras) by E. MARCZEWSKI [15], [16], [17].

Besides other questions, the problem of an axiomatization of a dependence relation comprises the two following important points:

(i) to cover various particular concepts of dependence (as linear dependence in vector spaces, algebraic dependence in the theory of fields, dependence in abelian groups etc.) and

(ii) to establish a general invariant — rank or dimension of the structure under consideration.

Therefore, in order to compare different axiomatic systems, the following rough criterion might be suggested: The success of the alleged axiomatic system in satisfying these two requirements (i) and (ii). From this point of view, the systems listed above have a common feature: those of them, which enable to fulfil the condition (ii), are, up to slight modifications (including the extension to infinite sets), equivalent to the VAN DER WAERDEN's axiomatic system in [28]. In particular, the corresponding dependence relations satisfy the third VAN DER WAERDEN's axiom stating the transitive property of the introduced (linear) dependence. Thus, no one of them covers the important dependence relation in abelian groups studied independently by T. SZELE [26] and the author [4] and exploited extensively in the L. FUCHS' monograph [6].

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[1]) The formulation in terms of sequences instead of sets is often used.

The present paper represents an attempt to introduce a dependence relation generalizing the previous concepts and, especially, including the group dependence. It might be understood as a solution of the problem indicated by T. SZELE in [26]. The primitive notion is the relation „an element depends on a set", i. e. a binary relation between the given set and its power-set. This relation determines uniquely the class of all the independent sets; the converse does not hold, for two different relations can correspond to the same class of independent sets. In this sense, our approach is a priori more general than that involving another choice of the primitive term. These and some related questions will be the subject of a separate paper.

The definition of a GA-dependence structure — a set S whith a GA-dependence relation, i. e. a binary relation $\delta \subseteq S \times \mathfrak{P}S$ between S and its power-set $\mathfrak{P}S$ satisfying six conditions (i) — (vi) — is given in § 2. Besides the first two conditions describing the behaviour of so-called neutral and singular elements and expressing the fact that our dependence is a property of finite character (and guaranteeing the existence of maximal independent sets), respectively, the other are related to those of VAN DER WAERDEN [28]. Thus, (iii) together with (vi) correspond to the first and (iv) is a slight generalization of the second VAN DER WAERDEN's postulate. The most typical feature of our axiomatic system consists in the absence of the „transitive axiom"; nevertheless, there is a weaker substitute (v) of it. This condition is of an existential character; but, we shall see that all our results will be indifferent to the particular choice of the canonic subset S^c (Remark 7. 20). As a consequence of the weak form of (v), the operation $X \to \mathrm{cl}(X)$ defined in § 7 is not idempotent (comp. J. SCHMIDT [25]); the corresponding least idempotent closure operation (see e. g. G. BIRKHOFF [1]) $X \to \mathrm{Cl}(X) = \bigcup_{n=1}^{\infty} \mathrm{cl}^{(n)}(X)$ giving, in general, only a very rough information on the situation is for our study nearly useless (for the related questions in abelian groups see the author's paper [5]).

Whilst the axiom (i)—(vi) are presented in a weak form suitable for proving whether a given relation is a GA-dependence relation or not, § 4 contains an equivalent, formally stronger, system (1)—(6) designed for the further study. The next § 5 shows that the postulates of both axiomatic systems are independent. Besides, Theorem 3. 3 of § 3 states that the axiom (vi) is not essential in our problem. In fact, reading „an element x either depends or belongs to a set X" instead of „x depends on X" one can, modifying the respective formulations, carry out the whole theory without (vi).

Some properties of independent sets and, in particular, the invariance of the concept of the rank (defined as a cardinality of a certain maximal independent system) are derived in §§ 6 and 7. Lemmas 7. 7 and 7. 9 (applied in the proof of Lemma 7. 12) appear here as a generalization of the STEINITZ's Exchange Theorem. Theorem 7. 15 states the basic property of the rank. § 8 deals with decompositions and compositions of GA-dependence structures preserving „linear property" of the rank function. Finally, in the last § 9 some applications are discussed.

Throughout the paper, capital letters stand for sets, small letters for their elements and the gothic ones for families of sets. The symbols $\cup$, $\cap$ and $\setminus$ denote the set-theoretical union, intersection and difference, respectively, $\emptyset$ denotes the empty set and card (X) the cardinality of X. The logical operations of conjunction, disjunction, implication, equivalence, universal and existential quantifications are denoted by $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$, $\forall$ and $\exists$, respectively. Though not exclusively, we shall use this notation for the sake of brevity.

§ 2. GA-dependence structures

Let $\bar{S}$ be a non-empty set and δ a binary relation defined between $\bar{S}$ and its power-set $\mathfrak{P}\bar{S}$, i. e. a subset of the cartesian product $\bar{S}\times\mathfrak{P}\bar{S}$:

$$\delta\subseteqq\bar{S}\times\mathfrak{P}\bar{S}.^{2)}$$

Denote by S_δ^N the subset of $\bar{S}$ of all the *δ-neutral* and by S_δ^S the subset of all the *δ-singular* elements defined by

$$(\mathfrak{N})\qquad x\in S_\delta^N\leftrightarrow x\in\bar{S}\wedge\forall X\quad(X\in\mathfrak{P}\bar{S}\rightarrow[x,X]\notin\delta)$$

and

$$(\mathfrak{S})\qquad x\in S_\delta^S\leftrightarrow x\in\bar{S}\wedge\forall X\quad(X\in\mathfrak{P}\bar{S}\rightarrow[x,X]\in\delta),$$

respectively. Clearly,

$$S_\delta^N\cap S_\delta^S=\emptyset.$$

Putting

$$S_\delta=\bar{S}\setminus(S_\delta^N\cup S_\delta^S),$$

we have

$$\bar{S}=S_\delta\cup S_\delta^N\cup S_\delta^S$$

with mutually disjoint summands.

Further, the symbol $\bar{\mathfrak{J}}_{\bar{S},\delta}$ denotes the family defined by

$$(\bar{\mathfrak{J}})\qquad \bar{I}\in\bar{\mathfrak{J}}_{\bar{S},\delta}\leftrightarrow\bar{I}\in\mathfrak{P}\bar{S}\wedge\forall x\quad(x\in\bar{I}\rightarrow[x,\bar{I}\setminus(x)]\notin\delta);$$

moreover, put

$$(\mathfrak{J})\qquad \mathfrak{J}_{S,\delta}=\bar{\mathfrak{J}}_{\bar{S},\delta}\cap\mathfrak{P}S_\delta;$$

in particular, $\emptyset\in\mathfrak{J}_{S,\delta}\subseteqq\bar{\mathfrak{J}}_{\bar{S},\delta}$. Finally, by $\bar{\mathscr{F}}_{\bar{S}}$ and $\mathscr{F}_S$ we denote the family of all the finite subsets of $\bar{S}$ and S, respectively; assume $\emptyset\in\mathscr{F}_S\subseteqq\bar{\mathscr{F}}_{\bar{S}}$.[3]

Now, we are ready to introduce the basic definition.

Definition 2. 1. The pair $(\bar{S},\delta)$ is said to be a GA-*dependence structure* and δ a GA-*dependence relation* (*general algebraic dependence relation*) defined on $\bar{S}$ if the following 6 conditions are satisfied:

(i) $x\in S\wedge X\in\mathfrak{P}\bar{S}\rightarrow([x,X]\in\delta\leftrightarrow[x,X\cap S]\in\delta)$;

(ii) $x\in S\wedge X\in\mathfrak{P}S\wedge[x,X]\in\delta\rightarrow\exists F(F\subseteqq X\wedge F\in\mathscr{F}\wedge[x,F]\in\delta)$;

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[2]) The elements of $\bar{S}\times\mathfrak{P}\bar{S}$ will be denoted, in usual way, by $[x, X]$ with $x\in\bar{S}$ and $X\subseteq\bar{S}$.

[3]) As far as there is no danger of confusion, we shall use the simpler notation $\bar{S}^N, \bar{S}^S, S, \bar{\mathfrak{J}}, \mathfrak{J}, \bar{\mathscr{F}}$ and $\mathscr{F}$ instead of $S_\delta^N, S_\delta^S, S_\delta, \bar{\mathfrak{J}}_{\bar{S},\delta}, \mathfrak{J}_{S,\delta}, \bar{\mathscr{F}}_{\bar{S}}$ aud $\mathscr{F}_S$, respectively.

(iii) $x \in S \wedge F \in \mathscr{F} \wedge F \subseteqq X \subseteqq S \wedge [x, F] \in \delta \rightarrow [x, X] \in \delta$;

(iv) $x \in S \wedge y \in S \wedge I \in \mathscr{F} \cap \mathfrak{J} \wedge x \neq y \wedge x \notin I \wedge y \notin I \wedge$

$$\wedge [x, I \cup (y)] \in \delta \wedge [x, I] \notin \delta \rightarrow [y, I \cup (x)] \in \delta;$$

(v) $\exists S_\delta^C [S_\delta^C \subseteqq S \wedge \{x \in S \setminus S_\delta^C \rightarrow \exists I (I \in \mathfrak{J} \cap \mathfrak{P} S_\delta^C \wedge [x, I] \in \delta)\} \wedge$

$$\wedge \{x \in S \wedge I \in \mathscr{F} \cap \mathfrak{J} \wedge C \in \mathscr{F} \cap \mathfrak{J} \cap \mathfrak{P} S_\delta^C \wedge x \notin I \wedge [x, C] \in \delta \wedge$$

$$\wedge \forall y (y \in C \wedge y \notin I \rightarrow [y, I] \in \delta) \rightarrow [x, I] \in \delta\}];$$

(vi) $x \in S \rightarrow [x, (x)] \in \delta$.

The elements of $\bar{\mathfrak{J}}$, $\mathfrak{J}$ and

($\mathfrak{C}$) $$\mathfrak{C}_{S_\delta^C} = \mathfrak{J} \cap \mathfrak{P} S_\delta^C \text{ [4]}$$

are said to be *δ-independent sets*, *δ-independent systems* and *δ-canonic systems*, respectively.

The subset $S^c \subseteqq S$ is not, in general, unique. Let us denote by $\mathfrak{S}_{\bar{S}, \delta}$[4]) the family of all the subsets of $\bar{S}$ having properties of S^c stated in (v); call them canonic subsets. We shall see in § 7 that $\mathfrak{S}_{\bar{S}, \delta}$ has minimal and maximal elements.

The family $\mathfrak{D}_{\bar{S}}$ of all the possible GA-dependence relations on a fixed set $\bar{S}$ is (partly) ordered: $\delta' \leqq \delta''$ means simply that $\delta' \subseteqq \delta''$ in $\bar{S} \times \mathfrak{P}\bar{S}$. One can deduce from $\delta' \leqq \delta''$ easily the inclusion $\bar{I}_{\bar{S}, \delta'} \supseteqq \bar{I}_{\bar{S}, \delta''}$. For given subsets $A \subseteqq \bar{S}$ and $B \subseteqq \bar{S}$, denote by $\mathfrak{D}_{\bar{S}, A, *}$, $\mathfrak{D}_{\bar{S}, *, B}$ and, in the case that $A \cap B = \emptyset$, $\mathfrak{D}_{\bar{S}, A, B}$ the subfamilies of those GA-dependence relations δ which satisfy the equalities $S_\delta^N = A$, $S_\delta^S = B$ and $S_\delta^N = A$ with $S_\delta^S = B$, respectively. Being subfamilies of $\mathfrak{D}_{\bar{S}}$, they are again (partly) ordered. For two GA-dependence relations $\delta'_{A,B}$, $\delta''_{A,B}$ of $\mathfrak{D}_{\bar{S},A,B}$ such that $\delta'_{A,B} \leqq \delta''_{A,B}$ we have besides $\bar{\mathfrak{J}}_{\bar{S},\delta'_{A,B}} \supseteqq \bar{\mathfrak{J}}_{\bar{S},\delta''_{A,B}}$ also $\mathfrak{J}_{S,\delta'_{A,B}} \supseteqq \mathfrak{J}_{S,\delta''_{A,B}}$. In particular, the following GA-dependence relations $\delta_{A,B}^{(0)}$ and $\delta_{A,B}^{(1)}$ are the least and the greatest elements of $\mathfrak{D}_{\bar{S},A,B}$, respectively:

$$x \in A \wedge X \in \mathfrak{P}\bar{S} \rightarrow [x, X] \notin \delta_{A,B}^{(0)} \wedge [x, X] \notin \delta_{A,B}^{(1)};$$

$$x \in B \wedge X \in \mathfrak{P}\bar{S} \rightarrow [x, X] \in \delta_{A,B}^{(0)} \wedge [x, X] \in \delta_{A,B}^{(1)};$$

$$x \in \bar{S} \setminus (A \cup B) \wedge X \in \mathfrak{P}\bar{S} \rightarrow ([x, X] \in \delta_{A,B}^{(0)} \leftrightarrow x \in X) \wedge ([x, X] \in \delta_{A,B}^{(1)} \leftrightarrow X \neq \emptyset).$$

The GA-dependence relations of these types will be called *zero-* and *unit-GA-dependence relations*, respectively. The zero- GA-dependence relation $\delta_{\bar{S},\emptyset}^{(0)}$ is evidently the least and the unit- GA-dependence relation $\delta_{\emptyset,\bar{S}}^{(1)}$ the greatest element of $\mathfrak{D}_{\bar{S}}$. Both $\delta_{\bar{S},\emptyset}^{(0)}$ and $\delta_{\emptyset,\bar{S}}^{(1)}$ are particular cases of what we shall call trivial GA-dependence relations, i. e. of GA-dependence relations δ such that $S_\delta^N \cup S_\delta^S = \bar{S}$; we shall exclude them from our further consideration.

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[4]) Again, we shall usually write briefly S^C, $\mathfrak{C}$ and $\mathfrak{S}$ instead of S_δ^C, $\mathfrak{C}_{S_\delta^C}$ and $\mathfrak{S}_{\bar{S},\delta}$, respectively.

Finally, let us remark that having constructed $\delta^{(0)}_{A,B}$ and $\delta^{(1)}_{A,B}$ we have shown simultaneously that our axiomatic system (i)—(vi) is consistent.

Definition 2. 2. By a GA-dependence table $\mathbf{T}(\bar{S}, \delta)$ of a structure $(\bar{S}, \delta)$ we shall mean a rectangular array of the form

$$\mathbf{T}(\bar{S}, \delta) \equiv \begin{array}{c|ccccccc} & . & . & . & X_\beta & . & . & . \\ \hline \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_\alpha & . & . & . & a_{\alpha\beta} & . & . & . \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{array},$$

where x_α runs through all the elements of $\bar{S}$, X_β through all the subsets of $\bar{S}$ and $a_{\alpha\beta}$ is equal to 0 or 1 according to as $[x_\alpha, X_\beta] \notin \delta$ or $[x_\alpha, X_\beta] \in \delta$, respectively.

