A note on the maximum term of an entire Dirichlet series

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1. Consider the Dirichlet series

$$f(s) = \sum_{n=1}^{\infty} a_n e^{s\lambda_n}$$

where $\lambda_{n+1} > \lambda_n$, $\lambda_1 \geqq 0$ $\lim_{n\to\infty} \lambda_n = \infty$, $s = \sigma + it$ and

$$\limsup_{n\to\infty} \frac{\log n}{\lambda_n} = 0.$$

Let σ_c and σ_a be, respectively, the abscissa of convergence and the abcissa of absolute convergence of $f(s)$. If $\sigma_c = \infty$, $\sigma_a = \infty$, $f(s)$ defines an entire function. Let $\lambda_{N(\sigma)}$ be the λ_n corresponding to the maximum term

$$\mu(\sigma) = \max \{|a_n| e^{\sigma\lambda_n}\}$$

of the series for $\mathrm{Re}\,(s) = \sigma$. Then evidently $\lambda_N(\sigma)$ is a non-decreasing function and $N(\sigma)$ is called the index or rank of the maximum term $\mu(\sigma)$. Since the functions $|a_{N(\sigma)}|$ and $\lambda_{N(\sigma)}$ are constants in intervals, have a denumerable number of discontinuities only, their differential coefficients vanish almost everywhere. Therefore we can differentiate $\mu(\sigma) = |a_{N(\sigma)}| e^{\sigma\lambda_{N(\sigma)}}$ everywhere except at a set of measure zero, getting

$$\frac{\mu'(\sigma)}{\mu(\sigma)} = \lambda_{N(\sigma)} \tag{1.1}$$

where $\mu'(\sigma)$ is the first derivative of $\mu(\sigma)$.

The object of this note is to investigate certain inequalities involving the maximum term and its derivative of an entire function representable by Dirichlet series.

2. Theorem 1. *Let* $f(s) = \sum_{n=1}^{\infty} a_n e^{s\lambda_n}$ *be an entire function,* $N(\sigma)$ *be the rank of its maximum term* $\mu(\sigma)$ *for* $\mathrm{Re}\,(s) = \sigma$ *and* $\mu'(\sigma)$ *the derivative of the maximum term. If*

$$\lim_{\sigma\to\infty} \begin{cases} \sup \\ \inf \end{cases} \frac{\lambda_{N(\sigma)}}{e^{\sigma}} = \begin{cases} \nu \\ \delta \end{cases} \tag{2.1}$$

then

$$\liminf_{\sigma\to\infty} \exp(-\lambda_{N(\sigma)}) \int_{\sigma_0}^{\sigma} \exp\left\{\frac{\mu'(r)}{\mu(r)}\right\} dr \leqq \frac{1}{\nu} \leqq \tag{2. 2}$$

$$\leqq \frac{1}{\delta} \leqq \limsup_{\sigma\to\infty} \exp(-\lambda_{N(\sigma)}) \int_{\sigma_0}^{\sigma} \exp\left\{\frac{\mu'(r)}{\mu(r)}\right\} dr.$$

The proof is based on the following:

Lemma. *If $\Phi(x)$ is a positive, real function continuous almost everywhere in (r_0, ∞) and*

$$\lim_{r\to\infty} \begin{Bmatrix} \sup \\ \inf \end{Bmatrix} \frac{\Phi(r)}{e^r} = \begin{Bmatrix} \alpha \\ \beta \end{Bmatrix} \tag{2. 3}$$

then

$$\liminf_{r\to\infty} \exp\{-\Phi(r)\} \int_{r_0}^{r} \exp\{\Phi(x)\}\, dx \leqq \frac{1}{\alpha} \leqq \tag{2. 4}$$

$$\leqq \frac{1}{\beta} \leqq \limsup_{r\to\infty} \exp\{-\Phi(r)\} \int_{r_0}^{r} \exp\{\Phi(x)\}\, dx.$$

For, let

$$I(r) = \int_{r_0}^{r} \exp\{\Phi(x)\}\, dx.$$

Suppose the first inequality in (2. 4) does not hold. Then for every value of $r > r_0$,

$$I(r) > C\cdot\exp\{\Phi(r)\}$$

where C is a positive constant greather than $\frac{1}{\alpha}$.

As $I'(r)$, the derivative of $I(r)$, exists and is equal to $\exp\{\Phi(r)\}$ almost everywhere, we have

$$\frac{I'(r)}{I(r)} < \frac{\exp\{\Phi(r)\}}{C\cdot\exp\{\Phi(r)\}} = \frac{1}{C}.$$

Therefore, for all $r > r_0$,

$$\log I(r) = \log I(r_0) + \int_{r_0}^{r} \{I'(x)/I(x)\}\, dr$$

$$< \log I(r_0) + \frac{1}{C}(r - r_0).$$

Hence

$$\log C+\Phi(r)<\log I(r)<\log I(r_0)+\frac{1}{C}(r-r_0)$$

and so

$$\limsup_{r\to\infty}\frac{\Phi(r)}{e^r}\leqq\frac{1}{C}<\alpha$$

which contradicts the hypothesis. Hence the first inequality in (2. 4) is true. Similarly we can prove the last inequality in (2. 4).

