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The generalized Hyers-Ulam stability of a class of functional equations

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Abstract. In the paper we generalize some results concerning Hyers–Ulam stability of functional equations. Our intention is to include a possibly most general class of functional equations, whose proof of stability runs by classical Hyers' method. The idea of this generalization comes from G. L. Forti and Z. Kominek.

1. Introduction

Questions concerning the stability of functional equations seem to have originated with S. M. Ulam and D. H. Hyers in the 1940s (see [20]). One of the first assertions to be proved in this direction is the following result, essentially due to Hyers (see [12]), that answered a question of Ulam. We present its final version devoid of some needless assumptions.

Hyers' Theorem. *Let $(X,+)$ be a commutative semigroup and let E be a real Banach space. If $f : X \to E$ and there exists $\varepsilon \geq 0$ such that $\|f(x+y) - f(x) - f(y)\| \leq \varepsilon$ for all $x, y \in X$, then the limit $g(x) := \lim(\frac{1}{2})^n f(2^n x)$ exists for every $x \in X$ and $g : X \to E$ is the unique additive function satisfying the condition $\|f(x) - g(x)\| \leq \varepsilon$ for every $x \in X$.*

This assertion is usually summarized by saying that the Cauchy functional equation is stable (in the sense of Hyers–Ulam). In [17] Th. M. Rassias gave the following generalization of Hyers' Theorem (we omit here some regularity assumptions strengthening the proposition).

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Rassias' Theorem. *Let E_1, E_2 be Banach spaces and let $f : E_1 \to E_2$. Assume that there exists $\varepsilon \geq 0$ and $p \in [0,1)$ such that $\|f(x+y) - f(x) - f(y)\| \leq \varepsilon(\|x\|^p + \|y\|^p)$ for all $x, y \in E_1$. Then there exists a unique additive mapping $T : E_1 \to E_2$ such that $\|f(x) - T(x)\| \leq \frac{2\varepsilon}{2-2^p}\|x\|^p$ for $x \in E_1$.*

The proof presented in [17] also works for $p < 0$. In [9] Z. Gajda following a similar approach obtained an analoguos theorem for $p > 1$. For $p = 1$ the similar theorem does not hold (see [9],[18]). Some additional assumptions, which guarantee stability also in this case were proposed by R. Ger (see [10],[11]). Further generalizations in this direction initiated by Th. M. Rassias lead to considering the following inequality

$$\|f(x+y) - f(x) - f(y)\| \leq \varphi(x,y) \quad \text{for all } x, y \in X, \tag{1.1}$$

where $f : X \to Y$, $\varphi : X^2 \to [0, +\infty)$, X is a semigroup and Y is a normed space. The problem of stability in this case is reduced to proving the existence of an additive function $T : X \to Y$ such that $\|f(x) - T(x)\| \leq \lambda(x)$ for every $x \in X$, where $\lambda : X \to [0, +\infty)$ is independent of the function f. The stability in this case, by suitable assumptions on φ, X, Y, was consider among others by R. Ger (see [10],[11]), G. Isac and Th. M. Rassias (see [13],[14]), Th. M. Rassias and P. Šemrl (see [19]) also G. L. Forti (see [8]) and Z. Kominek. Ger's results were based on the method of invariant means. Proofs of stability theorems included in other papers ran by classical Hyers' method. The inspiration of this paper was the following result obtained by G. L. Forti (see [8]) and Z. Kominek, independently (Forti deals with the stability of a wider class of functional equations; however, since the assumptions of this theorem are quite complex we only present its version for the Cauchy equation).

Forti-Kominek's Theorem. *Assume $(X, +)$ is a commutative semigroup and E is a Banach space. If the function $f : X \to E$ and $\varphi : X^2 \to [0, +\infty)$ satisfy the inequality (1.1) and for all $x, y \in X$*

$$\sum_{n=0}^{\infty} \left(\tfrac{1}{2}\right)^n \varphi(2^n x, 2^n x) < \infty \quad \text{and} \quad \lim_{n\to\infty} \left(\tfrac{1}{2}\right)^n \varphi(2^n x, 2^n y) = 0,$$

then there exists a unique additive function $T : X \to E$ such that $\|f(x) - T(x)\| \leq \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n \varphi(2^n x, 2^n x)$ for every $x \in X$.

In [4] C. Borelli and G. L. Forti proved two general theorems generalizing many of the well-known results concerning the stability in the Hyers–Ulam sense.

2. The main result

We introduce the following notations. Denote by $\mathbb{N}_0$ the set of non-negative integers, by $\mathbb{R}_+$ the set of non-negative reals and by $\boldsymbol{K}$ the field of real or complex numbers. Let $\mathbb{N} := \mathbb{N}_0 \setminus \{0\}$. In the set $\mathbb{R}_+ \times \mathbb{R}_+$ we define an order "$\leq^*$" by: $(x_1, y_1) \leq^* (x_2, y_2)$ if and only if $x_1 \leq x_2$ and $y_1 \leq y_2$. Let further X be a nonvoid set and Y be a normed space. By the symbol $\mathcal{F}(X, Y)$ we denote the set of all functions defined on X and taking values in Y, in which the addition of functions and, in the case $\mathcal{F}(X, X)$, the composition of functions are defined in the usual way. By $\|f\|$ we understand the composition of f and $\|\cdot\|$. Moreover, for $f, g \in \mathcal{F}(X, \mathbb{R}_+)$ put $f \leq g$ if and only if $f(x) \leq g(x)$ for every $x \in X$. The mapping $\sigma : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is said to be subadditive if $\sigma(x + y) \leq \sigma(x) + \sigma(y)$ for all $x, y \in \mathbb{R}_+ \times \mathbb{R}_+$. Now let $\alpha, \beta : X \to X$ be fixed functions. For $H : Y^2 \to Y$ we define a sequence of operators $\mathcal{G}_H^n : \mathcal{F}(X, Y) \to \mathcal{F}(X, Y)$ (and similarly $\mathcal{G}_H^n : \mathcal{F}(X^2, Y) \to \mathcal{F}(X^2, Y)$), $n \in \mathbb{N}_0$, as follows

$$\mathcal{G}_H^0 h(x) := h(x),$$
$$\mathcal{G}_H^{n+1} h(x) := H(\mathcal{G}_H^n h(\alpha(x)), \mathcal{G}_H^n h(\beta(x))), \quad h \in \mathcal{F}(X, Y),\ x \in X$$

(or $h \in \mathcal{F}(X^2, Y)$, $x \in X^2$, then for $x = (x_1, x_2) \in X^2$ denote $\alpha(x) := (\alpha(x_1), \alpha(x_2))$ and $\beta(x) := (\beta(x_1), \beta(x_2)))$, where for $n \in \mathbb{N}_0$ the symbol $\mathcal{G}_H^n h(x)$ stands for the value of the function $\mathcal{G}_H^n(h)$ on an element $x \in X$. We say that the functions α, β determine the operators $\mathcal{G}_H^n$.