The GA-dependence table $\mathbf{T}(\bar{S}, \delta)$ of $(\bar{S}, \delta)$, being dependent on the ordering of $\bar{S}$ and $\mathfrak{P}\bar{S}$, is not uniquely determined. But, if $\mathbf{T}'(\bar{S}, \delta)$ and $\mathbf{T}''(\bar{S}, \delta)$ are two different GA-dependence tables of the same GA-dependence structure $(\bar{S}, \delta)$, then $\mathbf{T}'(\bar{S}, \delta)$ can be brought by suitable permutations of its rows and columns to the form of $\mathbf{T}''(\bar{S}, \delta)$. Because of the properties of a GA-dependence relation resulting from (i)—(vi) the forms of GA-dependence tables are rather special. Nevertheless, we shall find GA-dependence tables very convenient to define a GA-dependence relation on a given set $\bar{S}$ (especially, for a finite set $\bar{S}$).

Remark. 2. 3. Let us notice that (ii) and (iii) can be expressed together in the form

$$x \in S \wedge X \in \mathfrak{P}S \rightarrow \{[x, X] \in \delta \leftrightarrow \exists F(F \subseteq X \wedge F \in \mathscr{F} \wedge [x, F] \in \delta)\}.$$

On the other hand, assuming (ii), (iii) can be formulated as follows:

(iii)° $\quad x \in S \wedge X \subseteq Y \subseteq S \wedge [x, X] \in \delta \rightarrow [x, Y] \in \delta$:

for, by (ii) and (iii)

$$x \in S \wedge X \subseteq Y \subseteq S \wedge [x, X] \in \delta \rightarrow$$

$$\rightarrow x \in S \wedge \exists F(F \in \mathscr{F} \wedge F \subseteq X \subseteq Y \subseteq S \wedge [x, F] \in \delta) \rightarrow [x, Y] \in \delta.$$

Then, the axioms (i) and (iii)° are obviously equivalent to the following two axioms

$\vec{\text{(i)}}$ $\quad x \in S \wedge X \in \mathfrak{P}\bar{S} \wedge [x, X] \in \delta \rightarrow [x, X \cap S] \in \delta$

and

(iii)′ $\quad x \in S \wedge X \subseteq Y \subseteq \bar{S} \wedge [x, X] \in \delta \rightarrow [x, Y] \in \delta.$

Of course, there are many other possibilities how to formulate the first three axioms to obtain an equivalent system; e. g. we can restrict (i) by the condition $X \in \bar{\mathscr{F}}$ to the form

(i)′ $\quad x \in S \wedge X \in \bar{\mathscr{F}} \rightarrow ([x, X] \in \delta \leftrightarrow [x, X \cap S] \in \delta)$

if simultaneously a weaker condition $X \in \mathfrak{P}\bar{S}$ instead of $X \in \mathfrak{P}S$ is put in (ii)[5]). The related question of bringing the whole axiomatic system (i)—(vi) to a formally stronger form will be considered in § 4.

§ 3. Axiom (vi) and GA-dependence relation $\bar{\delta}$

In this section, we shall show that the axiom (vi) is not essential in our study.[6]) Our aim is to define the δ-rank of the set $\bar{S}$ as the cardinality of certain elements (maximal δ-canonic systems) of $\bar{\mathfrak{J}}$. In fact, it will be the maximal cardinality of elements of $\mathfrak{J}$. Theorem 3. 3 together with Theorem 3. 2 imply that the property (vi) has no effect on the structure of $\bar{\mathfrak{J}}$ and $\mathfrak{J}$. Thus, the use of (vi) in the following sections will only simplify the formulations and proofs of our statements.

At the end, a similar consideration of the role of the subset S^N will be carried out.

First, let us remark that, making use of (iii)′, the axiom (vi) can be stated as follows

(vi)′ $x \in S \wedge X \in \mathfrak{P}\bar{S} \wedge x \in X \rightarrow [x, X] \in \delta.$

Now, introduce the definition of pseudosimilarity and similarity; let us point out that $\bar{S}$ is always a fixed set and „a relation ϱ on $\bar{S}$" means $\varrho \subseteqq \bar{S} \times \mathfrak{P}\bar{S}$ throughout this section.

Definition 3. 1. Two relations ϱ_1 and ϱ_2 on $\bar{S}$ are said to be *pseudosimilar* if

$$(\mathscr{P}) \qquad x \in \bar{S} \wedge X \in \mathfrak{P}\bar{S} \wedge x \notin X \rightarrow ([x, X] \in \varrho_1 \leftrightarrow [x, X] \in \varrho_2).$$

They are said to be *similar*, if they are pseudosimilar and $S_{\varrho_1} = S_{\varrho_2}$.

Both pseudosimilarity and similarity of relations on a set just defined are, obviously, equivalences. Further, if ϱ_1 and ϱ_2 are pseudosimilar, than

$$(3.\ 1) \qquad x \in \bar{S} \rightarrow ([x, \emptyset] \in \varrho_1 \leftrightarrow [x, \emptyset] \in \varrho_2)$$

Thus, for two similar relations ϱ_1 and ϱ_2 we have, besides $S_{\varrho_1} = S_{\varrho_2}$, also

$$(3.\ 2) \qquad S^N_{\varrho_1} = S^N_{\varrho_2} \quad \text{and} \quad S^S_{\varrho_1} = S^S_{\varrho_2}$$

Theorem 3. 2. *Let ϱ_1 and ϱ_2 be two relations on $\bar{S}$.* a) *If they are pseudosimilar, then $\bar{\mathfrak{J}}_{\bar{S}, \varrho_1} = \bar{\mathfrak{J}}_{\bar{S}, \varrho_2}$.* b) *If they are similar, then even $\mathfrak{J}_{S, \varrho_1} = \mathfrak{J}_{S, \varrho_2}$.*

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[5]) The system (i)′, (ii) and (iii)′ is not equivalent to (i), (ii) and (iii) (even if (iv), (v) and (vi) are supposed): If $\bar{S} = (s) \cup A$ with an infinite set A and a relation ϱ is defined by

$$[x, X] \in \varrho \leftrightarrow x = s \wedge (x \in X \vee X \notin \bar{\mathscr{F}}),$$

then, besides (ii), (iv), (v) and (vi) (with $S^c = S = (s)$), also (i)′ and (iii)′ are satisfied, but (i) is not (for, $[s, A] \in \varrho \wedge [s, \emptyset] \notin \varrho$).

[6]) A detailed consideration of this question will be given in another paper.

PROOF. The assertion a) follows immediately from the definitions ($\mathfrak{I}$) and ($\mathscr{P}$), the other part b) from the fact that $S_{\varrho_1} = S_{\varrho_2}$.

Theorem 3. 3. *Let ϱ be a relation on $\bar{S}$ satisfying the conditions* (i)—(v). *Then there exists a* GA-*dependence relation δ on $\bar{S}$ which is similar to ϱ.*

PROOF. Define the relation δ in the following way:

(a) $x \in \bar{S} \setminus S_\varrho \wedge X \in \mathfrak{P}\bar{S} \rightarrow ([x, X] \in \delta \leftrightarrow [x, X] \in \varrho)$;

(b) $x \in S_\varrho \wedge X \in \mathfrak{P}\bar{S} \wedge x \notin X \rightarrow ([x, X] \in \delta \leftrightarrow [x, X] \in \varrho)$;

(c) $x \in S_\varrho \wedge X \in \mathfrak{P}\bar{S} \wedge x \in X \rightarrow [x, X] \in \delta$.

Clearly, according to (a) and (b), $S_\varrho = S_\delta$ and ϱ and δ are similar. Then, by Theorem 3. 2, $\bar{\mathfrak{I}}_{\bar{S}, \varrho} = \bar{\mathfrak{I}}_{\bar{S}, \delta}$ and $\mathfrak{I}_{S, \varrho} = \mathfrak{I}_{S, \delta}$. Further, obviously,

$$[x, X] \in \varrho \rightarrow [x, X] \in \delta.$$

Now, using arguments of a routine nature we can easily prove that δ is a GA-dependence relation, i. e. that it satisfies (i)—(vi); let us remark only that, in consequence of the definitions (b) and (c), each of the properties (i)—(v) of δ follows already from the corresponding property of ϱ (taking $S_\varrho^c = S_\delta^c$) and that (c) is just the stronger form (vi) of the last condition for δ.

Thus, the proof of our theorem is completed.

Within the family $\mathfrak{D}_{\bar{S}}$ of all the GA-dependence relations on $\bar{S}$ the concept of similarity is unnecessary. For, any two similar GA-dependence relations are, in view of (3. 2), ($\mathscr{P}$) and (vi), identical. However, the other concept of pseudosimilarity enables us to introduce a „closure operation" in $\mathfrak{D}_{\bar{S}}$. Notice that the implication (3. 1) yields the equality $S_{\delta_1}^S = S_{\delta_2}^S$ for two pseudosimilar GA-dependence relations δ_1 and δ_2. Thus, two pseudosimilar GA-dependence relations δ_1 and δ_2 are different (i. e. are not similar) if, and only if, $S_{\delta_1}^N \neq S_{\delta_2}^N$.

Theorem 3. 4. *Let $\delta \in \mathfrak{D}_{\bar{S}}$. Then there exists a* GA-*dependence relation $\bar{\delta} \in \mathfrak{D}_{\bar{S}}$ such that*

(α) *δ and $\bar{\delta}$ are pseudosimilar and*

(β) $S_{\bar{\delta}}^N = \emptyset$.

Moreover, $\bar{\delta}$ is uniquely determined by (α) *and* (β).

PROOF. Define the relation $\bar{\delta}$ on $\bar{S}$ by

$$x \in \bar{S} \wedge X \in \mathfrak{P}\bar{S} \rightarrow ([x, X] \in \bar{\delta} \leftrightarrow [x, X] \in \delta \vee x \in S_\delta^N \cap X). \tag{3. 3}$$

First, we see immediately that δ and $\bar{\delta}$ are pseudosimilar, $S_{\bar{\delta}}^N = \emptyset$ and thus,

$$S_{\bar{\delta}} = S_\delta \cup S_\delta^N \quad \text{and} \quad \bar{\mathfrak{I}}_{\bar{S}, \delta} = \bar{\mathfrak{I}}_{\bar{S}, \bar{\delta}} = \mathfrak{I}_{S, \bar{\delta}}.$$

In order to show that $\bar{\delta}$ is a GA-dependence relation, we take $S_{\bar{\delta}}^c = S_\delta^c \cup S_\delta^N$ and notice that (i)—(vi) are satisfied in a trivial way provided x or y belong to S_δ^N; but, if x and y lie in S_δ, then the validity of (i)—(vi) for $\bar{\delta}$ follows from the corresponding property, and eventually the property (i), of δ.

The second part stating the uniqueness of $\bar{\delta}$ is a consequence of the assertion we have mentioned before formulating Theorem 3. 4.

REMARK. 3. 5. In a similar fashion, altering the definition (3. 3) for

$$x \in \bar{S} \wedge X \in \mathfrak{P}\bar{S} \to ([x, X] \in \delta^N \leftrightarrow [x, X] \in \delta \vee x \in (S_\delta^N \setminus N) \cap X),$$

the following generalization of Theorem 3. 4 can be proved:

Let $\delta \in \mathfrak{D}_{\bar{S}}$ and $N \subseteqq S_\delta^N$. Then there exists a GA-dependence relation $\delta^N \in \mathfrak{D}_{\bar{S}}$ such that

$(\alpha)'$ δ and δ^N are pseudosimilar and

$(\beta)'$ $S_{\delta^N}^N = N$.

The conditions $(\alpha)'$ and $(\beta)'$ determine δ^N uniquely.

REMARK. 3. 6. According to the foregoing, the operation $\delta \to \bar{\delta}$ in $\mathfrak{D}_{\bar{S}}$ is extensive ($\delta \leqq \bar{\delta}$) and idempotent ($\bar{\delta} = \bar{\bar{\delta}}$). Since it is also isotone, i. e., for δ_1 and δ_2 of $\mathfrak{D}_{\bar{S}}$, $\delta_1 \leqq \delta_2$ implies $\bar{\delta}_1 \leqq \bar{\delta}_2$, we can speak (in the sense very similar to that of E. H. MOORE [18]) of a closure operation $\delta \to \bar{\delta}$ in $\mathfrak{D}_{\bar{S}}$. The class $\mathfrak{C}_\delta \subseteqq \mathfrak{D}_{\bar{S}}$ of all the GA-dependence relations pseudosimilar to δ is just the subfamily of all the GA-dependence relations on $\bar{S}$ mapped in this operation onto $\bar{\delta}$; moreover, $\bar{\delta}$ belongs to $\mathfrak{C}_\delta$ and is the greatest element of it.

§ 4. An equivalent system of axioms

As throughout the whole paper, $(\bar{S}, \delta)$ is a fixed GA-dependence structure. First, let us state explicitely some easy consequences of (i)—(vi).

Lemma 4. 1. $x \in \bar{S} \setminus S^S \wedge X \subseteqq S^N \cup S^S \to [x, X] \notin \delta$.

PROOF. This follows immediately from $(\mathfrak{N})$ and (i).

Lemma 4. 2. $x \in S \wedge [x, (y)] \in \delta \to [y, (x)] \in \delta$. [7])

PROOF. By Lemma 4. 1, we have $y \in S$. The implication is trivial, if $x = y$. In the other case, we apply (iv) with $I = \emptyset$.

Lemma 4. 3. $x \in \bar{S} \wedge X \in \mathfrak{P}\bar{S} \wedge [x, X] \in \delta \to \exists F(F \subseteqq X \wedge F \in \mathscr{F} \wedge [x, F] \in \delta)$.

PROOF. By $(\mathfrak{N})$ and $(\mathfrak{S})$, the statement is obvious for $x \in S^N \cup S^S$, taking $F = \emptyset$. If $x \in S$, then, according to (i) and (ii), we deduce

$$[x, X \cap S] \in \delta \wedge X \cap S \in \mathfrak{P}S \to \exists F(F \subseteqq X \cap S \subseteqq X \wedge F \in \mathscr{F} \wedge [x, F] \in \delta).$$

[7]) Or, in another form: $x \in S \wedge y \in S \to ([x, (y)] \in \delta \leftrightarrow [y, (x)] \in \delta)$.

Lemma 4. 4. a) $\bar{I} \in \bar{\mathfrak{J}} \to \bar{I} \cap S^S = \emptyset$.

b) $\bar{I} \in \bar{\mathfrak{J}} \to \bar{I} \setminus S^N \in \mathfrak{J}$.

c) $I \in \mathfrak{J} \wedge X \subseteqq S^N \to I \cup X \in \bar{\mathfrak{J}}$. [8])

PROOF. All three parts are easy consequences of $(\bar{\mathfrak{J}})$, $(\mathfrak{J})$ and (i).

