On the maximum term of an entire Dirichlet series

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1. Consider the Dirichlet Series

$$f(s)=\sum_{n=1}^{\infty} a_n e^{s\lambda_n} \tag{1.1}$$

where $\lambda_{n+1}>\lambda_n$, $\lambda_1\geqq 0$, $\lim_{n\to\infty}\lambda_n=\infty$, $s=\sigma+it$ and

$$\limsup_{n\to\infty}\frac{\log n}{\lambda_n}=0. \tag{1.2}$$

Let σ_c, σ_a be the abscissa of convergence and the abscissa of absolute convergence, respectively, of $f(s)$. If $\sigma_c=\sigma_a=\infty$, $f(s)$ is an entire function. We shall suppose throughout that (1. 2) holds and $\sigma_c=\sigma_a=\infty$.

The maximum term $\mu(\sigma)$ of the series

$$\sum_{n=1}^{\infty}|a_n|e^{\sigma\lambda_n}$$

for any $\sigma<\sigma_a$ is defined as

$$\mu(\sigma)\equiv\mu(\sigma,f)=\max_{n\geqq 1}\{|a_n|e^{\sigma\lambda_n}\}$$

and if $\nu(\sigma,f)$ denotes the rank of the maximum term, we have,

$$\mu(\sigma,f)=|a_{\nu(\sigma,f)}|e^{\sigma\lambda_{\nu(\sigma,f)}}. \tag{1.3}$$

It is well known that the derivative of an entire function is also an entire function and both have the same finite order. We shall denote here the mth derivative of $f(s)$ by $f^{(m)}(s)$, its maximum term by $\mu(\sigma,f^{(m)})$ and the rank of this maximum term by $\nu(\sigma,f^{(m)})$. It is known that [3]

$$\lambda_{\nu(\sigma,f)}\leqq\lambda_{\nu(\sigma,f^{(1)})}\leqq\ldots\leqq\lambda_{\nu(\sigma,f^{(m)})}\leqq\ldots. \tag{1.4}$$

Hence, $\chi(\sigma,m)\equiv[\lambda_{\nu(\sigma,f^{(m)})}-\lambda_{\nu(\sigma,f)}]$ is positive and nondecreasing for $m=1,2,\ldots$. Also, it has been proved that if ϱ be the linear Ritt-order and λ the lower order of $f(s)$, then ([2], 706)

$$\lim_{\sigma\to\infty}\left\{\begin{matrix}\sup\\ \inf\end{matrix}\right.\left[\frac{1}{\sigma}\int_{\sigma_0}^{\sigma}\chi(t,m)\,dt\right]=\left\{\begin{matrix}m\varrho\\ m\lambda\end{matrix}\right. \qquad \sigma_0\leqq t\leqq\sigma,\quad m=1,2,\ldots. \tag{1.5}$$

Further, with the help of the relation (1. 5) it has been shown that ([2], 707) if $f(s)$ is of linearly regular growth, i. e., $\varrho=\lambda$ and $\lim \chi(\varrho, m)$, as $\sigma\to\infty$, exists, then

$$\lim_{\sigma\to\infty}\chi(\sigma, m)=m\varrho. \tag{1. 6}$$

In this paper we study the behaviour of $\chi(\sigma, m)$ when the limit in (1. 6) does not exist and prove that limit superior of $\chi(\sigma, m)$ cannot be less than $m\varrho$ and limit inferior of $\chi(\sigma, m)$ cannot exceed $m\lambda$. It will appear that the condition that $f(s)$ should be of regular growth for (1. 6) to hold can be removed since it follows as a nencessary consequence of the existence of $\lim \chi(\sigma, m)$ when $\sigma\to\infty$. We also formulate a necessary and sufficient condition for the existence of the limit in (1. 6) and then use it to derive a relation connecting the coefficients in the Dirichlet series with the order of an entire function of linearly regular growth. Finally, we give an example to illustrate (1. 6).