For example, fix elements $a, b \in \boldsymbol{K}$, $ab \neq 0$ and consider "multiplications" $X \ni x \mapsto ax \in X$, $X \ni x \mapsto bx \in X$, such that $(st)x = s(tx)$ for all $s, t \in \{a, b\}$ and $x \in X$. Let $c, d \in \boldsymbol{K}$, $cd \neq 0$, and $\alpha(x) := ax$, $\beta(x) := bx$, $x \in X$, $H(y_1, y_2) := cy_1 + dy_2$, $y_1, y_2 \in Y$. Using the well-known combinatorial formulas one can prove that the operator $\mathcal{G}_H^n$ takes in this case the following form

$$\mathcal{G}_H^n h(x) := \sum_{i=0}^{n} \binom{n}{i} c^{n-i} d^i h(a^{n-i} b^i x), \qquad \text{for } n \in \mathbb{N}_0,\ h \in \mathcal{F}(X, Y) \text{ and } x \in X.$$

First we show some properties of $\mathcal{G}_H^n$.

Lemma 1. *Let X be a nonvoid set, Y be a normed space over $\boldsymbol{K}$, $H : Y^2 \to Y$, let $\sigma : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ be an increasing function and $\alpha, \beta : X \to X$ be functions determining the operators $\mathcal{G}_H^n : \mathcal{F}(X, Y) \to \mathcal{F}(X, Y)$ and $\mathcal{G}_\sigma^n : \mathcal{F}(X, \mathbb{R}_+) \to \mathcal{F}(X, \mathbb{R}_+)$. The following conditions hold*

(a) $\mathcal{G}_H^{n+m} f = \mathcal{G}_H^m(\mathcal{G}_H^n f)$ *for all* $n, m \in \mathbb{N}_0$ *and* $f \in \mathcal{F}(X, Y)$,

(b) *if* $\|H(x_1, y_1) - H(x_2, y_2)\| \le \sigma(\|x_1 - x_2\|, \|y_1 - y_2\|)$ *for all* $x_1, x_2, y_1, y_2 \in Y$, *then* $\|\mathcal{G}_H^n f(x) - \mathcal{G}_H^n g(x)\| \le \mathcal{G}_\sigma^n \|f - g\|(x)$ *for every* $n \in \mathbb{N}_0$, $f, g \in \mathcal{F}(X, Y)$ *and* $x \in X$,

(c) $\mathcal{G}_\sigma^n$ *is increasing for every* $n \in \mathbb{N}_0$,

(d) *if* σ *is a subadditive function, then* $\mathcal{G}_\sigma^n$ *is subadditive for every* $n \in \mathbb{N}_0$.

Proof. The verification of each condition runs by standard induction. For example we prove the condition (b). Obviously it holds for $n = 0$. Let us fix $f, g \in \mathcal{F}(X, Y)$ and an $x \in X$. We have

$$\begin{aligned}
&\|\mathcal{G}_H^1 f(x) - \mathcal{G}_H^1 g(x)\| = \|H(f(\alpha(x)), f(\beta(x)) - H(g(\alpha(x)), g(\beta(x))\| \\
&\le \sigma(\|f(\alpha(x)) - g(\alpha(x))\|, \|f(\beta(x)) - g(\beta(x))\|) = \mathcal{G}_\sigma^1 \|f - g\|(x).
\end{aligned}$$

Assume now the condition to hold for an $n \in \mathbb{N}_0$. For fixed $f, g \in \mathcal{F}(X, Y)$ and $x \in X$ we obtain

$$\begin{aligned}
&\|\mathcal{G}_H^{n+1} f(x) - \mathcal{G}_H^{n+1} g(x)\| \\
&\qquad = \|H(\mathcal{G}_H^n f(\alpha(x)), \mathcal{G}_H^n f(\beta(x))) - H(\mathcal{G}_H^n g(\alpha(x)), \mathcal{G}_H^n g(\beta(x)))\| \\
&\qquad \le \sigma(\|\mathcal{G}_H^n f(\alpha(x)) - \mathcal{G}_H^n g(\alpha(x))\|, \|\mathcal{G}_H^n f(\beta(x)) - \mathcal{G}_H^n g(\beta(x))\|) \\
&\qquad \le \sigma(\mathcal{G}_\sigma^n \|f - g\|(\alpha(x)), \mathcal{G}_\sigma^n \|f - g\|(\beta(x))) = \mathcal{G}_\sigma^{n+1} \|f - g\|(x).
\end{aligned}$$

This completes the inductive proof of (b).

The following theorem is the main result of the paper.

Theorem A. *Let X be a nonvoid set, Y be a Banach space over $\boldsymbol{K}$, $A, B : X^2 \to X$, $\varphi : X^2 \to \mathbb{R}_+$, $\psi : X \to \mathbb{R}_+$ and $H : Y^2 \to Y$. Assume that $F : Y^4 \to Y$ is continuous function, $\sigma : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is subadditive and increasing, functions $\alpha, \beta : X \to X$ determine the operators $\mathcal{G}_\sigma^n : \mathcal{F}(Z, \mathbb{R}_+) \to \mathcal{F}(Z, \mathbb{R}_+)$, $Z \in \{X, X^2\}$, and the following conditions hold*

(H_1) $\|H(x_1, y_1) - H(x_2, y_2)\| \le \sigma(\|x_1 - x_2\|, \|y_1 - y_2\|)$ *for all* $x_1, x_2, y_1, y_2 \in Y$,

(H$_2$) $A(\alpha(x), \alpha(y)) = \alpha(A(x, y)),\quad A(\beta(x), \beta(y)) = \beta(A(x, y)),$
$B(\alpha(x), \alpha(y)) = \alpha(B(x, y)),\quad B(\beta(x), \beta(y)) = \beta(B(x, y))$
for all $x, y \in X$,

(H$_3$) $\|F(H(x_1, x_2), H(x_3, x_4), H(x_5, x_6), H(x_7, x_8))\|$
$\leq \|H(F(x_1, x_3, x_5, x_7), F(x_2, x_4, x_6, x_8)) - H(0, 0)\|$
for all $x_1, \dots, x_8 \in Y$,

(H$_4$) $\sum_{n=0}^{\infty} \mathcal{G}_\sigma^n \psi(x) < \infty$ *for every* $x \in X$,

(H$_5$) $\lim_{n\to\infty} \mathcal{G}_\sigma^n \varphi(x, y) = 0$ *for all* $x, y \in X$.

If $f : X \to Y$ *satisfies the inequalities*

$$\|F(f(A(x, y)), f(B(x, y)), f(x), f(y))\| \leq \varphi(x, y) \quad \text{for all } x, y \in X, \tag{2.1}$$

$$\|H(f(\alpha(x)), f(\beta(x))) - f(x)\| \leq \psi(x) \quad \text{for every } x \in X \tag{2.2}$$

then there exists $T : X \to Y$ *such that*

$$F(T(A(x, y)), T(B(x, y)), T(x), T(y)) = 0 \quad \text{for all } x, y \in X, \tag{2.3}$$

$$\|T(x) - f(x)\| \leq \sum_{n=0}^{\infty} \mathcal{G}_\sigma^n \psi(x) \quad \text{for every } x \in X \tag{2.4}$$

and, if H *is continuous,*

$$H(T(\alpha(x)), T(\beta(x))) = T(x) \quad \text{for every } x \in X. \tag{2.5}$$

Let, additionally, σ *be continuous at zero,* $\sigma(0, 0) = 0$ *and* $\lambda : X \to \mathbb{R}_+$ *be a function such that* $\lim_{n\to\infty} \mathcal{G}_\sigma^n \lambda(x) = 0$ *for every* $x \in X$. *If* $S : X \to Y$ *is a solution of* (2.3) *and* (2.5) *such that* $\|S(x) - f(x)\| \leq \lambda(x)$ *for every* $x \in X$, *then* $S = T$.