Now, let us formulate some basic properties of δ-independent sets and systems.

Theorem 4. 5. a) $\bar{I} \in \bar{\mathfrak{J}} \wedge \bar{I}' \subseteqq \bar{I} \to \bar{I}' \in \bar{\mathfrak{J}}$; [9])

b) $I \in \mathfrak{J} \wedge I' \subseteqq I \to I' \in \mathfrak{J}$;

c) $C \in \mathfrak{C} \wedge C' \subseteqq C \to C' \in \mathfrak{C}$;

d) $\bar{I} \in \bar{\mathfrak{J}} \leftrightarrow \forall F(F \subseteqq \bar{I} \wedge F \in \bar{\mathscr{F}} \to F \in \bar{\mathfrak{J}})$;

e) $I \in \mathfrak{J} \leftrightarrow \forall F(F \subseteqq I \wedge F \in \mathscr{F} \to F \in \mathfrak{J})$;

f) $C \in \mathfrak{C} \leftrightarrow \forall F(F \subseteqq C \wedge F \in \mathscr{F} \to F \in \mathfrak{C})$.

PROOF. a) Suppose $\bar{I}' \notin \bar{\mathfrak{J}}$. Then, there is, by $(\bar{\mathfrak{J}})$, an element $x_0 \in \bar{I}'$, such that $[x_0, \bar{I}' \setminus (x_0)] \in \delta$; in view of Lemma 4. 4 a) and $(\mathfrak{N})$, $x_0 \in S$. Hence, taking into account the inclusions $\bar{I}' \setminus (x_0) \subseteqq \bar{I} \setminus (x_0) \subseteqq \bar{S}$, we get, according to (iii)',

$$[x_0, \bar{I} \setminus (x_0)] \in \delta,$$

i. e. a contradiction of $\bar{I} \in \bar{\mathfrak{J}}$.

b) and c) result from a) immediately by $(\mathfrak{J})$ and $(\mathfrak{C})$, respectively.

d) The implication $\to$ follows also readily from a). Let us prove the other implication $\leftarrow$ indirectly. Then,

$$\bar{I} \notin \bar{\mathfrak{J}} \to \exists x_0 (x_0 \in \bar{I} \wedge [x_0, \bar{I} \setminus (x_0)] \in \delta),$$

and thus, in accordance with Lemma 4. 3, there exists F_0 such that

$$F_0 \subseteqq \bar{I} \setminus (x_0) \wedge F_0 \in \mathscr{F} \wedge [x_0, F_0] \in \delta.$$

Since $x_0 \in F_0 \cup (x_0)$ and $[x_0, (F_0 \cup (x_0)) \setminus (x_0)] = [x_0, F_0] \in \delta$, we have

$$F_0 \cup (x_0) \subseteqq \bar{I} \wedge F_0 \cup (x_0) \in \bar{\mathscr{F}} \wedge F_0 \cup (x_0) \notin \bar{\mathfrak{J}},$$

a contradiction of our hypothesis.

Finally, e) and f) follow again easily from d).

Lemma 4. 6. $x \in \bar{S} \wedge y \in \bar{S} \wedge \bar{I} \in \bar{\mathfrak{J}} \wedge [x, \bar{I} \cup (y)] \in \delta \wedge [x, \bar{I}] \notin \delta \to [y, \bar{I} \cup (x)] \in \delta$.

PROOF. First, notice that $[x, \bar{I} \cup (y)] \in \delta$ and $[x, \bar{I}] \notin \delta$ imply that $x \notin S^N$ and $x \notin S^S$, respectively. Further, if y were an element of $S^N \cup S^S$, then, by (i),

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[8]) Lemma 4. 4. can be expressed as the following equivalence

$$\bar{I} \in \bar{\mathfrak{J}} \leftrightarrow \bar{I} \cap S^S = \emptyset \wedge \bar{I} \cap S \in \mathfrak{J}.$$

[9]) Or, equivalently, $X \notin \bar{\mathfrak{J}} \wedge X \subseteq X' \to X' \notin \bar{\mathfrak{J}}$; and, similarly, for b) and c).

$[x, \bar{I}\cup(y)]\in\delta\rightarrow[x, \bar{I}]\in\delta$. Thus, both x and y belong to S. Moreover, clearly $y\notin\bar{I}$ and, by (vi)′, $x\notin\bar{I}$. Since the statement is trivial for $x=y$, suppose $x\neq y$.

Then, in view of Lemma 4. 3,

$$[x, \bar{I}\cup(y)]\in\delta\rightarrow\exists F(F\subseteqq\bar{I}\cup(y)\wedge F\in\mathscr{F}\wedge[x, F]\in\delta),$$

and, by (iii)′,

$$[x, \bar{I}]\notin\delta\wedge[x, F]\in\delta\wedge F\subseteqq\bar{I}\cup(y)\rightarrow F=F'\cup(y)\wedge F'\subseteqq\bar{I}\cap S,$$

i. e. $F'\in\mathscr{F}\cap\mathfrak{J}$, by Theorem 4. 5. Thus, in the whole,

$$x\in S\wedge y\in S\wedge F'\in\mathscr{F}\cap\mathfrak{J}\wedge x\neq y\wedge x\notin F'\wedge y\notin F'\wedge[x, F'\cup(y)]\in\delta\wedge[x, F']\notin\delta;$$

therefore, by (iv), $[y, F'\cup(x)]\in\delta$ and, by (iii)′,

$$[y, \bar{I}\cup(x)]\in\delta, \quad \text{q. e. d.}$$

Lemma 4. 7.

$$\exists S^C[S^C\subseteqq S\wedge\{x\in S\setminus S^C\rightarrow\exists I(I\in\mathscr{F}\cap\mathfrak{C}\wedge[x, I]\in\delta)\}\wedge$$
$$\wedge\{x\in\bar{S}\wedge\bar{I}\in\bar{\mathfrak{J}}\wedge C\in\mathfrak{C}\wedge x\notin\bar{I}\wedge[x, C]\in\delta\wedge$$
$$\wedge\forall y(y\in C\wedge y\notin\bar{I}\rightarrow[y, \bar{I}]\in\delta)\rightarrow[x, \bar{I}]\in\delta\}].$$

PROOF. The first part follows immediately from (v), Lemma 4. 3 and Theorem 4. 5 c). The second one is trivial for $x\in S^N\cup S^S$. Thus, let $x\in S$. Then, applying Lemma 4. 3 together with Theorem 4. 5 c), we see that there is a subset F such that

$$F\in\mathscr{F}\cap\mathfrak{C}\wedge F\subseteqq C\wedge[x, F]\in\delta;$$

let $F=(f_1, f_2, \ldots, f_n)$. Again, in view of Lemma 4. 3 and Theorem 4. 5, for any i, $1\leqq i\leqq n$, such that $f_i\notin\bar{I}$ there is F_i with the following properties

$$F_i\in\mathscr{F}\cap\mathfrak{J}\wedge F_i\subseteqq\bar{I}\wedge[f_i, F_i]\in\delta.$$

For the union F_0 of all these F_i we have, by Theorem 4. 5 and (iii),

$$F_0\subseteqq\bar{I}\wedge F_0\in\mathscr{F}\cap\mathfrak{J}\wedge\forall f(f\in F\wedge f\notin\bar{I}\rightarrow[f, F_0]\in\delta).$$

Then, making use of (v) together with (iii)′, we get our assertion $[x, \bar{I}]\in\delta$.

Now, we are ready to formulate our equivalent axiomatic system:

Theorem 4. 8. *The system of the axioms* (i)—(vi) *is equivalent to the following (formally stronger) system* (1)—(6):

(1) $x\in\bar{S}\wedge X\in\mathfrak{P}\bar{S}\rightarrow([x, X]\in\delta\leftrightarrow[x, X\cap S]\in\delta)$; [10])

(2) $x\in\bar{S}\wedge X\in\mathfrak{P}\bar{S}\wedge[x, X]\in\delta\rightarrow\exists F(F\subseteqq X\wedge F\in\mathscr{F}\wedge[x, F]\in\delta)$;

(3) $x\in\bar{S}\wedge X\subseteqq Y\subseteqq\bar{S}\wedge[x, X]\in\delta\rightarrow[x, Y]\in\delta$;

(4) $x\in\bar{S}\wedge y\in\bar{S}\wedge\bar{I}\in\bar{\mathfrak{J}}\wedge[x, \bar{I}\cup(y)]\in\delta\wedge[x, \bar{I}]\notin\delta\rightarrow[y, \bar{I}\cup(x)]\in\delta$;

[10]) The implication $[x, X\cap S]\in\delta\rightarrow[x, X]\in\delta$ follows already from (3).

(5) $\exists S^c[S^c \subseteqq S \wedge \{x \in S \rightarrow \exists I(I \in \mathscr{F} \cap \mathfrak{C} \wedge [x, I] \in \delta)\} \wedge$

$\wedge \{x \in \bar{S} \wedge \bar{I} \in \bar{\mathfrak{J}} \wedge C \in \mathfrak{C} \wedge [x, C] \in \delta \wedge \forall y(y \in C \rightarrow [y, \bar{I}] \in \delta) \rightarrow [x, \bar{I}] \in \delta\}]$;

(6) $X \in \mathfrak{P}\bar{S} \wedge x \in X \cap S \rightarrow [x, X] \in \delta$.

PROOF. Clearly, (1) → (i), (2) → (ii), (3) → (iii), (4) → (iv), (5) ∧ (6) → (v) and (6) → → (vi). On the other hand, (1) is an easy consequence of (i), (2) is the statement of Lemma 4. 3, (3) follows immediately from (iii)′, (4) is the assertion of Lemma 4. 6, Lemma 4. 7 together with (vi)′ imply (5) and (6) is the property (vi)′.

REMARK. 4. 9. It is quite obvious that neither in the first part of (5) nor in (6) the assumption $x \in S$ can be replaced merely by $x \in \bar{S}$. Also, (4) cannot be strenghtened by omitting the assumption $\bar{I} \in \bar{\mathfrak{J}}$ and as to (5), there does not exist, in general, a canonic subset S^c auch that any element of S depends on a suitable single element of S^c [11]). In order to prove these facts, let us consider two examples of GA-dependence structures.

1. Let $\bar{S} = (a, b, c, d)$ and δ' be given by the following GA-dependence table

	∅	(a)	(b)	(c)	(d)	(a, b)	(a, c)	(a, d)	(b, c)	(b, d)	(c, d)	(a, b, c)	(a, b, d)	(a, c, d)	(b, c, d)	(a, b, c, d)
a	0	1	1	0	0	1	1	1	1	1	0	1	1	1	1	1
b	0	1	1	0	0	1	1	1	1	1	0	1	1	1	1	1
c	0	0	0	1	0	0	1	0	1	0	1	1	1	1	1	1
d	0	0	0	0	1	0	0	1	0	1	1	0	1	1	1	1

Thus, $S = (a, b, c, d)$, $\bar{\mathfrak{J}} = \mathfrak{J} = \{\emptyset, (a), (b), (c), (d), (a, c), (a, d), (b, c), (b, d), (c, d), (a, c, d), (b, c, d)\}$. All the properties (i)—(vi) can easily be verified; moreover, $\bar{\mathfrak{S}} = \{\bar{S}, (a, c, d), (b, c, d)\}$. But,

$$[c, (a, b, d)] \in \delta' \wedge [c, (a, b)] \notin \delta' \wedge [d, (a, b, c)] \notin \delta',$$

showing the necessity of the assumption $\bar{I} \in \bar{\mathfrak{J}}$ in (4).

Let us point out that, taking $S^c = S$, we have

$$[[c, (a, b, d)] \in \delta' \wedge [a, (a, d)] \in \delta' \wedge [b, (a, d)] \in \delta' \wedge [d, (a, d)] \in \delta' \wedge [c, (a, d)] \notin \delta',$$

i. e. also the assumption $C \in \mathfrak{C}$ (in fact, $C \in \mathfrak{J}$) of (5) is here essential. However, we shall see that, as in this particular example, there exist, in general, canonic subsets such that this assumption can be omitted. But, because of rather limited choice of such canonic subsets we shall find it important that (v) (and thus also (5)) has been introduced in a weaker form.

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[11]) For the dependence in abelian groups (see also the last § 9) both these stronger properties are satisfied.

2. Now, consider the GA-dependence structure $(\bar{S}, \delta'')$ defined by the following table

	Ø	(a)	(b)	(c)	(d)	(a, b)	(a, c)	(a, d)	(b, c)	(b, d)	(c, d)	(a, b, c)	(a, b, d)	(a, c, d)	(b, c, d)	(a, b, c, d)
a	0	1	1	1	0	1	1	1	1	1	1	1	1	1	1	1
b	0	1	1	0	0	1	1	1	1	1	1	1	1	1	1	1
c	0	1	0	1	0	1	1	1	1	1	1	1	1	1	1	1
d	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1

Thus, $\bar{S} = S = (a, b, c, d)$ and $\bar{\mathfrak{J}} = \mathfrak{J} = \{\emptyset, (a), (b), (c), (d), (a, d), (b, c), (b, d), (c, d)\}$. Again, it is easy to verify the properties (i)—(vi). Since

$$[b, (a)] \in \delta'' \wedge [a, (c)] \in \delta'' \wedge [b, (c)] \notin \delta''$$

and

$$[d, (b, c)] \in \delta'' \wedge [b, (a)] \in \delta'' \wedge [c, (a)] \in \delta'' \wedge [d, (a)] \notin \delta'',$$

we conclude that

$$\hat{\mathfrak{S}} = \{(b, d), (c, d)\}.$$

But, if S^c is (b, d) or (c, d), then there is no single element in S^c such that c or b depends on it, respectively.

Remark. 4. 10. Let us conclude this section by the following characterization of the subsets of neutral and singular elements in a GA-dependence structure:

$$x \in S^N \leftrightarrow [x, (x)] \notin \delta;$$
$$x \in S^S \leftrightarrow [x, \emptyset] \in \delta.$$

§ 5. The independence of the axiomatic systems

In this short paragraph, we shall prove the independence of our systems by constructions of suitable models. We use here the notation $\overrightarrow{(\mathrm{i})}$ and $\overleftarrow{(\mathrm{i})}$ in order to distinguish two parts of the first axiom (i): $[x, X] \in \delta \rightarrow [x, X \cap S] \in \delta$ and $[x, X \cap S] \in \delta \rightarrow [x, X] \in \delta$, respectively.