2. Theorem 1. *Let* $f(s)=\sum_{n=1}^{\infty}a_n e^{s\lambda_n}$ *be an entire function of linear order* ϱ *and lower order* λ; $\nu(\sigma, f)$, $\nu(\sigma, f^{(m)})$ *denote respectively the ranks of the maximum terms* $\mu(\sigma, f)$ *and* $\mu(\sigma, f^{(m)}(s))$, *for* Re $(s)=\sigma$ *in the series for* $f(s)$ *and its m-th derivative* $f^{(m)}s)$. *If*

$$\lim_{\sigma\to\infty}\begin{Bmatrix}\sup\\ \inf\end{Bmatrix}[\lambda_{\nu(\sigma, f^{(m)})}-\lambda_{\nu(\sigma, f)}]=\begin{Bmatrix}\alpha_m\\ \beta_m\end{Bmatrix} \tag{2. 1}$$

then

$$\beta_m\leqq m\lambda\leqq m\varrho\leqq\alpha_m \tag{2. 2}$$

for $m=1, 2, 3, \ldots, \alpha_m, \beta_m$ *being sequences of positive constants.*

PROOF. Writing $\chi(\sigma, m)$ for $\lambda_{\nu(\sigma, f^{(m)})}-\lambda_{\nu(\sigma, f)}$, which is positive and nondecreasing for each $m=1, 2, \ldots,$ it follows from (2. 1) that for any $\varepsilon>0$, we have for $\sigma>\sigma_0$

$$\beta_m-\varepsilon<\chi(\sigma, m)<\alpha_m+\varepsilon$$

Therefore,

$$\frac{1}{\sigma}(\beta_m-\varepsilon)\int_{\sigma_0}^{\sigma}dt<\frac{1}{\sigma}\int_{\sigma_0}^{\sigma}\chi(t, m)\,dt<\frac{1}{\sigma}(\alpha_m+\varepsilon)\int_{\sigma_0}^{\sigma}dt.$$

Or,

$$(\beta-\varepsilon)[1-o(1)]<\frac{1}{\sigma}\int_{\sigma_0}^{\sigma}\chi(t, m)\,dt<(\alpha_m+\varepsilon)[1-o(1)].$$

Proceeding to limits and using the relation (1. 5), we therefore get,

$$\beta_m\leqq m\lambda\leqq m\varrho\leqq\alpha_m$$

Corollary. *If* $\alpha_m=\beta_m$, *(i)* $f(s)$ *is of linearly regular growth, (ii) its order* $\varrho=\alpha_m/m$ and (iii) $\lambda_{\nu(\sigma f^{(m)})}\sim\lambda_{\nu(\sigma, f)}$ as $\sigma\to\infty$ *for every* $m=1, 2, \ldots$.

We remark that even if $f(s)$ is of linearly regular growth, i. e., $\varrho=\lambda$, it does not follow necessarily from (2. 2) that $\alpha_m=m\lambda=m\varrho$. But if we assume the existence of $\lim\,\chi(\sigma, m)$ as $\sigma\to\infty$, i. e., take $\alpha_m=\beta_m$, then it is obvious from (2. 2) that

$\alpha_m = m\lambda = m\varrho$ so that $f(s)$ is of linearly regular growth and of order $\varrho = \alpha_m/m$. Thus, sucessive terms, of the sequence (α_m) become integral multiples of the constant ϱ, viz., the order of the function. Further, since

$$\lim_{\sigma\to\infty} [\lambda_{\nu(\sigma, f^{(m)})} - \lambda_{\nu(\sigma, f)}] = \alpha_m = m\varrho,$$

dividing both sides by $\lambda_{\nu(\sigma,f)}$ and proceeding to limit, we get $\lambda_{\nu(\sigma,f^{(m)})} \sim \lambda_{\nu(\sigma,f)}$ since $m\varrho$ is finite and $\lambda_{\nu(\sigma,f)} \to \infty$ as $\sigma \to \infty$.

Theorem 2. *Let* $f(s) = \sum_{n=1}^{\infty} a_n e^{s\lambda_n}$ *be an entire function of linearly regular growth and of order* ϱ; $\nu(\sigma, f)$, $\nu(\sigma, f^{(m)})$ *denote the ranks of the maximum terms in the series for* $f(s)$ *and its m-th derivative* $f^{(m)}(s)$. *Further, let*

$$\chi(\sigma, m) \equiv [\lambda_{\nu(\sigma, f^{(m)})} - \lambda_{\nu(\sigma, f)}], \qquad m = 1, 2, \dots$$

then a necessary and sufficient condition that

$$\lim_{\sigma\to\infty} \chi(\sigma, m) = m\varrho \tag{2. 3}$$

is that

$$\int_{\sigma_0}^{\sigma} t\, d\{\chi(t, m)\} = o(\sigma) \tag{2. 4}$$

as $\sigma \to \infty$.