First prove an auxiliary lemma.

Lemma 2. *Let* X *be a nonvoid set,* Y *be a normed space over* $\boldsymbol{K}$, $H : Y^2 \to Y$, $\psi : X \to \mathbb{R}_+$, *let* $\sigma : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ *be a subadditive and increasing function,* $\alpha, \beta : X \to X$ *be functions determining the operators*

$\mathcal{G}_H^n : \mathcal{F}(X,Y) \to \mathcal{F}(X,Y)$ and $\mathcal{G}_\sigma^n : \mathcal{F}(X,\mathbb{R}_+) \to \mathcal{F}(X,\mathbb{R}_+)$ and let (H_1) *holds. If $f : X \to Y$ satisfies* (2.2)*, then for every $n \in \mathbb{N}$*

$$\|\mathcal{G}_H^n f(x) - f(x)\| \leq \sum_{i=0}^{n-1} \mathcal{G}_\sigma^i \psi(x) \quad \text{for every } x \in X. \tag{2.6}$$

Proof. Obviously (2.6) holds for $n = 1$. Assume now (2.6) to hold for an $n \in \mathbb{N}$ and fix an $x \in X$. From (H_1), (2.2), the monotonicity and the subadditivity of σ we get

$$\begin{aligned}
\|\mathcal{G}_H^{n+1} f(x) - f(x)\| &= \|H(\mathcal{G}_H^n f(\alpha(x)), \mathcal{G}_H^n f(\beta(x))) - f(x)\| \\
&\leq \|H(\mathcal{G}_H^n f(\alpha(x)), \mathcal{G}_H^n f(\beta(x))) - H(f(\alpha(x)), f(\beta(x)))\| \\
&\quad + \|H(f(\alpha(x)), f(\beta(x))) - f(x)\| \\
&\leq \sigma(\|\mathcal{G}_H^n f(\alpha(x)) - f(\alpha(x))\|, \|\mathcal{G}_H^n f(\beta(x)) - f(\beta(x))\|) + \psi(x) \\
&\leq \sigma\left(\sum_{i=0}^{n-1} \mathcal{G}_\sigma^i \psi(\alpha(x)), \sum_{i=0}^{n-1} \mathcal{G}_\sigma^i \psi(\beta(x))\right) + \psi(x) \\
&\leq \sum_{i=0}^{n-1} \sigma(\mathcal{G}_\sigma^i \psi(\alpha(x)), \mathcal{G}_\sigma^i \psi(\beta(x))) + \psi(x) = \sum_{i=0}^{n} \mathcal{G}_\sigma^i \psi(x),
\end{aligned}$$

which completes the inductive proof of (2.6).

Proof of Theorem A. First we show that for every $x \in X$ the sequence $\mathcal{G}_H^n f(x)$ fulfils Cauchy's condition. Namely, applying the assertions (a),(b),(c),(d) of Lemma 1 one by one and (2.6) we have, for all $n, m \in \mathbb{N}$, $n > m$ and $x \in X$,

$$\begin{aligned}
\|\mathcal{G}_H^n f(x) - \mathcal{G}_H^m f(x)\| &= \|\mathcal{G}_H^m (\mathcal{G}_H^{n-m} f)(x) - \mathcal{G}_H^m f(x)\| \leq \mathcal{G}_\sigma^m \|\mathcal{G}_H^{n-m} f - f\|(x) \\
&\leq \mathcal{G}_\sigma^m \left(\sum_{i=0}^{n-m-1} \mathcal{G}_\sigma^i \psi\right)(x) \leq \sum_{i=0}^{n-m-1} \mathcal{G}_\sigma^m (\mathcal{G}_\sigma^i \psi)(x) \\
&= \sum_{i=0}^{n-m-1} \mathcal{G}_\sigma^{i+m} \psi(x) = \sum_{i=m}^{n-1} \mathcal{G}_\sigma^i \psi(x),
\end{aligned}$$

which with (H_4) means that the sequence $\mathcal{G}_H^n f(x)$ satisfies Cauchy's condition. Since Y is a complete space, we may define a function T by

$$T(x) := \lim_{n \to \infty} \mathcal{G}_H^n f(x), \quad x \in X.$$

We will show that T fulfils the equation (2.3). First we prove that

$$\|F(\mathcal{G}_H^n f(A(x,y)), \mathcal{G}_H^n f(B(x,y)), \mathcal{G}_H^n f(x), \mathcal{G}_H^n f(y))\| \le \mathcal{G}_\sigma^n \varphi(x,y) \quad \text{for all } x,y \in X,\ n \in \mathbb{N}. \tag{2.7}$$

Fix $x, y \in X$ and denote $x_1 := \alpha(A(x,y))$, $x_2 := \beta(A(x,y))$, $x_3 := \alpha(B(x,y))$, $x_4 := \beta(B(x,y))$, $x_5 := \alpha(x)$, $x_6 := \beta(x)$, $x_7 := \alpha(y)$, $x_8 := \beta(y)$. From (H_3), (H_1), (H_2) and (2.1) we obtain

$$\begin{aligned}
&\|F(\mathcal{G}_H^1 f(A(x,y)), \mathcal{G}_H^1 f(B(x,y)), \mathcal{G}_H^1 f(x), \mathcal{G}_H^1 f(y))\| \\
&\quad = \|F(H(f(x_1), f(x_2)), H(f(x_3), f(x_4)), \\
&\qquad H(f(x_5), f(x_6)), H(f(x_7), f(x_8)))\| \\
&\quad \le \|H(F(f(x_1), f(x_3), f(x_5), f(x_7)), \\
&\qquad F(f(x_2), f(x_4), f(x_6), f(x_8))) - H(0,0)\| \\
&\le \sigma(\|F(f(x_1), f(x_3), f(x_5), f(x_7))\|, \|F(f(x_2), f(x_4), f(x_6), f(x_8))\|) \\
&\quad \le \sigma(\varphi(\alpha(x), \alpha(y)), \varphi(\beta(x), \beta(y))) = \mathcal{G}_\sigma^1 \varphi(x,y).
\end{aligned}$$

Assume now (2.7) to hold for an $n \in \mathbb{N}$. Fix $x, y \in X$ and define $x_1, \dots, x_8$ in the same way. As for $n = 1$ we get

$$\begin{aligned}
&\|F(\mathcal{G}_H^{n+1} f(A(x,y)), \mathcal{G}_H^{n+1} f(B(x,y)), \mathcal{G}_H^{n+1} f(x), \mathcal{G}_H^{n+1} f(y))\| \\
&\quad = \|F(H(\mathcal{G}_H^n f(x_1), \mathcal{G}_H^n f(x_2)), H(\mathcal{G}_H^n f(x_3), \mathcal{G}_H^n f(x_4)), \\
&\qquad H(\mathcal{G}_H^n f(x_5), \mathcal{G}_H^n f(x_6)), H(\mathcal{G}_H^n f(x_7), \mathcal{G}_H^n f(x_8)))\| \\
&\quad \le \|H(F(\mathcal{G}_H^n f(x_1), \mathcal{G}_H^n f(x_3), \mathcal{G}_H^n f(x_5), \mathcal{G}_H^n f(x_7)), \\
&\qquad F(\mathcal{G}_H^n f(x_2), \mathcal{G}_H^n f(x_4), \mathcal{G}_H^n f(x_6), \mathcal{G}_H^n f(x_8))) - H(0,0)\| \\
&\quad \le \sigma(\|F(\mathcal{G}_H^n f(x_1), \mathcal{G}_H^n f(x_3), \mathcal{G}_H^n f(x_5), \mathcal{G}_H^n f(x_7))\|, \\
&\qquad \|F(\mathcal{G}_H^n f(x_2), \mathcal{G}_H^n f(x_4), \mathcal{G}_H^n f(x_6), \mathcal{G}_H^n f(x_8))\|) \\
&\quad \le \sigma(\mathcal{G}_\sigma^n \varphi(\alpha(x), \alpha(y)), \mathcal{G}_\sigma^n \varphi(\beta(x), \beta(y))) = \mathcal{G}_\sigma^{n+1} \varphi(x,y).
\end{aligned}$$