1. a) $\overrightarrow{(\mathrm{i})}$ does not depend on $\overleftarrow{(\mathrm{i})}$, (ii)—(vi). Consider the set $\bar{S} = (a, b)$ and the relation ϱ_{11} on it given by the following table[12])

	Ø	(a)	(b)	(a, b)
a	0	1	1	1
b	1	1	1	1

[12]) In this section, we extend the concept of a GA-dependence table (Definition 2. 2) to the case of a general relation on a set.

Thus, $S^N = \emptyset$, $S^S = (b)$, $S = (a)$ and $\bar{\mathfrak{I}} = \mathfrak{I} = \{\emptyset, (a)\}$. It is easy to verify the validity of $(\overleftarrow{\text{i}})$, (ii)—(vi). But, $[a, (b)] \in \varrho_{11} \wedge [a, \emptyset] \notin \varrho_{11}$. Simultaneously, we see that (1) does not depend on (2)—(6).

b) $(\overleftarrow{\text{i}})$ does not depend on $(\overrightarrow{\text{i}})$, (ii)—(vi). Let the relation ϱ_{12} on $\bar{S} = (a, b)$ be defined by the table

	$\emptyset$	(a)	(b)	(a, b)
a	0	1	0	0
b	1	1	1	1

It is $S^N = \emptyset$, $S^S = (b)$, $S = (a)$ and $\bar{\mathfrak{I}} = \mathfrak{I} = \{\emptyset, (a)\}$ again; now, $(\overrightarrow{\text{i}})$, (ii)—(vi) are satisfied, but $[a, (a)] \in \varrho_{12} \wedge [a, (a, b)] \notin \varrho_{12}$.

Here, $[a, (a, b)] \notin \varrho_{12}$, i. e. (6) does not hold for ϱ_{12}; let us give an example showing that $(\overleftarrow{\text{i}})$ does not depend even on $(\overrightarrow{\text{i}})$, (2), (iii), (4)—(6):[13]) The relation ϱ_{13} on $\bar{S} = (a, b, c)$ be given by

	$\emptyset$	(a)	(b)	(c)	(a, b)	(a, c)	(b, c)	(a, b, c)
a	0	1	1	0	1	1	0	1
b	0	1	1	0	1	1	1	1
c	1	1	1	1	1	1	1	1

Hence, $S^N = \emptyset$, $S^S = (c)$, $S = (a, b)$ and $\bar{\mathfrak{I}} = \mathfrak{I} = \{\emptyset, (a), (b)\}$. The properties $(\overrightarrow{\text{i}})$, (2), (iii), (4)—(6) are clearly satisfied (taking $S^c = (a)$), but $[a, (b)] \in \varrho_{13} \wedge [a, (b, c)] \notin \varrho_{13}$.

2. (ii) does not depend on (i), (iii)—(vi). Consider an infinite set $\bar{S}$ with the relation ϱ_2 defined by

$$a \in \bar{S} \wedge A \subseteqq \bar{S} \rightarrow ([a, A] \in \varrho_2 \leftrightarrow a \in A \vee A \notin \bar{\mathscr{F}}).\ ^{14})$$

Clearly, $S = \bar{S}$ and $\bar{\mathfrak{I}} = \mathfrak{I} = \mathscr{F}$. Taking $S^c = S$, we can verify all the properties (i), (1), (iii)—(vi) and (3)—(6); but $[a, \bar{S} \setminus (a)] \in \varrho_2$ and there is no finite subset F of $\bar{S} \setminus (a)$ such that $[a, F] \in \varrho_2$. Simultaneously, we have shown that (2) does not depend on (1), (3)—(6).

3. (iii) does not depend on (i), (ii), (iv)—(vi). This can be established very simply by consideration of the set $\bar{S} = (a, b)$ and the relation ϱ_{31} given by the table

	$\emptyset$	(a)	(b)	(a, b)
a	0	1	0	0
b	0	0	1	0

[13]) On the other hand, as we have pointed out, it is a consequence of (3).

[14]) This example might be generalized in the following way: If card $(\bar{S}) = \mathfrak{a} \geqq \aleph_0$, and $\mathfrak{a} \geqq \mathfrak{b} \geqq \aleph_0$ then define ϱ_2 by

$$a \in \bar{S} \wedge A \subseteq \bar{S} \rightarrow ([a, A] \in \varrho_2 \leftrightarrow a \in A \vee \text{card}\,(A) \geqq \mathfrak{b}).$$

In order to show that also (3) does not depend on (1), (2), (4)—(6), let us consider the following general type of relations on a set $\bar{S}$:

Let card $(\bar{S}) = \mathfrak{a} \geqq 2$ and $2 \leqq \mathfrak{b} \leqq \mathfrak{a}$ be a given cardinal number; define the relation ϱ_{32} by

$$a \in \bar{S} \wedge A \subseteqq \bar{S} \rightarrow ([a, A] \in \varrho_{32} \leftrightarrow a \in A \vee 1 \leqq \text{card}(A) < \mathfrak{b}).$$

Clearly, $S = \bar{S}$ and $\bar{\mathfrak{I}} = \mathfrak{I} = \emptyset \cup \{(a)\}_{a \in \bar{S}}$. Taking $S^c = (a_0)$, where a_0 is an arbitrary element of $\bar{S}$, the properties (1), (2), (4)—(6) can easily be verified. If $\mathfrak{b} = \mathfrak{a}$ is finite (especially, if $\mathfrak{a} = 2$), then even (3) is satisfied, i. e. ϱ_{32} is a GA-dependence relation on $\bar{S}$. Otherwise, ϱ_{32} fails to satisfy (3). [15])

4. (iv) does not depend on (i)—(iii), (v), (vi). Consider the relation ϱ_4 on the set $\bar{S} = (a, b)$ given by

	Ø	(a)	(b)	(a, b)
a	0	1	0	1
b	0	1	1	1

Then, $S = (a, b)$ and $\bar{\mathfrak{I}} = \mathfrak{I} = \{\emptyset, (a), (b)\}$. (i), (ii), (iii), (v) (taking e. g. $S^c = (a)$) and (vi) are satisfied, but $[b, (a)] \in \varrho_4 \wedge [a, (b)] \notin \varrho_4$ implies, in view of Lemma 4. 2, that ϱ_4 does not possess the property (iv). The example shows, at the same time, that (4) does not depend on (1)—(3), (5) and (6).

5. (v) and (5) do not depend on (i)—(iv), (vi) and (1)—(4), (6). Define the relation ϱ_5 on the set $\bar{S} = (a, b, c, d)$ in the following way:

	Ø	(a)	(b)	(c)	(d)	(a, b)	(a, c)	(a, d)	(b, c)	(b, d)	(c, d)	(a, b, c)	(a, b, d)	(a, c, d)	(b, c, d)	(a, b, c, d)
a	0	1	1	1	0	1	1	1	1	1	1	1	1	1	1	1
b	0	1	1	0	1	1	1	1	1	1	1	1	1	1	1	1
c	0	1	0	1	1	1	1	1	1	1	1	1	1	1	1	1
d	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1

We have $S = \bar{S}$ and $\bar{\mathfrak{I}} = \mathfrak{I} = \{\emptyset, (a), (b), (c), (d), (a, d), (b, c)\}$. The relation ϱ_5 has evidently all the properties in question with the exception of (v) and (5). Since

$$[a, (b)] \in \varrho_5, \quad [b, (d)] \in \varrho_5, \quad [d, (c)] \in \varrho_5, \quad [c, (a)] \in \varrho_5,$$

and

$$[a, (d)] \notin \varrho_5, \quad [b, (c)] \notin \varrho_5, \quad [d, (a)] \notin \varrho_5, \quad [c, (b)] \notin \varrho_5,$$

no one of the elements b, d, c, a can be in S^c. Thus, neither (v) nor (5) is valid.

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[15]) Note that, if $\aleph_0 \leqq \mathfrak{a} \leqq \mathfrak{b}$, the relation ϱ_{32} is „nearly complementary" to the relation ϱ_2 introduced above in the footnote[14]).

6. (vi) and (6) do not depend on (i)—(v) and (1)—(5). Let ϱ_6 be the relation on a set $\bar{S}$, $3 \leqq \text{card}(\bar{S}) < \aleph_0$, defined by

$$a \in \bar{S} \wedge A \subseteqq \bar{S} \rightarrow ([a, A] \in \varrho_6 \leftrightarrow A \supseteqq \bar{S} \setminus (a)).$$

Hence, $S = \bar{S}$ and $\bar{\mathfrak{J}} = \mathfrak{J} = \mathfrak{P}\bar{S} \setminus \{\bar{S}\}$. Taking $S^c = S$, the validity of (i)—(v) and (1)—(5) can readily be proved. However,

$$[a, (a)] \notin \varrho_6 \text{ for every } a \in \bar{S}.^{16)}$$

§ 6. δ-independent sets

The purpose of this section is to prove the existence of maximal δ-independent sets, and especially, of maximal δ-canonic systems. Let us start with some preparatory lemmas.

Lemma 6. 1. $x \in \bar{S} \wedge \bar{I} \in \bar{\mathfrak{J}} \wedge [x, \bar{I}] \notin \delta \rightarrow \bar{I} \cup (x) \in \bar{\mathfrak{J}}$. [17])

PROOF. Let $\bar{I} \cup (x) \notin \bar{\mathfrak{J}}$. Then, in view of our assumption, there is $y \in \bar{I}$ such that

$$[y, (\bar{I} \setminus (y)) \cup (x)] \in \delta.$$

Since, by Theorem 4. 5, $\bar{I} \setminus (y) \in \bar{\mathfrak{J}}$ and since $[y, \bar{I} \setminus (y)] \notin \delta$, we get, by Theorem 4. 8 (4), a contradiction.

Lemma 6. 2. *Let* $\{I_\gamma\}_{\gamma \in \Gamma}$ *be a directed system*[18]) *of elements of* $\bar{\mathfrak{J}}$, *or* $\mathfrak{J}$, *or* $\mathfrak{C}$ *and let*

$$I = \bigcup_{\gamma \in \Gamma} I_\gamma.$$

Then $I \in \bar{\mathfrak{J}}$, *or* $I \in \mathfrak{J}$, *or* $I \in \mathfrak{C}$, *respectively.*

PROOF. We give an indirect proof again; suppose $I \notin \bar{\mathfrak{J}}$. Then there are an element $x \in I$ and a finite subset $F \subseteqq I \setminus (x)$ such that $[x, F] \in \delta$; thus, the finite set $F \cup (x)$ does not belong to $\bar{\mathfrak{J}}$. According to the properties of a directed system, there

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[16] Also the relation ϱ_{61} on $\bar{S} = (a, b)$ defined by means of the table

	Ø	(a)	(b)	(a, b)
a	0	0	1	1
b	0	1	1	1

shows the independence of the axioms (vi) and (6) on the others.

[17]) This can be expressed in another form as follows:

$$x \in \bar{S} \wedge \bar{I} \in \bar{\mathfrak{J}} \wedge x \notin \bar{I} \rightarrow ([x, \bar{I}] \in \delta \leftrightarrow \bar{I} \cup (x) \notin \bar{\mathfrak{J}}).$$

[18]) Or, in another terminology (see e. g. J. KELLEY [11]) a directed set.

is an index $\gamma_0 \in \Gamma$ such that

$$F \cup (x) \subseteq I_{\gamma_0},$$

and since $I_{\gamma_0} \in \bar{\mathfrak{J}}$, we have a contradiction of Theorem 4. 5.

If, for every $\gamma \in \Gamma$, $I_\gamma \subseteq S$ or $I_\gamma \subseteq S^c$, then obviously also $I \subseteq S$ or $I \subseteq S^c$, respectively. The proof is completed.

As a particular case of Lemma 6. 2 let us formulate

Lemma 6. 2.' *Let*

$$I_1 \subseteq I_2 \subseteq \ldots \subseteq I_\alpha \subseteq \ldots, \qquad \alpha < \nu,$$

be a non-decreasing chain of elements of $\bar{\mathfrak{J}}$, *or* $\mathfrak{J}$, *or* $\mathfrak{C}$. *Then* $\bigcup_{\alpha<\nu} I_\alpha$ *is also in* $\bar{\mathfrak{J}}$, *or* $\mathfrak{J}$, *or* $\mathfrak{C}$, *respectively.*

REMARK 6. 3. Lemma 6. 2.' and Lemma 6. 1 can be used to prove the following condition for a set $\bar{I}$ to be independent:

Let $\bar{I} = (x_\alpha)_{1 \leq \alpha < \nu}$; then, $\bar{I} \in \bar{\mathfrak{J}}$ if, and only if,

$$[x_\beta, (x_\alpha)_{1 \leq \alpha < \beta}] \notin \delta \text{ for any } \beta,$$

(where by $(x_\alpha)_{1 \leq \alpha < 1}$ the empty set $\emptyset$ is to be understood).

Lemma 6. 4. $x \in \bar{S} \wedge \bar{I} \in \bar{\mathfrak{J}} \wedge C \in \mathfrak{C} \wedge [x, \bar{I}] \notin \delta \wedge [x, C] \in \delta \rightarrow \exists C'(C' \subseteq C \wedge C' \in \mathscr{F} \wedge \wedge \bar{I} \cup C' \in \bar{\mathfrak{J}} \wedge [x, \bar{I} \cup C'] \in \delta$.

PROOF. First, there is a finite subset $F \subseteq C$ such that $[x, F] \in \delta$. Let card $(F) = k$.

Now, there should be $x_1 \in F$ such that $[x_1, \bar{I}] \notin \delta$; for, otherwise, by Theorem 4. 8 (5), we would get $[x, \bar{I}] \in \delta$. Thus, in view of Lemma 6. 1, we have $\bar{I} \cup (x_1) \in \bar{\mathfrak{J}}$. If $[x, \bar{I} \cup (x_1)] \in \delta$, we put $C' = (x_1)$. In the other case, we must reach, by induction in at most k steps, a subset $C' = (x_i)_{i=1,2,\ldots,l} \subseteq F \subseteq C$, $l \leq k$, such that

$$\bar{I} \cup C' \in \bar{\mathfrak{J}} \quad \text{and} \quad [x, \bar{I} \cup C'] \in \delta, \qquad \text{q. e. d.}$$

On the basis of the preceding lemma, we can prove

Lemma 6. 5. $x \in S \wedge \bar{I} \ni \bar{\mathfrak{J}} \wedge [x, \bar{I}] \notin \delta \rightarrow \exists y(y \in S^C \wedge [y, \bar{I}] \notin \delta)$.

PROOF. Theorem 4. 8 (5) guarantees the existence of $C \in \mathfrak{C}$ such that $[x, C] \in \delta$. Then, making use of Lemma 6. 4, we can find a subset $C' \subseteq C$ with

$$\bar{I} \cup C' \in \bar{\mathfrak{J}} \quad \text{and} \quad [x, \bar{I} \cup C'[\in \delta.$$

Thus, $C' \neq \emptyset$ and, for any $y \in C'$, we have $[y, \bar{I}] \notin \delta$.