We now prove our theorem.

From (1. 1) we have

$$\exp\{\mu'(\sigma)/\mu(\sigma)\}=\exp(\lambda_{N(\sigma)}).$$

Hence

$$\lim_{\sigma\to\infty}\begin{cases}\sup\\ \inf\end{cases}\exp(-\lambda_{N(\sigma)})\int_{\sigma_0}^{\sigma}\exp\{\mu'(r)/\mu(r)\}\,dr= \tag{2. 5}$$

$$=\lim_{\sigma\to\infty}\begin{cases}\sup\\ \inf\end{cases}\exp(-\lambda_{N(\sigma)})\int_{\sigma_0}^{\sigma}\exp(\lambda_{N(r)})\,dr.$$

Putting $\Phi(\sigma)=\lambda_{N(\sigma)}$ in (2. 3) we get (2. 1) with $\alpha=\nu$ and $\beta=\delta$ and then (2. 4) gives

$$\liminf_{\sigma\to\infty}\exp(-\lambda_{N(\sigma)})\int_{\sigma_0}^{\sigma}\exp(\lambda_{N(r)})\,dr\leqq\frac{1}{\nu}\leqq$$

$$\leqq\frac{1}{\delta}\leqq\limsup_{\sigma\to\infty}\exp(-\lambda_{N(\sigma)})\int_{\sigma_0}^{\sigma}\exp(\lambda_{N(r)})\,dr.$$

which in view of (2. 5) becomes

$$\liminf_{\sigma\to\infty}\exp(-\lambda_{N(\sigma)})\int_{\sigma_0}^{\sigma}\exp\{\mu'(r)/\mu(r)\}\,dr\leqq\frac{1}{\nu}\leqq$$

$$\leqq\frac{1}{\delta}\leqq\limsup_{\sigma\to\infty}\exp(-\lambda_{N(\sigma)})\int_{\sigma_0}^{\sigma}\exp\{\mu'(r)/\mu(r)\}\,dr.$$

3. Theorem 2. *If* $0<\sigma_1<\sigma_2$, *then*

$$\lambda_{N(\sigma_1)}\leqq\frac{\log\mu(\sigma_2)-\log\mu(\sigma_1)}{\sigma_2-\sigma_1}\leqq\lambda_{N(\sigma_2)}. \tag{3. 1}$$

PROOF. We have

$$\mu(\sigma_1)=|a_{N(\sigma_1)}|\exp(\sigma_1\lambda_{N(\sigma_1)})$$

and

$$\mu(\sigma_2)=|a_{N(\sigma_2)}|\exp(\sigma_2\lambda_{N(\sigma_2)}).$$

Since $\sigma_1 < \sigma_2$, $N(\sigma_2) \geqq N(\sigma_1)$. Now

$$(3.2) \qquad \mu(\sigma_2) = \exp\{(\sigma_2-\sigma_1)\lambda_{N(\sigma_2)}\}|a_{N(\sigma_2)}| \exp(\sigma_1\lambda_{N(\sigma_2)}) \leqq \leqq \exp\{(\sigma_2-\sigma_1)\lambda_{N(\sigma_2)}\}\,\mu(\sigma_1).$$

Also,

$$\mu(\sigma_2) \geqq |a_{N(\sigma_1)}| \exp(\sigma_2\lambda_{N(\sigma_1)}) = \exp\{(\sigma_2-\sigma_1)\lambda_{N(\sigma_1)}\}|a_{N(\sigma_1)}| \exp(\sigma_1\lambda_{N(\sigma_1)})$$

or

$$(3.3) \qquad \mu(\sigma_2) \geqq \exp\{(\sigma_2-\sigma_1)\lambda_{N(\sigma_1)}\}\,\mu(\sigma_1).$$

From (3. 2) and (3. 3), therefore

$$\exp\{(\sigma_2-\sigma_1)\lambda_{N\sigma(_1)}\} \leqq \mu(\sigma_2)/\mu(\sigma_1) \leqq \exp\{(\sigma_2-\sigma_1)\lambda_{N\sigma(_2)}\}$$

and the result follows on taking logarithms and dividing by $(\sigma_2-\sigma_1)$.

Corollary 1.

$$(3.4) \qquad \frac{d}{d\sigma}\{\log\mu(\sigma)\} = \lambda_{N(\sigma)}$$

for almost all values of $\sigma > \sigma_0 \geqq 0$.