Letting n increase to ∞ in (2.7) and taking into account the continuity of F and (H_5) we obtain immediately (2.3). Moreover, T satisfies (2.4) which results on letting $n \to \infty$ in (2.6). Next observe that for any $n \in \mathbb{N}$ and $x \in X$ we have

$$\|H(\mathcal{G}_H^n f(\alpha(x)), \mathcal{G}_H^n f(\beta(x))) - \mathcal{G}_H^n f(x)\| = \|\mathcal{G}_H^{n+1} f(x) - \mathcal{G}_H^n f(x)\|.$$

Since the right-hand of this equality tends to zero as $n \to \infty$, it becomes apparent that if H is continuous, then T satisfies (2.5).

Now assume that σ is continuous at zero, $\sigma(0,0)=0$ and $\lambda : X \to \mathbb{R}_+$ is a function such that $\lim\limits_{n\to\infty} \mathcal{G}_\sigma^n \lambda(x) = 0$ for every $x \in X$. Let $S : X \to Y$ be a solution of (2.3) and (2.5) such that $\|S(x) - f(x)\| \leq \lambda(x)$ for every $x \in X$. First we will show that for every $n \in \mathbb{N}$

$$\mathcal{G}_\sigma^n\left(\sum_{i=0}^{\infty} \mathcal{G}_\sigma^i \psi\right)(x) \leq \sum_{i=n}^{\infty} \mathcal{G}_\sigma^i \psi(x) \quad \text{for every } x \in X. \tag{2.8}$$

Fix an $x \in X$ and $\varepsilon > 0$. By (H_4) there exist $N \in \mathbb{N}$ such that for every $k \geq N$

$$\sum_{i=0}^{\infty} \mathcal{G}_\sigma^i \psi(\alpha(x)) \leq \varepsilon + \sum_{i=0}^{k} \mathcal{G}_\sigma^i \psi(\alpha(x)) \text{ and}$$
$$\sum_{i=0}^{\infty} \mathcal{G}_\sigma^i \psi(\beta(x)) \leq \varepsilon + \sum_{i=0}^{k} \mathcal{G}_\sigma^i \psi(\beta(x)).$$

Using the above inequalities, the monotonicity and the subadditivity of σ we obtain

$$\begin{aligned}
\mathcal{G}_\sigma^1\left(\sum_{i=0}^{\infty} \mathcal{G}_\sigma^i \psi\right)(x) &= \sigma\left(\sum_{i=0}^{\infty} \mathcal{G}_\sigma^i \psi(\alpha(x)), \sum_{i=0}^{\infty} \mathcal{G}_\sigma^i \psi(\beta(x))\right) \\
&\leq \sigma\left(\varepsilon + \sum_{i=0}^{N} \mathcal{G}_\sigma^i \psi(\alpha(x)), \varepsilon + \sum_{i=0}^{N} \mathcal{G}_\sigma^i \psi(\beta(x))\right) \\
&\leq \sigma(\varepsilon,\varepsilon) + \sigma\left(\sum_{i=0}^{N} \mathcal{G}_\sigma^i \psi(\alpha(x)), \sum_{i=0}^{N} \mathcal{G}_\sigma^i \psi(\beta(x))\right) \\
&\leq \sigma(\varepsilon,\varepsilon) + \sum_{i=0}^{N} \sigma(\mathcal{G}_\sigma^i \psi(\alpha(x)), \mathcal{G}_\sigma^i \psi(\beta(x))) \\
&\leq \sigma(\varepsilon,\varepsilon) + \sum_{i=0}^{\infty} \sigma(\mathcal{G}_\sigma^i \psi(\alpha(x)), \mathcal{G}_\sigma^i \psi(\beta(x))) \\
&= \sigma(\varepsilon,\varepsilon) + \sum_{i=1}^{\infty} \mathcal{G}_\sigma^i \psi(x).
\end{aligned}$$

Since σ is continuous at zero and $\sigma(0,0)=0$, we can make the expression $\sigma(\varepsilon,\varepsilon)$ arbitrarily small, which implies (2.8) to hold for $n=1$. Claim further that (2.8) holds for an $n \in \mathbb{N}$. Fix an $x \in X$ and $\varepsilon > 0$. Take

$N \in \mathbb{N}$ such that for every $k \geq N$

$$\sum_{i=n}^{\infty} \mathcal{G}_{\sigma}^{i}\psi(\alpha(x)) \leq \varepsilon + \sum_{i=n}^{k} \mathcal{G}_{\sigma}^{i}\psi(\alpha(x)) \text{ and}$$
$$\sum_{i=n}^{\infty} \mathcal{G}_{\sigma}^{i}\psi(\beta(x)) \leq \varepsilon + \sum_{i=n}^{k} \mathcal{G}_{\sigma}^{i}\psi(\beta(x)).$$

We get

$$\mathcal{G}_{\sigma}^{n+1}\left(\sum_{i=0}^{\infty} \mathcal{G}_{\sigma}^{i}\psi\right)(x) = \sigma\left(\mathcal{G}_{\sigma}^{n}\left(\sum_{i=0}^{\infty} \mathcal{G}_{\sigma}^{i}\psi\right)(\alpha(x)), \mathcal{G}_{\sigma}^{n}\left(\sum_{i=0}^{\infty} \mathcal{G}_{\sigma}^{i}\psi\right)(\beta(x))\right)$$
$$\leq \sigma\left(\sum_{i=n}^{\infty} \mathcal{G}_{\sigma}^{i}\psi(\alpha(x)), \sum_{i=n}^{\infty} \mathcal{G}_{\sigma}^{i}\psi(\beta(x))\right)$$
$$\leq \sigma\left(\varepsilon + \sum_{i=n}^{N} \mathcal{G}_{\sigma}^{i}\psi(\alpha(x)), \varepsilon + \sum_{i=n}^{N} \mathcal{G}_{\sigma}^{i}\psi(\beta(x))\right)$$
$$\leq \sigma(\varepsilon, \varepsilon) + \sum_{i=n}^{N} \sigma(\mathcal{G}_{\sigma}^{i}\psi(\alpha(x)), \mathcal{G}_{\sigma}^{i}\psi(\beta(x))) \leq \sigma(\varepsilon, \varepsilon) + \sum_{i=n+1}^{\infty} \mathcal{G}_{\sigma}^{i}\psi(x).$$

Seeing that σ is continuous at zero, $\sigma(0,0) = 0$ and x, ε are arbitrarily fixed, the condition (2.8) is true for $n+1$.