Now, let us introduce the following definitions concerning maximal subsets:

Definition 6. 6. Define the classes $\mathfrak{M}$, $\bar{\mathfrak{J}}^*$, $\mathfrak{J}^*$ and $\mathfrak{C}^*$ [19]) of *maximal sets, maximal δ-independent sets, maximal δ-independent systems* and *maximal δ-canonic*

[19]) In full notation, $\mathfrak{M}_{\bar{S},\delta}$, $\bar{\mathfrak{J}}^*_{\bar{S},\delta}$, $\mathfrak{J}^*_{S,\delta}$ and $\mathfrak{C}^*_{S\underset{\circ}{C}}$.

systems by

$(\mathfrak{M})\quad M\in\mathfrak{M} \leftrightarrow M\in\mathfrak{P}\bar{S}\wedge\forall x\ \ (x\in\bar{S}\setminus M\to[x, M]\in\delta),$

$(\bar{\mathfrak{J}}^*)\quad \bar{\mathfrak{J}}^*=\mathfrak{M}\cap\bar{\mathfrak{J}},$

$(\mathfrak{J}^*)\quad I\in\mathfrak{J}^*\leftrightarrow I\in\mathfrak{J}\wedge\forall x\ (x\in S\setminus I\to[x, I]\in\delta)$ and

$(\mathfrak{C}^*)\quad \mathfrak{C}^*=\mathfrak{J}^*\cap\mathfrak{P}S^C,$

respectively.

The following propositions can be deduced readily:

$\bar{I}^*\in\bar{\mathfrak{J}}^*\leftrightarrow\bar{I}^*\in\bar{\mathfrak{J}}\wedge\forall x\ \ (x\in\bar{S}\setminus\bar{I}^*\to\bar{I}^*\cup(x)\notin\bar{\mathfrak{J}});$

$\bar{I}^*\in\bar{\mathfrak{J}}^*\leftrightarrow\bar{I}^*\in\bar{\mathfrak{J}}\wedge\forall x\ \ (x\in\bar{S}\to[x,\bar{I}^*]\in\delta\vee x\in\bar{I}^*);$

$I^*\in\mathfrak{J}^*\leftrightarrow I^*\in\mathfrak{J}\wedge\forall x\ \ (x\in S\setminus I^*\to I^*\cup(x)\notin\mathfrak{J});$

$I^*\in\mathfrak{J}^*\leftrightarrow I^*\in\mathfrak{J}\wedge\forall x\ \ (x\in S\to[x, I^*]\in\delta);$

$I^*\in\bar{\mathfrak{J}}^*\to S^N\subseteq\bar{I}^*\wedge S^S\cap\bar{I}^*=\emptyset;$

$I\subseteq S\to(I\in\mathfrak{J}^*\leftrightarrow I\cup S^N\in\bar{\mathfrak{J}}^*).$

And now, formulate the main result of this section.

Theorem 6. 7. a) $\bar{I}\in\bar{\mathfrak{J}}\to\exists\bar{I}^*(\bar{I}^*\in\bar{\mathfrak{J}}^*\wedge\bar{I}\subseteq\bar{I}^*).$

b) $I\in\mathfrak{J}\to\exists I^*(I^*\in\mathfrak{J}^*\wedge I\subseteq I^*).$

c) $I\in\mathfrak{J}\to\exists I^*(I^*\in\mathfrak{J}^*\wedge I\subseteq I^*\wedge I^*\setminus I\subseteq S^C).$

d) $\bar{\mathfrak{J}}^*\neq\emptyset.$

e) $\mathfrak{J}^*\neq\emptyset.$

f) $\mathfrak{C}^*\neq\emptyset.$

Proof. Assume, for a moment, the validity of the assertion c); then, adding S^N to I^*, we get a); b) follows immediately; and, taking $\bar{I}=I=\emptyset$, we have d), e) and f).

In order to prove c), consider the family $\mathfrak{X}$ of all the subsets X such that

$$X\in\mathfrak{J}\wedge X\supseteq I\wedge X\setminus I\subseteq S^C.$$

Clearly, the union of a chain of such

$$X_1\subseteq X_2\subseteq\ldots\subseteq X_\alpha\subseteq\ldots,\qquad \alpha<\nu,$$

is an element of $\mathfrak{X}$ again; for, by Lemma 6. 2', $\bigcup_{\alpha<\nu}X_\alpha\in\mathfrak{J}$ and $I\subseteq\bigcup_{\alpha<\nu}X_\alpha$ with

$$\Big(\bigcup_{\alpha<\nu}X_\alpha\Big)\setminus I=\bigcup_{\alpha<\nu}(X_\alpha\setminus I)\subseteq S^C.$$

Hence, by Zorn's lemma, pick a maximal $I^*\in\mathfrak{X}$.

It remains to prove that $I^* \in \mathfrak{J}^*$. We suppose the contrary. Then, there is $x \in S \setminus I^*$ with $[x, I^*] \notin \delta$. According to Lemma 6. 5, an element $y \in S^c$ exists such that $[y, I^*] \notin \delta$, i. e., in view of Lemma 6. 1, $I^* \cup (y) \neq I^*$ belongs to $\mathfrak{Y}$. This contradicts the maximality of I^* and concludes the proof of Theorem 6. 7.

§ 7. The δ-rank of a GA-dependence structure

Let us start with the following definitions.

Definition 7. 1. For any $X \subseteq \bar{S}$ or $X \subseteq S$, define the *complete δ-closure* $\bar{\mathrm{cl}}(X)$ or *δ-closure* $\mathrm{cl}(X)$ of X by

$$(\bar{\mathrm{cl}}) \qquad x \in \bar{\mathrm{cl}}(X) \leftrightarrow x \in \bar{S} \wedge ([x, X] \in \delta \vee x \in X)$$

or

$$(\mathrm{cl}) \qquad x \in \mathrm{cl}(X) \leftrightarrow x \in S \wedge [x, X] \in \delta,$$

respectively. A subset X of $\bar{S}$ is said to be *completely δ-closed* if $\bar{\mathrm{cl}}(X) = X$ and $X \subseteq S$ to be *δ-closed* if $\mathrm{cl}(X) = X$.

Clearly, for any $X \subseteq \bar{S}$,

$$\bar{\mathrm{cl}}(X) = \mathrm{cl}(X \cap S) \cup X \cup S^S = \mathrm{cl}(X \cap S) \cup (X \cap S^N) \cup S^S.$$

Also, X is completely δ-closed if, and only if, $X \supseteq S^S$ and $X \cap S$ is δ-closed.

Further, we see immediately that both operations of the δ-closures are extensive and isotone, i. e.

$$X_1 \subseteq X_2 \subseteq \bar{S} \to X_1 \subseteq \bar{\mathrm{cl}}(X_1) \wedge \bar{\mathrm{cl}}(X_1) \subseteq \bar{\mathrm{cl}}(X_2)$$

and

$$X_1 \subseteq X_2 \subseteq S \to X_1 \subseteq \mathrm{cl}(X_1) \wedge \mathrm{cl}(X_1) \subseteq \mathrm{cl}(X_2).$$

Even more, for I_1 and I_2 from $\bar{\mathfrak{J}}$, $I_1 \subset I_2$, we have obviously

$$\bar{\mathrm{cl}}(I_1) \subset \bar{\mathrm{cl}}(I_2)$$

and similarly for the δ-closure provided I_1 and I_2 belong to $\mathfrak{J}$.

Remark 7. 2. The present example shows that these operations are not, in general, idempotent, i. e., in general, $\mathrm{cl}^{(2)}(X) = \mathrm{cl}(\mathrm{cl}(X)) = \mathrm{cl}(X)$ does not hold.

Let

$$\bar{S}_n = (a_i)_{1 \leq i \leq n+1}, \; n \geq 3, \text{[20]}$$

be a given set; define the relation δ on it in the following way (X being a subset of $\bar{S}_n$):

$$[a_1, X] \in \delta \leftrightarrow [a_2, X] \leftrightarrow a_1 \in X \vee a_2 \in X \vee (a_n, a_{n+1}) \subseteq X;$$

$$[a_i, X] \in \delta \leftrightarrow a_i \in X \vee (a_1, a_2, \dots, a_{i-1}) \subseteq X \quad \text{for} \quad 3 \leq i \leq n-1;$$

$$[a_n, X] \in \delta \leftrightarrow a_n \in X \vee (a_1, a_{n+1}) \subseteq X \vee (a_2, a_{n+1}) \subseteq X \vee (a_1, a_2, \dots, a_{n-1}) \subseteq X;$$

$$[a_{n+1}, X] \in \delta \leftrightarrow a_{n+1} \in X \vee (a_1, a_n) \subseteq X \vee (a_2, a_n) \subseteq X.$$

[20]) A similar GA-dependence structure for $n = 2$ can also be constructed.

Thus, $S_n = \bar{S}_n$ and, putting $S_n^c = (a_i)_{3 \leq i \leq n+1}$, it can easily be verified that δ is a GA-dependence relation on $\bar{S}_n$ (in fact, all the properties (i)–(vi), besides—perhaps—(iv), are evident).

Now, considering the subsets $A_1 = (a_1)$ and $A_2 = (a_n, a_{n+1})$, we deduce that

$$\overline{\mathrm{cl}}^{(j)}(A_1) = \mathrm{cl}^{(j)}(A_1) = (a_1, a_2, \ldots, a_{j+1}) \quad \text{for} \quad 0 \leq j \leq n,$$

and

$$\overline{\mathrm{cl}}^{(j)}(A_2) = \mathrm{cl}^{(j)}(A_2) = A_2 \cup (a_1, a_2, \ldots, a_{j+1}) \quad \text{for} \quad 1 \leq j \leq n-2.$$ [21])

Hence,

$$A_1 \neq \mathrm{cl}(A_1) \neq \ldots \neq \mathrm{cl}^{(n-1)}(A_1) \neq \mathrm{cl}^{(n)}(A_1) = \mathrm{cl}^{(n+1)}(A_1) = \ldots$$

and

$$A_2 \neq \mathrm{cl}(A_2) \neq \ldots \neq \mathrm{cl}^{(n-3)}(A_2) \neq \mathrm{cl}^{(n-2)}(A_2) = \mathrm{cl}^{(n-1)}(A_2) = \ldots$$

In the next section (Remark 8. 12) we give a more general example of a GA-dependence structure $(\bar{S}_\omega, \delta)$ containing, for any natural n_0, a subset $A \subseteq S_\omega$ such that

$$A \neq \mathrm{cl}(A) \neq \ldots \neq \mathrm{cl}^{(n_0-1)}(A) \neq \mathrm{cl}^{(n_0)}(A) = \mathrm{cl}^{(n_0+1)}(A) = \ldots$$

We have seen that the operations $X \to \overline{\mathrm{cl}}(X)$ and $X \to \mathrm{cl}(X)$ are „mehrstufigen Hüllenoperatoren" in the sense of J. SCHMIDT [25]. Due to this fact, our concept of a GA-dependence relation is an essential generalization of the „classical" dependence relation of VAN DER WAERDEN.

REMARK 7. 3. Let us mention that in terms of the δ-closure the second part of (5) of Theorem 4. 8 (or, also of the axiom (v)) can be formulated as follows:

$$(5)' \quad I \in \bar{\mathfrak{I}} \wedge C \in \mathfrak{C} \wedge C \subseteq \overline{\mathrm{cl}}(I) \to \overline{\mathrm{cl}}(C) \subseteq \overline{\mathrm{cl}}(I)$$

(or,

$$(\mathrm{V})' \quad I \in \mathscr{F} \cap \mathfrak{I} \wedge C \in \mathscr{F} \cap \mathfrak{C} \wedge C \subseteq \mathrm{cl}(I) \to \mathrm{cl}(C) \subseteq \mathrm{cl}(I)).$$

Thus, in particular, for $C_1 \in \mathfrak{C}$ and $C_2 \in \mathfrak{C}$ satisfying $C_1 \subseteq \mathrm{cl}(C_2)$ and $C_2 \subseteq \mathrm{cl}(C_1)$, we have $\mathrm{cl}(C_1) = \mathrm{cl}(C_2)$.

Also the classes $\mathfrak{M}$, $\bar{\mathfrak{I}}^*$, $\mathfrak{I}^*$ and $\mathfrak{C}^*$ can easily be defined as the classes of subsets M such that $\overline{\mathrm{cl}}(M) \supseteq \bar{S}$, etc.

REMARK 7. 4. Consider for a moment the subfamilies $\bar{\mathfrak{D}}$ and $\mathfrak{D}$ (of $\mathfrak{P}\bar{S}$) of all the completely δ-closed and δ-closed subsets, respectively. Since the intersection of elements of $\bar{\mathfrak{D}}$ (or $\mathfrak{D}$) belongs to $\bar{\mathfrak{D}}$ (or $\mathfrak{D}$) again, there exists, to any subset $X \subseteq \bar{S}$ (or $X \subseteq S$) the least completely δ-closed (or δ-closed) subset containing X; let us denote it by $\overline{\mathrm{Cl}}(X)$ (or $\mathrm{Cl}(X)$). The operations

$$X \to \overline{\mathrm{Cl}}(X) \quad \text{and} \quad X \to \mathrm{Cl}(X)$$

[21]) For a natural j, $\overline{\mathrm{cl}}^{(j)}(X)$ is defined, by induction, as $\overline{\mathrm{cl}}(\overline{\mathrm{cl}}^{(j-1)}(X))$ and $\overline{\mathrm{cl}}^{(0)}(X) = X$. Similarly for $\mathrm{cl}^{(j)}(X)$.

are proper closure operations, i. e. they are also idempotent („einstufigen Hüllen-operatoren" of J. SCHMIDT in [25]); unfortunately, in general, the knowledge of these operations does not contribute very much to the study of the GA-dependence structures. In fact,

$$\overline{\mathrm{Cl}}(X) = \bigcup_{n=1}^{\infty} \overline{\mathrm{cl}}^{(n)}(X) \quad \text{and} \quad \mathrm{Cl}(X) = \bigcup_{n=1}^{\infty} \mathrm{cl}^{(n)}(X).$$

Now, let us introduce the relation ε by

Definition 7. 5. Define the binary relation ε on $\mathfrak{J}$ (i. e. $\varepsilon \subseteqq \mathfrak{J} \times \mathfrak{J}$) by

$$(\varepsilon) \qquad [I_1, I_2] \in \varepsilon \leftrightarrow I_1 \subseteqq \mathrm{cl}\,(I_2) \wedge I_2 \subseteqq \mathrm{cl}\,(I_1).$$

The relation ε is obviously reflexive and symmetric. It induces a relation ε_C on $\mathfrak{C} \subseteqq \mathfrak{J}$ (i. e. if restricted to $\mathfrak{C} \times \mathfrak{C}$) which is, by (5) of Remark 7. 3, also transitive; let us call it the *δ-equivalence*.