For, taking $\sigma_1 = \sigma$ and $\sigma_2 = \sigma + h$, we get from (3. 1)

$$\lambda_{N(\sigma)} \leqq \frac{\log\mu(\sigma+h) - \log\mu(\sigma)}{h} \leqq \lambda_{N(\sigma+h)}$$

and the result follows on taking limits as $h \to 0$.

Corollary 2. *If* $0 \leqq \sigma_0 < \infty$, *then*

$$(3.5) \qquad \log\mu(\sigma) = A + \int_{\sigma_0}^{\sigma} \lambda_{N(t)}\,dt.$$

We get (3. 5) on integrating both sides of (3. 4) in the interval (σ_0, σ).

Corollary 3. *If* k *be a constant,* $0 < k < 1$, *then*

$$(3.6) \qquad \lim_{\sigma\to\infty}\{\mu(k\sigma)/\mu(\sigma)\} = 0$$

provided $f(s)$ *is not an exponential polynomial.*

Take $\sigma_1 = k\sigma$ and $\sigma_2 = \sigma$, then from (3. 1) we get

$$\exp\{\sigma(1-k)\lambda_{N(k\sigma)}\} \leqq \mu(\sigma)/\mu(k\sigma) \leqq \exp\{\sigma(1-k)\lambda_{N(\sigma)}\}$$

or,

$$\exp\{-\sigma(1-k)\lambda_{N(k\sigma)}\} \geqq \mu(k\sigma)/\mu(\sigma) \geqq \exp\{-\sigma(1-k)\lambda_{N(\sigma)}\}.$$

As $\sigma \to \infty$, $\lambda_{N(k\sigma)}$ and $\lambda_{N(\sigma)}$ both tend to infinity and (3. 6) follows.

4. We now give an alternative proof, using the notion of type, of the following results of R. P. Srivastav ([1], 84) which he established with the help of the notion of order.

Let $f(s)=\sum_{n=1}^{\infty} a_n e^{s\lambda_n}$ be an integral function of order ϱ, lower order λ and type τ $(0<(\varrho, \lambda, \tau)<\infty)$. Let $N(\sigma)$ be the rank of the maximum term $\mu(\sigma)$ of $f(s)$. Then

$$\limsup_{\sigma\to\infty} \frac{\log \mu(\sigma)}{\lambda_{N(\sigma)} \log \lambda_{N(\sigma)}} \leqq \frac{1}{\lambda}-\frac{1}{\varrho}, \tag{4. 1}$$

$$\limsup_{\sigma\to\infty} \frac{\log \mu(\sigma)}{\sigma\lambda_{N(\sigma)}} \leqq 1-\frac{\lambda}{\varrho}. \tag{4. 2}$$

PROOF. We know ([2] Th. 2.2, p. 71) that

$$\limsup_{\sigma\to\infty} \frac{\lambda_n}{\varrho e} |a_n|^{\frac{\varrho}{\lambda_n}}=\tau.$$

Therefore, for all $n>n_0$ and any $\varepsilon>0$,

$$\log |a_n| < \frac{\lambda_n}{\varrho} \log \left\{\frac{\varrho e}{\lambda_n}(\tau+\varepsilon)\right\}.$$

Hence

$$\log \mu(\sigma) = \log |a_{N(\sigma)}| + \sigma\lambda_{N(\sigma)} <$$

$$< \sigma\lambda_{N(\sigma)} + \frac{\lambda_{N(\sigma)}}{\varrho} \log \{\varrho e(\tau+\varepsilon)\} - \frac{\lambda_{N(\sigma)}}{\varrho} \log \lambda_{N(\sigma)} \leqq$$

$$\leqq \sigma\lambda_{N(\sigma)} + (\tau+\varepsilon)\lambda_{N(\sigma)} - \frac{\lambda_{N(\sigma)}}{\varrho} \log \lambda_{N(\sigma)}$$

or

$$\frac{\log \mu(\sigma)}{\lambda_{N(\sigma)} \log \lambda_{N(\sigma)}} < \frac{\sigma}{\log \lambda_{N(\sigma)}} + \frac{\tau+\varepsilon}{\log \lambda_{N(\sigma)}} - \frac{1}{\varrho}. \tag{4. 3}$$

The result now follows on proceeding to limits since ([1] p. 84)

$$\lim_{\sigma\to\infty} \begin{cases}\sup \\ \inf\end{cases} \frac{\log \lambda_{N(\sigma)}}{\sigma} = \begin{cases}\varrho \\ \lambda\end{cases}.$$

Similarly dividing $\log \mu(\sigma)$ by $\sigma\lambda_{N(\sigma)}$ and proceeing to limits we get (4. 2) from (4. 3).

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Bibliography

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