Let us observe now that if a function $h : X \to Y$ satisfies (2.5), then it satisfies (2.2) with $\psi_0 \equiv 0$, too. Since $\sigma(0,0) = 0$, we have $\mathcal{G}_{\sigma}^{n}\psi_0 \equiv 0$ for every $n \in \mathbb{N}_0$. Hence, in virtue of Lemma 2, we obtain, for every $x \in X$ and $n \in \mathbb{N}$, $\|\mathcal{G}_{H}^{n}h(x) - h(x)\| \leq \sum_{i=0}^{n-1} \mathcal{G}_{\sigma}^{i}\psi_0(x) = 0$, whence $\mathcal{G}_{H}^{n}h = h$ for every $n \in \mathbb{N}$. Applying this equality to T and S and using the assertions (b),(c),(d) of Lemma 1 one by one and (2.8), for every $x \in X$, we get

$$\|T(x) - S(x)\| = \|\mathcal{G}_{H}^{n}T(x) - \mathcal{G}_{H}^{n}S(x)\| \leq \mathcal{G}_{\sigma}^{n}\|T - S\|(x)$$
$$\leq \mathcal{G}_{\sigma}^{n}(\|T - f\| + \|f - S\|)(x) \leq \mathcal{G}_{\sigma}^{n}\left(\sum_{i=0}^{\infty} \mathcal{G}_{\sigma}^{i}\psi + \lambda\right)(x)$$
$$\leq \mathcal{G}_{\sigma}^{n}\left(\sum_{i=0}^{\infty} \mathcal{G}_{\sigma}^{i}\psi\right)(x) + \mathcal{G}_{\sigma}^{n}\lambda(x) \leq \sum_{i=n}^{\infty} \mathcal{G}_{\sigma}^{i}\psi(x) + \mathcal{G}_{\sigma}^{n}\lambda(x).$$

From (H_4) and the fact that $\lim_{n\to\infty} \mathcal{G}_{\sigma}^{n}\lambda(x) = 0$ for an $x \in X$ the right-hand side of the above inequality becomes arbitrarily small as $n \to \infty$, which means that $T = S$. This proves the theorem.

Remark 1. If F is a linear function, $a, b \in \boldsymbol{K}$ and $H(y_1, y_2) := ay_1 + by_2$ for $y_1, y_2 \in Y$, then (H_3) holds.

Remark 2. If σ is continuous at zero and $\sigma(0,0) = 0$, then, by (H_1), H is continuous.

Remark 3. It is easy to verify that if X is a topological space and the functions f, H, α and β are continuous, then the operators $\mathcal{G}_H^n f$ are continuous for every $n \in \mathbb{N}$. Thus if the series $\sum_{i=0}^{\infty} \mathcal{G}_\sigma^i \psi(x)$ coverges uniformly on X, then, by the inequality

$$\|\mathcal{G}_H^n f(x) - \mathcal{G}_H^m f(x)\| \le \sum_{i=m}^{n-1} \mathcal{G}_\sigma^i \psi(x),$$

the function T is continuous.

Remark 4. If F does not depend on the first variable (on the second variable, respectively), then we may omit in (H_2) the equalities concerning A (B, resp.).

3. Some applications of the main result

3.1. The stability of some case of the general linear functional equation

Let X be a linear space over $\boldsymbol{K}$ and Y a Banach space over $\boldsymbol{K}$. Consider the equation

$$f(ax + by) = cf(x) + df(y) \quad \text{for all } x, y \in X, \tag{3.1}$$

where $f : X \to Y$, $a, b, c, d \in \boldsymbol{K}$, $abcd \neq 0$, which is a particular case of the general linear functional equation (see p. 66 in [1], also p. 339 in [15]). Notice that if for $\varphi : X \to \mathbb{R}_+$ and $f : X \to Y$ the following inequality holds

$$\|f(ax + by) - cf(x) - df(y)\| \le \varphi(x, y) \quad \text{for all } x, y \in X, \tag{3.2}$$

then

$$(\mathrm{L}_1) \quad \left\| \frac{1}{c+d} f((a+b)x) - f(x) \right\| \le \frac{1}{|c+d|} \varphi(x, x) \quad \text{for every } x \in X, \text{ if } c + d \neq 0,$$

$$(\mathrm{L}_2) \quad \left\| (c+d) f\left(\frac{x}{a+b}\right) - f(x) \right\| \le \varphi\left(\frac{x}{a+b}, \frac{x}{a+b}\right) \quad \text{for every } x \in X, \text{ if } a + b \neq 0,$$

(L$_3$) $\left\|\frac{1}{c}f(ax)-f(x)\right\| \leq \frac{1}{|c|}\varphi(x,0)+\left|\frac{d}{c}\right| \, \|f(0)\|$
for every $x\in X$,

(L$_4$) $\left\|cf\left(\frac{x}{a}\right)-f(x)\right\| \leq \varphi\left(\frac{x}{a},0\right)+|d| \, \|f(0)\|$
for every $x\in X$,

(L$_5$) $\left\|\frac{1}{d}f(bx)-f(x)\right\| \leq \frac{1}{|d|}\varphi(0,x)+\left|\frac{c}{d}\right| \, \|f(0)\|$
for every $x\in X$,

(L$_6$) $\left\|df\left(\frac{x}{b}\right)-f(x)\right\| \leq \varphi(0,\frac{x}{b})+|c| \, \|f(0)\|$
for every $x\in X$.

In compliance with (L$_1$)–(L$_6$) define a function $\delta : X \to \mathbb{R}_+$, constants $s,t,z\in \boldsymbol{K}$ and $x_0\in X$ as follows

(P$_1$) $s:=\frac{1}{c+d}, \quad t:=a+b, \quad z:=0,$
x_0 at will, $\quad \delta(x):=|s|\varphi(x,x),\ x\in X,$

(P$_2$) $s:=c+d, \quad t:=\frac{1}{a+b}, \quad z:=0,$
x_0 at will, $\quad \delta(x):=\varphi(tx,tx),\ x\in X,$

(P$_3$) $s:=\frac{1}{c}, \quad t:=a, \quad z:=-\frac{d}{c},$
$x_0=0, \quad \delta(x):=|s|\varphi(x,0),\ x\in X,$

(P$_4$) $s:=c, \quad t:=\frac{1}{a}, \quad z:=d,$
$x_0=0, \quad \delta(x):=\varphi(tx,0),\ x\in X,$

(P$_5$) $s:=\frac{1}{d}, \quad t:=b, \quad z:=-\frac{c}{d},$
$x_0=0, \quad \delta(x):=|s|\varphi(0,x),\ x\in X,$

(P$_6$) $s:=d, \quad t:=\frac{1}{b}, \quad z:=c,$
$x_0=0, \quad \delta(x):=\varphi(0,tx),\ x\in X.$

From the main theorem we obtain the following corollary concerning the stability of the equation (3.1).

Corollary 1. *Suppose that X is a linear space over $\boldsymbol{K}$, Y a Banach space over $\boldsymbol{K}$, $\varphi : X^2 \to \mathbb{R}_+$, $a, b, c, d \in \boldsymbol{K}$ and $abcd \neq 0$. Assume that $s, t, z \in \boldsymbol{K}$, $x_0 \in X$ and $\delta : X \to \mathbb{R}_+$ are defined as in one of the conditions* (P_1)–(P_6),

$$\sum_{n=0}^{\infty} |s|^n \delta(t^n x) < \infty \quad \text{for every } x \in X, \tag{3.3}$$

$$\lim_{n\to\infty} |s|^n \varphi(t^n x, t^n y) = 0 \quad \text{for all } x, y \in X, \tag{3.4}$$

and $zf(x_0) = 0$ or $c + d = 1$ or $|s| < 1$.