We deduce immediately

Lemma 7. 6. a) $I_1^* \in \mathfrak{J}^* \wedge I_2^* \in \mathfrak{J}^* \rightarrow [I_1^*, I_2^*] \in \varepsilon.$

b) $C_1^* \in \mathfrak{C}^* \wedge C_2^* \in \mathfrak{C}^* \rightarrow [C_1^*, C_2^*] \in \varepsilon_C.$

Lemma 7. 7. $I \in \mathfrak{J} \wedge C \in \mathfrak{C} \wedge I \subseteqq \mathrm{cl}\,(C) \wedge I' \subset I \rightarrow C \nsubseteq \mathrm{cl}\,(I').$

PROOF. For, $C \subseteqq \mathrm{cl}(I')$ together with our assumptions imply, by (5)' of Remark 7. 3,

$$I \subseteqq \mathrm{cl}(C) \subseteqq \mathrm{cl}(I'),$$

a contradiction of $I \in \mathfrak{J}$.

Especially, making use of Lemma 7. 6, we have

Lemma 7. 8. a) $I \in \mathfrak{J} \wedge C \in \mathfrak{C} \wedge [I, C] \in \varepsilon \wedge I' \subset I \rightarrow [I', C] \notin \varepsilon.$

b) $I^* \in \mathfrak{J}^* \wedge C^* \in \mathfrak{C}^* \wedge I^{*\prime} \subset I^* \rightarrow C^* \nsubseteq \mathrm{cl}\,(I^{*\prime}).$

Further, we shall need also

Lemma 7. 9. $I_1 \in \mathfrak{J} \wedge I_2 \in \mathfrak{J} \wedge I_2 \subseteqq \mathrm{cl}\,(I_1) \rightarrow$

$\rightarrow \exists I_0 (I_0 \subseteqq I_1 \wedge I_2 \cup I_0 \in \mathfrak{J} \wedge [I_2 \cup I_0, I_1] \in \varepsilon).$

PROOF. Consider the family $\mathfrak{X}$ of all X such that

$$I_2 \subseteqq X \subseteqq I_2 \cup I_1 \wedge X \in \mathfrak{J}.$$

First, $\mathfrak{X}$ is non-empty (for $I_2 \in \mathfrak{X}$) and the union of a chain

$$X_1 \subseteqq X_2 \subseteqq \ldots \subseteqq X_\alpha \subseteqq \ldots, \alpha < \nu,$$

of elements of $\mathfrak{X}$ belongs, by Lemma 6. 2', to $\mathfrak{X}$, as well. This implies the existence of a maximal element X_0 of $\mathfrak{X}$:

$$X_0 = I_2 \cup I_0 \quad \text{with} \quad I_0 \subseteqq I_1.$$

Hence, for each element $x \in I_1$, we have either $x \in I_0$ or $X_0 \cup (x) \notin \mathfrak{J}$, i. e. $[x, X_0] \in \delta$. The assertion $[I_2 \cup I_0, I_1] \in \varepsilon$ follows.

The following three lemmas deal with cardinalities of maximal δ-independent and δ-canonic systems.

Lemma 7. 10. $I \in \mathfrak{J} \wedge C \in \mathfrak{C} \cap \mathscr{F} \wedge [I, C] \in \varepsilon \rightarrow I \in \mathscr{F}$.

Proof. For every $c \in C$, there is a finite subset $I_c \subseteq I$ such that $[c, I_c] \in \delta$. Thus, the union $\bigcup_{c \in C} I_c$ is a finite subset of I and clearly

$$[\bigcup_{c \in C} I_C, C] \in \varepsilon.$$

Hence, from Lemma 7. 8 a) we deduce $\bigcup_{c \in C} I_c = I$, i. e. I is really finite.

Lemma 7. 11. $I \in \mathfrak{J} \wedge C \in \mathfrak{C} \wedge C \notin \mathscr{F} \wedge [I, C] \in \varepsilon \rightarrow \operatorname{card}(I) \leqq \operatorname{card}(C)$.

Proof. Following the line of the previous proof (of Lemma 7. 10) we get again that

$$I = \bigcup_{c \in C} I_C \quad \text{with} \quad I_C \in \mathscr{F}.$$

Thus, because of card $(C) \geqq \aleph_0$,

$$\operatorname{card}(I) \leqq \aleph_0 \operatorname{card}(C) \leqq \operatorname{card}(C),$$

as required.

Lemma 7. 12. $I \in \mathfrak{J} \wedge C \in \mathfrak{C} \wedge [I, C] \in \varepsilon \rightarrow \operatorname{card}(I) \leqq \operatorname{card}(C)$.

Proof. By Lemma 7. 11, the conclusion holds if $C \notin \mathscr{F}$. Suppose therefore $C \in \mathscr{F}$; then, in view of Lemma 7. 10, also $I \in \mathscr{F}$. Thus, let

$$I = I_0 \cup I' \quad \text{and} \quad C = I_0 \cup C' \quad \text{with} \quad I' \cap C' = \emptyset$$

and

$$\operatorname{card}(I)' = k, \quad \operatorname{card}(C') = l.$$

We are going to prove the statement of our lemma (i. e. $k \leqq l$) indirectly. Let, on the contrary, be $k > l$. Let us suppose that, for a natural number n, $0 \leqq n < k$, there exists $I^{(n)}$ such that

$$I^{(n)} \subseteq I' \cup C' \wedge \operatorname{card}(I^{(n)} \cap I') = k - n \wedge \operatorname{card}(I^{(n)} \cap C') \geqq n \wedge I_0 \cup I^{(n)} \in \mathfrak{J}$$

and, moreover,

$$[I_0 \cup I^{(n)}, C] \in \varepsilon.$$

Thus,

$$\operatorname{card}(I^{(n)}) \geqq k - n + n = k > l = \operatorname{card}(C').$$

Now, take an element $x_{n+1} \in I^{(n)} \cap I'$. By Lemma 7. 7, there is an element $y_{n+1} \in C'$ such that

$$[y_{n+1}, (I_0 \cup I^{(n)}) \setminus (x_{n+1})] \notin \delta.$$

Applying Lemma 7. 9 to C and $(I_0 \cup I^{(n)}) \setminus (x_{n+1})$, we deduce that there is a non-empty subset $I'^{(n+1)} \subseteq C'$ such that $I_0 \cup I^{(n+1)}$, where $I^{(n+1)} = (I^{(n)} \setminus (x_{n+1})) \cup I'^{(n+1)}$, satisfies

$$I_0 \cup I^{(n+1)} \in \mathfrak{J} \quad \text{and} \quad [I_0 \cup I^{(n+1)}, C] \in \varepsilon.$$

Furthermore,

$$I^{(n+1)} \subseteq I' \cup C' \wedge \operatorname{card}(I^{(n+1)} \cap I') = k-(n+1) \wedge \operatorname{card}(I^{(n+1)} \cap C') \geqq n+1,$$

i. e.

$$\operatorname{card}(I^{(n+1)}) > \operatorname{card}(C').$$

Clearly, for $n=0$, there exists $I^{(0)}=I'$. Hence, by induction,

$$\operatorname{card}(I^{(m)} \cap C') \geqq k > l = \operatorname{card}(C')$$

for $m(\leqq k)$ large enough, — a contradiction.

Thus, in fact, there is $k \leqq l$, i. e. $\operatorname{card}(I) \leqq \operatorname{card}(C)$.

Now, we are ready to formulate

Theorem 7. 13. $I^* \in \mathfrak{J}^* \wedge C_1^* \in \mathfrak{C}^* \wedge C_2^* \in \mathfrak{C} \to \operatorname{card}(I^*) \leqq \operatorname{card}(C_1^*) = \operatorname{card}(C_2^*)$.

PROOF. The inequality $\operatorname{card}(I^*) \leqq \operatorname{card}(C_1^*)$ follows immediately from Lemma 7. 12, taking into account Lemma 7. 6.

The validity of this inequality in both directions in the case of C_1^* and C_2^* yields then the required equality $\operatorname{card}(C_1^*) = \operatorname{card}(C_2^*)$.

Definition 7. 14. Let us define the sets $\bar{\mathfrak{R}}_{\bar{S}}$ and $\mathfrak{R}_{\bar{S}}$ of cardinal numbers $\mathfrak{r}$ by

$$(\bar{\mathfrak{R}}) \qquad \mathfrak{r} \in \bar{\mathfrak{R}}_{\bar{S}} \leftrightarrow \exists \bar{I}^* (\bar{I}^* \in \bar{\mathfrak{J}}^* \wedge \operatorname{card}(\bar{I}^*) = \mathfrak{r})$$

and

$$(\mathfrak{R}) \qquad \mathfrak{r} \in \mathfrak{R}_{\bar{S}} \leftrightarrow \exists I^* (I^* \in \mathfrak{J}^* \wedge \operatorname{card}(I^*) = \mathfrak{r}).$$

Then, Theorem 7. 13 can be completed and read as follows:

Theorem 7. 15. a) $\exists \mathfrak{r}^* \{\mathfrak{r}^* \in \mathfrak{R}_{\bar{S}} \wedge \forall \mathfrak{r} (\mathfrak{r} \in \mathfrak{R}_{\bar{S}} \to \mathfrak{r} \leqq \mathfrak{r}^*)\}$;

b) $\forall C^* (C^* \in \mathfrak{C}^* \to \operatorname{card}(C^*) = \mathfrak{r}^*)$;

c) $\exists \bar{\mathfrak{r}}^* \{\bar{\mathfrak{r}}^* \in \bar{\mathfrak{R}}_{\bar{S}} \wedge \forall \mathfrak{r} (\mathfrak{r} \in \bar{\mathfrak{R}}_{\bar{S}} \to \mathfrak{r} \leqq \bar{\mathfrak{r}}^*)\}$;

d) $\forall \bar{I}^* (\bar{I}^* \in \bar{\mathfrak{J}}^* \wedge \bar{I}^* \cap S \subseteq S^C \to \operatorname{card}(\bar{I}^*) = \bar{\mathfrak{r}}^*)$;

e) $\bar{\mathfrak{r}}^* = \mathfrak{r}^* + \operatorname{card}(S^N)$.

Definition 7. 16. The cardinal numbers $\mathfrak{r}^*$ and $\bar{\mathfrak{r}}^*$ of Theorem 7. 15 will be called the *δ-rank* and the *complete δ-rank* of the set $\bar{S}$ and denoted by

$$\mathfrak{r}^* = r_\delta(\bar{S}) \quad \text{and} \quad \bar{\mathfrak{r}}^* = \bar{r}_\delta(\bar{S}),$$

respectively.

Let us notice that as far as $(\bar{S}, \delta)$ is a non-trivial GA-dependence structure, both $r_\delta(\bar{S})$ and $\bar{r}_\delta(\bar{S})$ are different from zero. In fact, the equality $r_\delta(\bar{S})=0$ is a necessary and sufficient condition for $(\bar{S}, \delta)$ to be trivial.

In some particular cases, the GA-dependence relation δ satisfies some additional conditions making possible to strenghten Theorem 7. 15. Let us introduce the following one applicable e. g. in the case of the dependence relation in abelian groups:

$$\text{(vii)} \qquad \forall I\{I \in \mathscr{F} \cap \mathfrak{J} \to \forall C(C \in \mathfrak{C} \wedge C \subseteq \operatorname{cl}(I) \to C \in \mathscr{F})\} \wedge$$
$$\wedge \forall I\{I \in \mathfrak{J} \wedge I \notin \mathscr{F} \to \forall C(C \in \mathfrak{C} \wedge C \subseteq \operatorname{cl}(I) \to \operatorname{card}(C) \leqq \operatorname{card}(I)\}.$$

Assuming (vii) we can prove

Lemma 7. 17. $I \in \mathfrak{I} \wedge C \in \mathcal{C} \wedge C \notin \mathcal{F} \wedge [I, C] \in \varepsilon \rightarrow \mathrm{card}(I) = \mathrm{card}(C)$.

Proof. According to the first part of (vii), $I \notin \mathcal{F}$. But then, making use of the second part, we get $\mathrm{card}(C) \leqq \mathrm{card}(I)$. Since, in view of Lemma 7. 11, $\mathrm{card}(C) \geqq \mathrm{card}(I)$, the required equality follows.

Thus, especially,

$$I^* \in \mathfrak{I}^* \wedge C^* \in \mathcal{C}^* \wedge \mathrm{card}(C^*) \geqq \aleph_0 \rightarrow \mathrm{card}(I^*) = \mathrm{card}(C^*).$$

or, expressing this fact as a supplement of Theorem 7. 15, we have

Theorem 7. 15. f) $\mathfrak{r}^* \geqq \aleph_0 \rightarrow (\mathfrak{R}_{\bar{S}} = (\mathfrak{r}^*) \wedge \bar{\mathfrak{R}}_{\bar{S}} = (\bar{\mathfrak{r}}^*))$.

Let us conclude this section by several remarks.

Remark 7. 18. In order to give an example showing that (vii) does not depend on (i)—(vi), consider the set

$$\bar{S}_{\mathfrak{a}} = (a_0) \cup (a_\gamma)_{\gamma \in \Gamma}, \quad \mathfrak{a} = \mathrm{card}(\Gamma) \geqq \aleph_0,$$

with the following relation δ on it ($X \subseteq \bar{S}_{\mathfrak{a}}$):

$$[a_0, X] \in \delta \leftrightarrow X \neq \emptyset,$$

$$[a_\gamma, X] \in \delta \leftrightarrow (a_0, a_\gamma) \cap X \neq \emptyset \quad \text{for every} \quad \gamma \in \Gamma.$$

Taking $S_{\mathfrak{a}}^c = (a_\gamma)_{\gamma \in \Gamma}$, we can easily verify the validity (i)—(vi) (obviously, e. g. $\mathfrak{I} = \{(a_0)\} \cup \mathfrak{P}(a_\gamma)_{\gamma \in \Gamma}$). Now, we have $(a_0) \in I^*$ and $S_{\mathfrak{a}}^c \in \mathcal{C}^*$; thus, $S_{\mathfrak{a}}^c \subseteq \mathrm{cl}(a)$, but $S_{\mathfrak{a}}^c$ is infinite.

This example shows, at the same time, that we cannot expect that Theorem 7. 15 could be improved, in general (in the corresponding set $\mathfrak{R}_{\bar{S}}$ we have 1 and simultaneously an (arbitrary) cardinal number $\mathfrak{a}$).

In the next section (Remark 8. 12) a more general example of this type will be introduced.