If $f : X \to Y$ satisfies (3.2)*, then there exists a solution $T : X \to Y$ of the equation* (3.1) *such that $\|T(x) - f(x)\| \leq \sum_{n=0}^{\infty} |s|^n \delta(t^n x) + \frac{\|zf(x_0)\|}{1-|s|}$ for every $x \in X$. If, additionally, $\lambda : X \to \mathbb{R}_+$ is a function such that $\lim_{n\to\infty} |s|^n \lambda(t^n x) = 0$ for every $x \in X$, $S : X \to Y$ is a solution of* (3.1) *fulfilling the inequality $\|S(x) - f(x)\| \leq \lambda(x)$ for every $x \in X$ and $z = 0$ or $S(x_0) = T(x_0)$, then $S = T$.*

Proof. Put $F(x_1, x_2, x_3, x_4) := x_1 - cx_3 - dx_4$, $x_1, x_2, x_3, x_4 \in Y$, $H(x, y) := sx$, $x, y \in Y$, $A(x, y) := ax + by$, $x, y \in X$, $\alpha(x) := tx$, $\beta(x) := x$, $x \in X$, $\sigma(u, v) := |s|u$, $u, v \in \mathbb{R}_+$ and $\psi(x) := \delta(x) + \|zf(x_0)\|$, $x \in X$. It is easy to observe that the functions defined this way satisfy $(\mathrm{H}_1), (\mathrm{H}_2), (\mathrm{H}_3)$.

I. First suppose that $c+d \neq 1$. Assume that s, t, z, x_0 and δ are defined by (P_i), $i \in \{1, \ldots, 6\}$. In view of (3.3), (3.4) and fact that $zf(x_0) = 0$ or $|s| < 1$, for all $x, y \in X$, we have

$$\sum_{n=0}^{\infty} \mathcal{G}_\sigma^n \psi(x) = \sum_{n=0}^{\infty} |s|^n \psi(t^n x) = \sum_{n=0}^{\infty} |s|^n \delta(t^n x) + \frac{\|zf(x_0)\|}{1 - |s|} < \infty \quad \text{and}$$
$$\lim_{n\to\infty} \mathcal{G}_\sigma^n \varphi(x, y) = \lim_{n\to\infty} |s|^n \varphi(t^n x, t^n y) = 0.$$

If f fulfils (3.2), then also it fulfils (L_i). From the main theorem there exists a solution $T : X \to Y$ of (3.1) such that $\|T(x) - f(x)\| \leq \sum_{n=0}^{\infty} |s|^n \delta(t^n x) + \frac{\|zf(x_0)\|}{1-|s|}$ for every $x \in X$. Additionally, T satisfies (2.5), thus $sT(tx) = T(x)$ for every $x \in X$. Next notice that if $h : X \to Y$ is a solution of (3.1), then

$$sh(tx) + zh(x_0) = h(x) \quad \text{for every } x \in X, \tag{3.5}$$

which with (2.5) gives $zT(x_0) = 0$.

Assume that $\lambda : X \to \mathbb{R}_+$ is a function such that $\lim_{n\to\infty} |s|^n \lambda(t^n x) = 0$ for every $x \in X$, $S : X \to Y$ is a solution of (3.1) fulfilling the inequality $\|S(x) - f(x)\| \le \lambda(x)$ for every $x \in X$ and $z = 0$ or $S(x_0) = T(x_0)$. Since S fulfils (3.5), for any $x \in X$, we obtain

$$H(S(\alpha(x)), S(\beta(x))) = sS(tx) = sS(tx) + zT(x_0) = sS(tx) + zS(x_0) = S(x),$$

thus S satisfies (2.5). As σ is continuous at zero and $\sigma(0, 0) = 0$, from Theorem A we get immediately $S = T$.

II. Now suppose that $c + d = 1$. Put $g(x) := f(x) - f(x_0)$, $x \in X$. Obviously g fulfils (3.2) and $g(x_0) = 0$. The first part of the proof implies that there exists a solution $T^* : X \to Y$ of (3.1), such that $\|T^*(x) - g(x)\| \le \sum_{n=0}^{\infty} |s|^n \delta(t^n x)$ for every $x \in X$ and $zT^*(x_0) = 0$. Let further $T(x) := T^*(x) + f(x_0)$, $x \in X$. T satisfies (3.1) and for every $x \in X$

$$\|T(x) - f(x)\| = \|T^*(x) + f(x_0) - f(x)\| = \|T^*(x) - g(x)\| \le \sum_{n=0}^{\infty} |s|^n \delta(t^n x).$$

Next assume that $\lambda : X \to \mathbb{R}_+$ is a function such that $\lim_{n\to\infty} |s|^n \lambda(t^n x) = 0$ for every $x \in X$, $S : X \to Y$ is a solution of (3.1) fulfilling the inequality $\|S(x) - f(x)\| \le \lambda(x)$ for every $x \in X$ and $z = 0$ or $S(x_0) = T(x_0)$. Notice that the function $S(x) - f(x_0)$ is a solution of (3.1). Moreover, for every $x \in X$, we have

$$\begin{aligned}\|S(x) - f(x_0) - g(x)\| &= \|S(x) - f(x_0) + f(x_0) - f(x)\| \\ &= \|S(x) - f(x)\| \le \lambda(x).\end{aligned}$$

Since $z = 0$ or $S(x_0) = T(x_0)$ and $zT^*(x_0) = 0$, we get

$$z(S(x_0) - f(x_0)) = z(T(x_0) - f(x_0)) = zT^*(x_0) = 0.$$

Hence and by (3.5), for every $x \in X$, we obtain

$$\begin{aligned}&H(S(\alpha(x)) - f(x_0), S(\beta(x)) - f(x_0)) = s[S(tx) - f(x_0)] \\ &= S(x) - f(x_0) - z[S(x_0) - f(x_0)] = S(x) - f(x_0),\end{aligned}$$

which means that $S(x) - f(x_0)$ satisfies (2.5). From the first part of the proof we get $T^* = S - f(x_0)$ and consequently $T = T^* + f(x_0) = S$. This proves the corollary.

Remark 5. The assumption that X is a linear space was made to simplify Corollary 1. In fact, it is sufficient to assume that $(X,+)$ is a groupoid, in which "multiplications" by some constants from $\boldsymbol{K}$ with suitable assumptions are defined. In further considerations we will make the structure of X as weak as possible.