Remark 7. 19. All through this section, we have restricted our study of relations among cardinalities of subsets of $\bar{S}$ on the independent sets only. Let us point out that the independence of the subsets was an essential condition. Consider e. g. the GA-dependence structure $(\bar{S}, \delta)$ given by the following table

	$\emptyset$	(a)	(b)	(c)	(d)	(a, b)	(a, c)	(a, d)	(b, c)	(b, d)	(c, d)	(a, b, c)	(a, b, d)	(a, c, d)	(b, c, d)	(a, b, c, d)
a	0	1	0	0	0	1	1	1	0	0	1	1	1	1	1	1
b	0	0	1	0	0	1	0	0	1	1	1	1	1	1	1	1
c	0	0	0	1	1	0	1	1	1	1	1	1	1	1	1	1
d	0	0	0	1	1	0	1	1	1	1	1	1	1	1	1	1

Clearly, $\bar{\mathfrak{J}} = \mathfrak{J} = \{\emptyset, (a), (b), (c), (d), (a, b), (a, c), (a, d), (b, c), (b, d), (a, b, c), (a, b, d)\}$. Taking $S^c = \bar{S}$, we can easily see that δ is a GA-dependence relation. Then, $\mathfrak{C}^* = \{(a, b, c), (a, b, d)\}$; thus,

$$r_\delta(\bar{S}) = \bar{r}_\delta(\bar{S}) = 3.$$

But, there is a subset of S^c with two elements only, viz. $(c, d) \subseteqq S^c$, which is maximal, i. e. $\mathrm{cl}(c, d) = \bar{S}$. The assumption of Theorem 7. 15 on the set C^* to be independent is therefore necessary.

It is possible, however, to reduce the subset S^c so that this assumption and the similar ones (e. g. already in the axiom (v)) could be omitted even in the general case. For that reason to take $S^c = C^*$, where C^* is an element of $\mathfrak{C}^*$, is sufficient (in our particular case e. g. $S^c = (a, b, c)$ or $S^c = (a, b, d)$). For, then every subset of S^c is automatically independent.

The fact that we have not introduced the postulate (v) in this form (i. e. without the condition $C \in \mathfrak{J}$) gives us the possibility of a wider choice of canonic subsets and thus enable us to describe canonic subsets for the whole classes of GA-dependence structures simultaneously.

Remark 7. 20. Finally, let us remark that the cardinal numbers $r_\delta(\bar{S})$ and $\bar{r}_\delta(\bar{S})$ are really invariants of the GA-dependence structure $(\bar{S}, \delta)$ (in spite of the fact that we have introduced them by means of a fixed (chosen) canonic subset $S^c \in \mathfrak{S}$). This follows immediately from Theorem 7. 13 (or Theorem 7. 15).

As to the properties of the family $\mathfrak{S}$, Theorem 6. 7 f) can be interpreted as the proof of the existence of minimal elements in $\mathfrak{S}$. It is easy to prove that there are also maximal elements in $\mathfrak{S}$. Nevertheless, in general, there is neither the least nor the greatest element in $\mathfrak{S}$ (see, e. g. the example of Remark 7. 19 and the second example of Remark 4. 9). The subfamily of all the minimal elements of $\mathfrak{S}$ coincides obviously with the subfamily of all the δ-independent canonic subsets. On the whole, we have proved that

$$I^* \in \mathfrak{J}^* \to \mathrm{card}\,(I^*) \leqq r_\delta(\bar{S}),$$

$$I^* \in \mathfrak{J}^* \wedge \exists S^C(S^C \in \mathfrak{S} \wedge I^* \subseteqq S^C) \to \mathrm{card}\,(I^*) = r_\delta(\bar{S}),$$

and, in particular for minimal elements of $\mathfrak{S}$,

$$I^* \in \mathfrak{J}^* \cap \mathfrak{S} \to \mathrm{card}\,(I^*) = r_\delta(\bar{S}). \tag{7. 1}$$

Thus, the δ-rank is the least cardinality of elements of $\mathfrak{S}$ and in the case that $r_\delta(\bar{S})$ is finite, an element of $\mathfrak{S}$ is minimal if, and only if, its cardinality equals $r_\delta(\bar{S})$. However, the converse of (7. 1), i. e.

$$I^* \in \mathfrak{J}^* \wedge \mathrm{card}(I^*) = r_\delta(\bar{S}) \to I^* \in \mathfrak{S}$$

is not, generally, true even in the finite case, as illustrated by the, already mentioned, example 2) of Remark 4. 9.

§ 8. Direct decompositions

Let $(\bar{S}, \delta)$ be a fixed given GA-dependence structure. If $\bar{T}$ is a subset of $\bar{S}$, then δ induces a binary relation δ_T between T and $\mathfrak{P}T$ (there will be no confusion if we denote it again simply by δ). Then, applying a similar notation to that in § 2, we see immediately that

$$(8.1) \qquad T^N = \bar{T} \cap S^N \quad \text{and} \quad T^S = \bar{T} \cap S^S.$$

Thus, $T = \bar{T} \setminus (T^N \cup T^S)$ is a subset of S. Moreover,

$$\bar{\mathfrak{J}}_{\bar{T}} = \bar{\mathfrak{J}}_{\bar{T}, \delta} = \bar{\mathfrak{J}}_{\bar{S}, \delta} \cap \mathfrak{P}\bar{T}$$

and

$$\mathfrak{J}_T = \mathfrak{J}_{T, \delta} = \mathfrak{J}_{S, \delta} \cap \mathfrak{P}T.$$

The axioms (i)—(iv) and (vi) are also satisfied by δ on $\bar{T}$.

If there is a subset $T^c \subseteq T$ satisfying the properties of (v) (i. e. a canonic subset), then $(\bar{T}, \delta)$ is also a GA-dependence structure — a *substructure* of $(\bar{S}, \delta)$: $(\bar{T}, \delta) \subseteq \subseteq (\bar{S}, \delta)$. Then, in view of Theorem 6. 7 and Theorem 7. 15, we have readily

Theorem 8. 1. $(\bar{T}, \delta) \subseteq (\bar{S}, \delta) \rightarrow r_\delta(\bar{T}) \leqq r_\delta(\bar{S}) \wedge \bar{r}_\delta(\bar{T}) \leqq \bar{r}_\delta(\bar{S})$.

REMARK 8. 2. Let us give an example that, in general, a subset $\bar{T}$ of $\bar{S}$ with the induced relation δ is not a GA-dependence structure of $(\bar{S}, \delta)$:

The GA-dependence structure $(\bar{S}, \delta)$ be given by the following table

	$\emptyset$	(a)	(b)	(c)	(d)	(e)	(f)	(a,b)	(a,c)	(a,d)	(a,e)	(a,f)	(b,c)	(b,d)	(b,e)	(b,f)	(c,d)	(c,e)	(c,f)	(d,e)	(d,f)	(e,f)	(a,b,c,d,e,f)
a	0	1	1	1	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1...1
b	0	1	1	0	1	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1...1
c	0	1	0	1	1	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1...1
d	0	0	1	1	1	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1...1
e	0	0	0	0	0	1	0	0	0	1	1	1	1	0	1	1	0	1	1	1	1	1	1...1
f	0	0	0	0	0	0	1	0	0	1	1	1	1	0	1	1	0	1	1	1	1	1	1...1

Then, $\bar{T} = (a, b, c, d)$, together with the induced relation, is not a GA-dependence structure (see § 5, section 5).

On the other hand, it may well happen that there exist canonic subsets T^c and S^c of a substructure $(\bar{T}, \delta) \subseteq (\bar{S}, \delta)$ and of $(\bar{S}, \delta)$ such that $T^c \subseteq S^c$. Let us call the substructure $(\bar{T}, \delta)$ in this case *normal* (in $(\bar{S}, \delta)$). Thus, in the class $\mathfrak{P}\bar{S}$ of all the subsets of $\bar{S}$ there are subclasses $\mathfrak{S}$ and $\mathfrak{N}$ corresponding to all the substructures and normal substructures of $(\bar{S}, \delta)$, respectively: $\mathfrak{N} \subseteq \mathfrak{S} \subseteq \mathfrak{P}\bar{S}$. Moreover, obviously $\bar{\mathfrak{J}} \subseteq \mathfrak{S}$.

Further, a GA-dependence substructure $(\bar{T}, \delta)$ of $(\bar{S}, \delta)$ is said to be a *proper kernel substructure* (of $(\bar{S}, \delta)$) if

$$\hat{\mathfrak{S}}_{\bar{T}, \delta} \cap \hat{\mathfrak{S}}_{\bar{S}, \delta} \neq \emptyset,$$

i. e. if there is a canonic subset S^c of $\bar{S}$ such that $S^c \subseteqq T$. Thus, if $(\bar{T}, \delta)$ is a proper kernel substructure of $(\bar{S}, \delta)$, then

$$\mathrm{cl}(T) \supseteqq S$$

and, especially,

$$r_\delta(\bar{T}) = r_\delta(\bar{S}).$$

Denoting the family of all the proper kernel substructures of $(\bar{S}, \delta)$ by $\mathfrak{K}$, we have

$$\mathfrak{C}^* \subseteqq \mathfrak{K} \subseteqq \mathfrak{N} \subseteqq \mathfrak{S} \subseteqq \mathfrak{P}\bar{S}.$$

A generalization of the latter concept is that of the *kernel substructure:* $(\bar{T}, \delta)$ is said to be a kernel substructure of $(\bar{S}, \delta)$ if there is a finite chain

$$(\bar{T}, \delta) = (\bar{T}_0, \delta) \subseteqq (\bar{T}_1, \delta) \subseteqq \ldots \subseteqq (\bar{T}_k, \delta) = (\bar{S}, \delta)$$

such that $(\bar{T}_i, \delta)$ is a proper kernel substructure of $(\bar{T}_{i+1}, \delta)$ for $0 \leqq i \leqq k-1$. Then, again

$$r_\delta(\bar{T}) = r_\delta(\bar{S}). \tag{8.2}$$

Let us notice that a kernel substructure of a GA-dependence structure need not be a normal substructure of it:

Let $(\bar{S}, \delta)$ be the GA-dependence structure of Remark 8. 2 and

$$\bar{T} = (c, d, e), \quad \bar{T}_1 = (b, c, d, e, f).$$

Then $(\bar{T}, \delta)$ is a kernel substructure of $(\bar{S}, \delta)$, for

$$\hat{\mathfrak{S}}_{\bar{T}, \delta} = \{(c, e), (d, e)\},$$

$$\hat{\mathfrak{S}}_{\bar{T}_1, \delta} = \{(b, e), (b, f), (c, e), (c, f), (e, f), (b, e, f), (c, e, f)\} \text{ and}$$

$$\hat{\mathfrak{S}}_{\bar{S}, \delta} = \{(e, f)\}.$$

Evidently, $(\bar{T}, \delta)$ is not a normal substructure of $(\bar{S}, \delta)$.

Definition 8. 3. A GA-dependence structure $(\bar{S}, \delta)$ is said to be a *pseudo-direct union* of its substructures $(\bar{S}_\gamma, \delta)$, $\gamma \in \Gamma$, if

(1) $\bigcup_{\gamma \in \Gamma} \bar{S}_\gamma = \bar{S}$ and

(2) $\gamma_0 \in \Gamma \wedge x \in \bar{S}_{\gamma_0} \wedge X \in \bar{\mathfrak{F}}_{\bar{S}} \cap \bar{\mathscr{F}}_{\bar{S}} \wedge [x, X] \in \delta \rightarrow [x, X \setminus \bigcup_{\gamma \in \Gamma \setminus (\gamma_0)} \bar{S}_\gamma] \in \delta.$ [22])

[22]) The condition $X \in \bar{\mathscr{F}}_{\bar{S}}$ is, of course, merely formal.

This fact is denoted by

$$(\bar{S}, \delta) = \sum_{\gamma \in \Gamma}^{P} (\bar{S}_\gamma, \delta).$$

If, moreover, all $\bar{S}\gamma$, $\gamma \in \Gamma$, are δ-closed in $\bar{S}$, then $(\bar{S}, \delta)$ is said to be a *direct union* of $(\bar{S}_\gamma, \delta)$:

$$(\bar{S}, \delta) = \sum_{\gamma \in \Gamma}^{D} (\bar{S}_\gamma, \delta).$$

If there is a kernel GA-dependence substructure $(\bar{T}, \delta)$ of $(\bar{S}, \delta)$ which is a pseudodirect union of $(\bar{S}_\gamma, \delta)$, $\gamma \in \Gamma$, then we shall call $(\bar{S}, \delta)$ a *pseudodirect product* of $(\bar{S}_\gamma, \delta)$:

$$(\bar{S}, \delta) = \prod_{\gamma \in \Gamma}^{P} (\bar{S}_\gamma, \delta),$$

and finally, if $\bar{S}_\gamma$ is, moreover, δ-closed in $\bar{S}$ for every $\gamma \in \Gamma$, then $(\bar{S}, \delta)$ is called a *direct product* of $(\bar{S}_\gamma, \delta)$:

$$(\bar{S}, \delta) = \prod_{\gamma \in \Gamma}^{D} (\bar{S}_\gamma, \delta).$$

The following properties of a pseudodirect union $(\bar{S}, \delta) = \sum_{\gamma \in \Gamma}^{P} (\bar{S}_\gamma, \delta)$ follow readily from Definition 8. 3.

Lemma 8. 4. $\gamma_1 \neq \gamma_2 \rightarrow \bar{S}_{\gamma_1} \cap \bar{S}_{\gamma_2} \subseteq S^N \cup S^S$.

Lemma 8. 5. a) $\gamma_1 \neq \gamma_2 \rightarrow S_{\gamma_1} \cap S_{\gamma_2} = \emptyset$.

b) $S = \bigcup_{\gamma \in \Gamma} S_\gamma$.

Lemma 8. 6. a) $I \in \mathfrak{J}_S \rightarrow I \cap S_\gamma \in \mathfrak{J}_{S_\gamma}$.

b) $I_\gamma \in \mathfrak{J}_{S_\gamma} \rightarrow \bigcup_{\gamma \in \Gamma} I_\gamma \in \mathfrak{J}_S$.

Lemma 8. 7. a) $S^C \in \hat{\mathfrak{S}}_{\bar{S}} \rightarrow S^C \cap S_\gamma \in \hat{\mathfrak{S}}_{\bar{S}_\gamma}$.

b) $S_\gamma^C \in \hat{\mathfrak{S}}_{\bar{S}_\gamma} \rightarrow \bigcup_{\gamma \in \Gamma} S_\gamma^C \in \hat{\mathfrak{S}}_{\bar{S}}$.