Remark 6. If $a=b=c=d=1$ and $(X,+)$ is a commutative semigroup (uniquely divisible by 2, respectively), then defining constants s,t,z and x_0, δ as in (P_1) ((P_2), resp.) we infer the generalized stability of the Cauchy functional equation. In the case corresponding with (P_1) we get exactly Forti–Kominek's Theorem. This results generalize well-known theorems concerning the stability of the Cauchy equation (we omit here some regularity assumptions, which guarantee the real homogeneity of a solution T), namely, Theorem 1 and Theorem 2 from [13], Theorem 1 from [14] and Theorem 1 from [19]. In particular it also implies the stability of the following functional equation (considered e.g. in [1])

$$
\begin{aligned}
&f\left(\sqrt[k]{x^k+y^k}\right)=f(x)+f(y),\\
&\qquad\text{where } k\in\mathbb{N} \text{ and } f:\mathbb{R}\to\mathbb{R} \text{ (or } f:\mathbb{R}_+\to\mathbb{R})\\
&f\left(\sqrt{x^2+y^2+1}\right)=f(x)+f(y),\\
&\qquad\text{where } f:\mathbb{R}\to\mathbb{R} \text{ (or } f:\mathbb{R}_+\to\mathbb{R}).
\end{aligned}
$$

Indeed, if we put $x*y:=\sqrt[k]{x^k+y^k}$, $x\star y:=\sqrt{x^2+y^2+1}$, $x,y\in\mathbb{R}$ (or $x,y\in\mathbb{R}_+$) and $X=\mathbb{R}$ (or $X=\mathbb{R}_+$), then $(X,*)$ and $(X,\star)$ are commutative semigroups. If $X=\mathbb{R}_+$, then $(X,*)$ is also uniquely divisible by 2.

Remark 7. If $a=b=c=d=\frac{1}{2}$, $f:D\to Y$, where D is a Jensen convex subset of a uniquely divisible by 2 commutative semigroup X with a neutral element 0 and $0\in D$, then defining constants s,t,z and x_0, δ as in (P_4) or (P_6) ((P_3) or (P_5), respectively) we infer the generalized stability of the Jensen functional equation.

Remark 8. We can apply Corollary 1 to considerations of the Pexider equation $f(x+y)=h(x)+g(y)$, where $f,g,h:X\to Y$ and X is a commutative semigroup with an neutral element 0 (uniquely divisible by 2, respectively). It is sufficient to observe that if a function $\varphi:X^2\to\mathbb{R}_+$ satisfies the inequality

$$\|f(x+y)-g(x)-h(y)\|\le\varphi(x,y)\quad\text{for all } x,y\in X, \tag{3.6}$$

then putting in (3.6) $y=0$ then $x=0$ and taking into account (3.6), for $x,y\in X$, we obtain

$$\begin{aligned}\|[f(x+y)-g(0)-h(0)]-[f(x)-g(0)-h(0)]-[f(y)-g(0)-h(0)]\|\\ \leq \varphi(x,y)+\varphi(x,0)+\varphi(0,y).\end{aligned}$$

Thus the functions $f^*(x):=f(x)-g(0)-h(0)$, $x\in X$ and $\varphi^*(x,y):=\varphi(x,y)+\varphi(x,0)+\varphi(0,y)$, $x,y\in X$ satisfy the inequality $\|f^*(x+y)-f^*(x)-f^*(y)\|\leq\varphi^*(x,y)$ for all $x,y\in X$. Under suitable assumptions on φ^*, in view of the corollary referring to the stability of the Cauchy equation, there exists an additive function $T:X\to Y$ "near" the function f^*. Putting $\tilde{f}:=T+g(0)+h(0)$, $\tilde{g}:=T+g(0)$, $\tilde{h}:=T+h(0)$ we obtain a solution of the Pexider equation "near" the functions f,g,h.

Remark 9. Let us observe that if a function $f:X\to Y$ fulfils the inequality

$$\|f(x+y)-f(x)-f(y)\|\leq\varepsilon(\|x\|^p+\|y\|^p)\ \text{ for all } x,y\in X,$$

where X is a normed space over $\boldsymbol{K}$, Y is a Banach space over $\boldsymbol{K}$, $\varepsilon\geq 0$ and $p\in[0,1)$, then, for every $k\in\mathbb{N}\setminus\{1\}$, we have

$$\|f(kx)-kf(x)\|\leq\varepsilon\|x\|^p\sum_{\nu=1}^{k-1}(\nu^p+1)\ \text{ for all } x,y\in X.$$

Hence, as a corollary from Theorem A, we obtain Theorem 2 in [16].

3.2. The stability of the quadratic functional equation

Let X be an abelian group and Y a Banach space over $\boldsymbol{K}$. For $f:X\to Y$ consider the equation

$$f(x+y)+f(x-y)=2f(x)+2f(y)\ \text{ for all } x,y\in X, \tag{3.7}$$

called the quadratic functional equation. It is easy to observe that if for a function $\varphi:X^2\to\mathbb{R}_+$ the following inequality holds

$$\|f(x+y)+f(x-y)-2f(x)-2f(y)\|\leq\varphi(x,y)\ \text{ for all } x,y\in X,$$

then, putting in (3.8) $x=y=0$ and then $x=y$, we get $\|f(0)\|\leq\frac{1}{2}\varphi(0,0)$ and $\|\frac{1}{4}f(2x)-f(x)\|\leq\frac{1}{4}\varphi(x,x)+\frac{1}{4}\|f(0)\|$ for every $x\in X$. Hence

$$\|\tfrac{1}{4}f(2x)-f(x)\|\leq\tfrac{1}{4}\varphi(x,x)+\tfrac{1}{8}\varphi(0,0)\ \text{ for every } x\in X. \tag{3.9}$$

Next, if X is uniquely divisible by 2, putting in (3.9) $\frac{x}{2}$ in place of x we have

$$\|4f(\tfrac{x}{2}) - f(x)\| \leq \varphi(\tfrac{x}{2}, \tfrac{x}{2}) + \tfrac{1}{2}\varphi(0,0) \quad \text{for every} \quad x \in X. \tag{3.10}$$

Notice also that if $T : X \to Y$ satisfies (3.7), then $T(0) = 0$. Define functions $F, H, \sigma, A, B, \alpha, \beta$ and ψ in compliance with the conditions (3.8) and (3.9) ((3.8) and (3.10), resp.) as follows

$$\begin{aligned}
&F(x_1, x_2, x_3, x_4) := x_1 + x_2 - 2x_3 - 2x_4,\ x_1, x_2, x_3, x_4 \in Y,\\
&H(x, y) := \tfrac{1}{4}x \quad (H(x, y) := 4x),\ x, y \in Y,\\
&\sigma(u, v) := \tfrac{1}{4}u \quad (\sigma(u, v) := 4u),\ u, v \in \mathbb{R}_+,\\
&A(x, y) := x + y,\ B(x, y) := x - y,\ x, y \in X,\\
&\alpha(x) := 2x, \quad (\alpha(x) := \tfrac{x}{2}), \quad \beta(x) := x,\ x \in X,\\
&\psi(x) := \tfrac{1}{4}\varphi(x, x) + \tfrac{1}{8}\varphi(0, 0),\\
&\qquad (\psi(x) := \varphi(\tfrac{x}{2}, \tfrac{x}{2}) + \tfrac{1}{2}\varphi(0, 0)),\ x \in \mathbb{R}_+.
\end{aligned}$$

Applying the main theorem we obtain the following corollary, which generalizes some results given in [5], [6] and [7].