Thus, making use of Lemma 8. 7 and Lemma 8. 6 together with Theorem 7. 15 and (8. 2), we get

Theorem 8. 8. $(\bar{S}, \delta) = \sum_{\gamma \in \Gamma}^{P} (\bar{S}_\gamma, \delta) \vee (\bar{S}, \delta) = \sum_{\gamma \in \Gamma}^{D} (\bar{S}_\gamma, \delta) \vee (\bar{S}, \delta) = \prod_{\gamma \in \Gamma}^{P} (\bar{S}_\gamma, \delta) \vee$ $\vee (\bar{S}, \delta) = \prod_{\gamma \in \Gamma}^{D} (\bar{S}_\gamma, \delta) \rightarrow r_\delta(\bar{S}) = \sum_{\gamma \in \Gamma} r_\delta(\bar{S}_\gamma)$.

Remark 8. 9. Let us mention that the definition of a pseudodirect product includes also trivial decompositions of the following type:

Let $C^* \in \mathfrak{C}_{S^c}^*$ and $\{\dot{C}_\gamma^*\}_{\gamma \in \Gamma}$ be an arbitrary partition of $S^N \cup C^*$; then

$$(\bar{S}, \delta) = \prod_{\gamma \in \Gamma}^{P} (C_\gamma^*, \delta).$$

As a matter of fact, if $(\bar{S}, \delta)=\prod_{\gamma\in\Gamma}^{P}(\bar{S}_\gamma, \delta)$, then, in view of Lemmas 8. 7 and 8. 6, there are substructures $(\bar{T}_\gamma, \delta)\subseteqq(\bar{S}_\gamma, \delta)$ such that

$$(\bar{S}, \delta)=\prod_{\gamma\in\Gamma}^{P}(\bar{T}_\gamma, \delta) \quad\text{and}\quad \bigcup_{\gamma\in\Gamma}\bar{T}_\gamma\in\mathfrak{J}_S^*.$$

REMARK 8. 10. For the purpose of this remark only, the symbol $\otimes$ replaces a (fixed) symbol out of the following four ones: $\sum^P$, $\sum^D$, $\prod^P$, $\prod^D$. We can easily see that if $(\bar{S}, \delta)=\bigotimes_{\gamma\in\Gamma}(\bar{S}_\gamma, \delta)$ and, for every $\gamma\in\Gamma$, $(\bar{S}_\gamma, \delta)=\bigotimes_{\eta\in H}(\bar{S}_{\gamma\eta}, \delta)$, then

$$(\bar{S}, \delta)=\bigotimes_{\substack{\gamma\in\Gamma\\ \eta\in H}}(\bar{S}_{\gamma\eta}, \delta).^{23)}$$

In the case of a (pseudo) direct product, this is due to a rather general definition of a kernel substructure. Also, splitting the class of „factors" $(\bar{S}_\gamma, \delta)$ in an arbitrary way into disjoint subclasses and replacing the factors $(\bar{S}_\gamma, \delta)$ that enter into each of these subclasses by their union we obtain a new decomposition of $(\bar{S}, \delta)$ (of the same kind as the original one).

So far we have dealt with decompositions of a given GA-dependence structure. In the sequel, we shall form a new GA-dependence structure from given structures by construction of their direct union.

Theorem 8. 11. *Let $(\bar{S}_\gamma, \delta_\gamma)$, $\gamma\in\Gamma$, be a family of GA-dependence structures. Let $\bar{S}$ be a disjoint union of sets $\bar{T}_\gamma$, $\gamma\in\Gamma$, such that there is a one-to-one mapping φ_γ of $\bar{T}_\gamma$ onto $\bar{S}_\gamma$ for every $\gamma\in\Gamma$. Define a binary relation δ on $\bar{S}$ in the following way*

$$\gamma\in\Gamma\wedge x\in\bar{T}_\gamma\wedge X\in\mathfrak{P}\bar{S}\rightarrow([x, X]\in\delta\leftrightarrow[\varphi_\gamma(x), \varphi_\gamma(X\cap\bar{T}_\gamma)]\in\delta_\gamma).$$

Then, $(\bar{S}, \delta)$ is a GA-*dependence structure — the direct union of $(\bar{S}_\gamma, \delta_\gamma)$*:

$$(\bar{S}, \delta)=\sum_{\gamma\in\Gamma}^{D}(\bar{S}_\gamma, \delta_\gamma). ^{24)}$$

PROOF. Clearly, for every $\gamma\in\Gamma$, the subset $\bar{T}_\gamma$ of $\bar{S}$ with the (restricted) relation δ on it is a GA-dependence structure $(\bar{T}_\gamma, \delta)$ („isomorphic" to $(\bar{S}_\gamma, \delta_\gamma)$). Furthermore,

$$S^N=\bigcup_{\gamma\in\Gamma}T_\gamma^N, \quad S^S=\bigcup_{\gamma\in\Gamma}T_\gamma^S,$$

and taking $S^c=\bigcup_{\gamma\in\Gamma}T_\gamma^c$, it is a routine to check the requisite postulates and we conclude that $(\bar{S}, \delta)$ is a GA-dependence structure.

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[23]) There is possible to consider also „refinements" which need not be of the same type as the original decomposition.

[24]) In fact, $(\bar{S}, \delta)$ is, in accordance with Definition 8. 3, a direct union of the GA-dependence substructures $(\bar{T}_\gamma, \delta)$ — which are „isomorphic" to $(\bar{S}_\gamma, \delta_\gamma)$ in the following sense:

$$x\in\bar{T}_\gamma\wedge X\subseteq\bar{T}_\gamma\rightarrow([x, X]\in\delta\leftrightarrow[\varphi_\gamma(x), \varphi_\gamma(X)]\in\delta_\gamma).$$

Finally, for an element $x \in \bar{T}_{\gamma_0}$ and an arbitrary $X \subseteqq \bar{S}$ (even not necessarily independent) such that $[x, X] \in \delta$, we have evidently

$$[x, X \setminus \bigcup_{\gamma \in \Gamma \setminus (\gamma_0)} \bar{T}_\gamma] = [x, X \cap \bar{T}_{\gamma_0}] \in \delta,$$

and $\bar{T}_\gamma$ is a δ-closed subset of $(\bar{S}, \delta)$ for every $\gamma \in \Gamma$. In view of Definition 8. 3, the theorem follows.

REMARK 8. 2. Now, we are in possession to construct the examples of GA-dependence structures mentioned in the previous § 7, Remark 7. 2 and Remark 7. 18.

In Remark 7. 2, we introduced, for any natural $n \geqq 3$, a GA-dependence structure $(\bar{S}_n, \delta_n)$. If we put

$$(\bar{S}_\omega, \delta) = \sum_{n \geqq 3}^{D} (\bar{S}_n, \delta_n),$$

then, on the basis of Remark 7. 2, it is easy te find, for any natural number n_0, a subset $A \subseteqq \bar{S}_\omega$ such that

$$\mathrm{cl}^{(n_0-1)}(A) \neq \mathrm{cl}^{(n_0)}(A) = \mathrm{cl}^{(n_0+1)}(A).$$

The following example illustrates that both parts of the condition (vii) do not depend on (i)—(vi) (in Remark 7. 18, it was shown that the first part does not depend). Consider, for a cardinal number $\mathfrak{a} > \aleph_0$, a countable family of GA-dependence structures $(\bar{S}_{\mathfrak{a},n}, \delta)$,

$$\bar{S}_{\mathfrak{a},n} = (a_0^n) \cup (a_\gamma^n)_{\gamma \in \Gamma}, \ \mathrm{card}\,(\Gamma) = \mathfrak{a},$$

defined in Remark 7. 18, and take the direct union

$$(\bar{S}, \delta) = \sum_{n \geqq 1}^{D} (\bar{S}_{\mathfrak{a},n}, \delta).$$

Then, besides the first part of (vii) (which is not satisfied in any „factor" $(\bar{S}_{\mathfrak{a},n}, \delta)$) also the second one fails to hold; for, $I = (a_0^n)_{n \geqq 1}$ is evidently a countable independent subset and $C = \bigcup_{n \geqq 1} (a_\gamma^n)_{\gamma \in \Gamma}$ is a canonic system such that $C \subseteqq \mathrm{cl}(I)$ and

$$\mathrm{card}\,(C) = \mathfrak{a} > \aleph_0 = \mathrm{card}\,(I).^{25)}$$

§ 9. Some applications

In this short final section, we shall deal with some illustrations of the introduced general concepts on concrete GA-dependence structures, mainly on abelian groups;[26]) for some particular results we shall refer to the author's paper [5].

[25]) The example can be slightly generalized so that, for arbitrary two cardinal numbers $\mathfrak{a} > \mathfrak{b}$, there are $I \in \mathfrak{J}$ and $C \in \mathfrak{C}$ such that $C \subseteq \mathrm{cl}(I)$ and card $(C) = \mathfrak{a} > \mathfrak{b} = \mathrm{card}\,(I)$.

[26]) Many results could be easily formulated for modules over principal ideal rings.

Let G be an abelian group and H a subgroup of G. Define the binary relation δ_H on G in the following way

$$(\delta_H) \qquad g \in G \wedge K \subseteq G \to ([g, K] \in \delta_H \leftrightarrow \langle g \rangle \cap \langle K \cup H \rangle \nsubseteq H).^{27})$$

Then, $G^N_{\delta H} = H$ and $G^S_{\delta H} = \emptyset$. Taking the set of all the elements g of G such that the corresponding cosets $g+H$ of G modulo H are of infinite or prime power order as a canonic subset $G^c_{\delta H}$,[28]) it is a routine to check that δ_H is a GA-dependence relation on G. Thus, for any group G and its subgroup H, we have a well-defined invariant — the δ_H-rank of G: $r_{\delta H}(G)$ (comp. [26], [4] and [6], where also some other invariants of G derived from it are considered). There is a very close relation between the GA-dependence structures (G, δ_H) and $(G/H, \delta_{(0)})$; in fact,

$$r_{\delta H}(G) = r_{\delta_{(0)}}(G/H).$$

Although this special case of a group dependence relation is a rather typical „representation" of our general concept of a GA-dependence structure, it has some additional particular properties. E. g., it is easy to see that the property described by (iv) is satisfied without the assumption $I \in \mathfrak{F}$, that — as to the first part of (v) — there is for every element g a single element in a canonic set such that g depends on it and that also (vii) is fulfilled. On the other hand, even in this particular case of the group dependence relation δ_H, the assumption $C \in \mathfrak{F}$ in the second part of (v) and, generally, the restriction to consider cardinalities of independent subsets only (as mentioned in Remark 7. 19) is essential.

Consider, for a moment, a fixed (group) GA-dependence structure (G, δ_H). If G' is a subgroup of G such that $H \subseteq G' \subseteq G$, then there corresponds a substructure (G', δ_H) of (G, δ_H) to G' (in the sense of § 8). There is, of course, no need for G' to be a subgroup of G in order to form (together with δ_H) a substructure of (G, δ_H), but every subset of G containing H does not form it. Thus, e. g. the subset $(6g_0, 10g_0, 21g_0, 35g_0)$ of the group GA-dependence structure $(G_0, \delta_{(0)})$, where $G_0 = \langle g_0 \rangle$ is a cyclic group of order 210, with the relation $\delta_{(0)}$ on it (being „isomorphic" to the example 5. in § 5) does not.

The properties of the closure operation $K \to \mathrm{cl}(K)$ were studied in detail in [5]; in particular, it was shown there that, in difference to the general case, $\mathrm{cl}^{(4)}(K)$ is δ_H-closed for any subset $K \subseteq G$.

Further, let G be a H-direct sum of its subgroups G_γ, $\gamma \in \Gamma$, i. e.

$$\langle G_\gamma \rangle_{\gamma \in \Gamma} = G \quad \text{and} \quad G_{\gamma_0} \cap \langle G_\gamma \rangle_{\gamma \in \Gamma \setminus (\gamma_0)} \subseteq H \quad \text{for every } \gamma_0.$$

Then,

$$(G, \delta_H) = \prod_{\gamma \in \Gamma}^{P} (G_\gamma, \delta_H)$$

and

$$r_{\delta_H}(G) = \sum_{\gamma \in \Gamma} r_{\delta_H}(G_\gamma).$$

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[27]) For anon-empty subset K of a group G, the symbol $\langle K \rangle$ denotes the subgroup of G generated by K.

[28]) The set of all $g \in G$ such that $g+H$ is of infinite or prime order in the quotient group G/H is another suitable choice of a canonic subset.

Also, denoting by T_H the subgroup of G of all the elements g such that a multiple ng belongs to H for a suitable natural n, we can easily see that (G, δ_H) is a pseudodirect union of (T_H, δ_H) and $(G \setminus T_H, \delta_H)$. As a matter of fact, this is the only non-trivial decomposition of this type of (G, δ_H) (see [5]).

If H is a neat subgroup of G[29]), then there is a maximal canonic system C^* of (G, δ_H) such that

$$g \in C^* \wedge ng \in H \rightarrow ng = 0.$$

Thus, $(G, \delta_{(0)})$ is a pseudodirect product of $(H, \delta_{(0)})$ and $(C^*, \delta_{(0)})$ and therefore

$$r_{\delta_{(0)}}(G) = r_{\delta_{(0)}}(H) + r_{\delta_H}(G) = r_{\delta_{(0)}}(H) + r_{\delta_{(0)}}(G/H).$$

In fact, if $r_{\delta_{(0)}}(G)$ is finite, then, on the contrary, the latter equality implies that H is neat in G (see [4]).

As an immediate consequence of these results we get the well-known equality

$$r_{\delta_{(0)}}(G) = r_{\delta_{(0)}}(G/T) + \sum_p r_{\delta_{(0)}}(T_p),$$

where T is the maximal torsion subgroup of G and T_p its primary components.

We have considered the dependence in abelian groups as a relation on the set of all their elements. Alternatively, it can be defined on the family $\bar{S}(G)$ of all the subgroups of G by

$$L \in \bar{S}(G) \wedge \mathfrak{L} \subseteq \bar{S}(G) \rightarrow ([L, \mathfrak{L}] \in \delta'_H \leftrightarrow L \cap \langle \bigcup_{K \in \mathfrak{L}} K \cup H \rangle \subsetneqq H),$$

thus expressing directly whether a group union of subgroups is their direct sum or not (comp. I. KAPLANSKY [10]).

The „linear" dependence in abelian groups, used e. g. in the KUROSH' monograph [13] (corresponding to our relation δ_T in G, where T is the maximal torsion subgroup of G) is, alike the dependence in vector spaces, the algebraic dependence, the inclusion etc., covered by the (generalized) VAN DER WAERDEN's axiomatic system, as mentioned in the introduction. Roughly speaking, these dependence relations are those GA-dependence relations which (in the notation of § 2) fulfil the property (v) without the restrictions $I \in \mathfrak{J}$ and $C \in \mathfrak{J} \cap \mathfrak{P}S^c$, i. e. for any finite $I \subseteq S$ and $C \subseteq S$.[30])

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