Corollarry 2. *Let X be an abelian group (uniformly divisible by 2, respectively), Y a Banach space over $\boldsymbol{K}$, $\varphi : X^2 \to \mathbb{R}_+$ and*

$$\sum_{n=0}^{\infty} (\tfrac{1}{4})^n \varphi(2^n x, 2^n x) < \infty \tag{3.11}$$

$$\left(\sum_{n=1}^{\infty} 4^n \varphi(\tfrac{x}{2^n}, \tfrac{x}{2^n}) < \infty \text{ and } \varphi(0,0) = 0\right) \text{ for every } x \in X,$$

$$\lim_{n\to\infty} (\tfrac{1}{4})^n \varphi(2^n x, 2^n y) = 0 \tag{3.12}$$

$$\left(\lim_{n\to\infty} 4^n \varphi(\tfrac{x}{2^n}, \tfrac{x}{2^n}) = 0\right) \text{ for all } x, y \in X.$$

If $f : X \to Y$ satisfies (3.8), *then there exists a solution $T : X \to Y$ of the equation* (3.7) *such that*

$$\|T(x) - f(x)\| \leq \tfrac{1}{4} \sum_{n=0}^{\infty} (\tfrac{1}{4}) \varphi(2^n x, 2^n x) \left(\|T(x) - f(x)\| \leq \tfrac{1}{4} \sum_{n=1}^{\infty} 4^n \varphi(\tfrac{x}{2^n}, \tfrac{x}{2^n}) \right)$$

for every $x \in X$. If, additionally, $\lambda : X \to \mathbb{R}_+$ is a function such that $\lim_{n\to\infty} (\frac{1}{4})^n \lambda(2^n x) = 0$ ($\lim_{n\to\infty} 4^n \lambda(\frac{x}{2^n}) = 0$) for every $x \in X$ and $S : X \to Y$

is a solution of (3.7) *such that* $\|S(x) - f(x)\| \le \lambda(x)$ *for every* $x \in X$*, then* $S = T$.

Remark 10. In the analogous way as for the quadratic functional equation we can obtain similar stability results for the following well-known equations

$$f(x+y) - f(x-y) = 2f(y) \text{ and } f(x+y) + f(x-y) = 2f(x).$$

3.3. The stability of an another type of functional equations

Consider the equation (see e.g. [2])

$$f\left(\tfrac{x+y}{2} - \sqrt{xy}\right) + f\left(\tfrac{x+y}{2} + \sqrt{xy}\right) = f(x) + f(y) \quad \text{for all } x, y \in \mathbb{R}_+, \tag{3.13}$$

where $f : \mathbb{R}_+ \to \mathbb{R}$. Applying the main theorem we get the following corollary concerning the stability of this equation.

Corollary 3. *Let* $\varphi : \mathbb{R}_+^2 \to \mathbb{R}_+$, $\emptyset \ne I \subset \{1, 2, 3\}$,

$$\sum_{n=0}^{\infty} (\tfrac{1}{2})^n \min_{i \in I} \psi_i(2^n x) < \infty$$

$$\left(\sum_{n=1}^{\infty} 2^n \min_{i \in I} \psi_i(\tfrac{x}{2^n}) < \infty, \ \textit{resp.} \right) \textit{ for every } x \in X,$$

where $\psi_1(x) := \frac{1}{2}\varphi(x, x)$, $\psi_2(x) := \frac{1}{2}\varphi(2x, 0)$, $\psi_3(x) := \frac{1}{2}\varphi(0, 2x)$, $x \in \mathbb{R}_+$, *and*

$$\lim_{n\to\infty} (\tfrac{1}{2})^n \varphi(2^n x, 2^n y) = 0 \ \left(\lim_{n\to\infty} 2^n \varphi(\tfrac{x}{2^n}, \tfrac{x}{2^n}) = 0, \ \textit{resp.}\right) \textit{ for all } x, y \in \mathbb{R}_+.$$

If $f : \mathbb{R}_+ \to \mathbb{R}$ *satisfies the inequality*

$$|f\left(\tfrac{x+y}{2} - \sqrt{xy}\right) + f\left(\tfrac{x+y}{2} + \sqrt{xy}\right) - f(x) - f(y)| \le \varphi(x, y) \quad \textit{for all } x, y \in \mathbb{R}_+, \tag{3.14}$$

then there exists a solution $T : \mathbb{R}_+ \to \mathbb{R}$ *of the equation* (3.13) *such that* $\|T(x) - f(x)\| \le \sum_{n=0}^{\infty} (\frac{1}{2})^n \min_{i\in I} \psi_i(2^n x)$ ($\|T(x) - f(x)\| \le \sum_{n=1}^{\infty} 2^n \min_{i\in I} \psi_i(\frac{x}{2^n})$, *resp.) for every* $x \in \mathbb{R}_+$. *If, additionally,* $\lambda : \mathbb{R}_+ \to \mathbb{R}_+$ *is a function such that* $\lim_{n\to\infty} (\frac{1}{2})^n \lambda(2^n x) = 0$ ($\lim_{n\to\infty} 2^n \lambda(\frac{x}{2^n}) = 0$) *for every* $x \in X$, $S : \mathbb{R}_+ \to \mathbb{R}$

is a solution of (3.13) *such that* $\|S(x) - f(x)\| \leq \lambda(x)$ *for every* $x \in X$ *and* $S(0) = T(0)$, *then* $S = T$.

Proof. Notice that if $f : \mathbb{R}_+ \to \mathbb{R}$ satisfies (3.14), then, for every $x \in \mathbb{R}_+$, we have

$$|\tfrac{1}{2}f(2x) + \tfrac{1}{2}f(0) - f(x)| \leq \min_{i\in\{1,2,3\}} \psi_i(x) \text{ and}$$
$$|2f(\tfrac{x}{2^n}) - f(0) - f(x)| \leq \min_{i\in\{1,2,3\}} 2\psi_i\left(\frac{x}{2}\right).$$

If $f(0) = 0$, then the existence and uniqueness assertion of a solution T results directly from the application of Theorem A. If $f(0) \neq 0$,then we follow as in the second part of the proof of Corollary 1.

The main theorem can also determine the stability of an equation in a single variable. For example take the equation

$$af(\alpha(x)) = f(x) \quad \text{for every } x \in X, \tag{3.15}$$

where $f : X \to Y$, $\alpha : X \to X$, $a \in \boldsymbol{K}$, X is a nonvoid set and Y is a Banach space over $\boldsymbol{K}$. The classical Hyers–Ulam stability of this equation was consider in [3]. In Theorem A put $F(x_1, x_2, x_3, x_4) := ax_1 - x_3$, $x_1, x_2, x_3, x_4 \in Y$, $A(x, y) := \alpha(x)$, $x \in X$, $H(x, y) := ax$, $x, y \in Y$, $\varphi(x, y) := \psi(x)$, $x, y \in X$, where $\psi : X \to \mathbb{R}_+$. Let α^n stands for an n-th iterate of α. Easily we obtain the following corollary.

Corollary 4. *If* $\|af(\alpha(x)) - f(x)\| \leq \psi(x)$ *and* $\sum_{n=0}^{\infty} |a|^n \psi(\alpha^n(x)) < \infty$ *for every* $x \in X$, *then there exists a solution* $T : X \to Y$ *of* (3.15) *such that* $\|T(x) - f(x)\| \leq \sum_{n=0}^{\infty} |a|^n \psi(\alpha^n(x))$ *for every* $x \in X$. *If, additionally,* $\lambda : X \to \mathbb{R}_+$ *is a function such that* $\lim_{n\to\infty} |a|^n \lambda(\alpha^n(x)) = 0$ *for every* $x \in X$ *and* $S : X \to Y$ *is a solution of* (3.15) *such that* $\|S(x) - f(x)\| \leq \lambda(x)$ *for every* $x \in X$, *then* $S = T$.

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