Proof. Since $f(s)$ is of regular growth and order ϱ, we have from (1. 5)

$$\lim_{\sigma\to\infty} \left[\frac{1}{\sigma} \int_{\sigma_0}^{\sigma} \chi(t, m)\, dt\right] = m\varrho.$$

Integrating by parts therefore gives

$$\lim_{\sigma\to\infty} \left[\chi(\sigma, m) - \frac{1}{\sigma} \int_{\sigma_0}^{\sigma} t\, d\{\chi(t, m)\}\right] = m\varrho. \tag{2. 5}$$

Hence, if (2. 3) holds (2. 4) follows from (2. 5). Again, if (2. 4) holds (2. 3) follows from (2. 5).

Theorem 3. *Let* $f(s) = \sum_{n=1}^{\infty} a_n e^{s\lambda_n}$ *be an entire function of regular growth and order* ϱ, $\nu(\sigma, f)$, $\nu(\sigma, f^{(m)})$ *be defined as in Theorem 2. If condition (2. 4) of Theorem 2 holds, then*

$$\log \left| \frac{a_{\nu(\sigma, f)}}{a_{\nu(\sigma, f^{(m)})}} \right| \sim m\varrho\sigma, \qquad m = 1, 2, \dots \tag{2. 6}$$

as $\sigma \to \infty$.

Proof. We have

$$\mu(\sigma, f) = |a_{\nu(\sigma, f)}| e^{\sigma\lambda_{\nu(\sigma, f)}}$$

and

$$\mu(\sigma, f^{(m)}) = \lambda^m_{\nu(\sigma, f^{(m)})} |a_{\nu(\sigma, f^{(m)})}| e^{\sigma\lambda_{\nu(\sigma, f^{(m)})}}.$$

Therefore,

$$\log \frac{\mu(\sigma, f^{(m)})}{\mu(\sigma, f)} = \log \left\{ \lambda^{m}_{\nu(\sigma, f^{(m)})} \left| \frac{a_{\nu(\sigma, f^{(m)})}}{a_{\nu(\sigma, f)}} \right| e^{\sigma(\lambda_{\nu(\sigma, f^{(m)})} - \lambda_{\nu(\sigma, f)})} \right\}.$$

Or

$$(2.7) \qquad \frac{1}{\sigma} \log \frac{\mu(\sigma, f^{(m)})}{\mu(\sigma, f)} - \frac{m}{\sigma} \log \lambda_{\nu(\sigma, f^{(m)})} = \frac{1}{\sigma} \log \left| \frac{a_{\nu(\sigma, f^{(m)})}}{a_{\nu(\sigma, f)}} \right| + \chi(\sigma, m).$$

The left side of the above relation approaches 0 as $\sigma \to \infty$, since for functions of regular growth and of finite order ϱ, we have ([1], 89)

$$\lim_{\sigma \to \infty} \log \left\{ \frac{\mu(\sigma, f^{(m)})}{\mu(\sigma, f)} \right\} \Big/ \sigma = m\varrho = m \lim_{\sigma \to \infty} \frac{\log \lambda_{\nu(\sigma, f^{(m)})}}{\sigma}.$$

Further, since (2. 4) holds it follows from theorem 2 that $\chi(\sigma, m) \to m\varrho$ as $\sigma \to \infty$. Hence, on proceeding to limits (2. 7) yields,

$$\lim_{\sigma \to \infty} \frac{1}{\sigma} \log \left| \frac{a_{\nu(\sigma, f)}}{a_{\nu(\sigma, f^{(m)})}} \right| = m\varrho$$

which gives (2. 6).

3. Example. Consider the function

$$f(s) = \exp(e^s) - 1 = \sum_{n=1}^{\infty} \frac{1}{n!} e^{ns}.$$

It can readily be shown that $f(s)$ is an entire function of linearly regular growth and order $\varrho = 1$. The mth derivative of $f(s)$ is

$$f^{(m)}(s) = \sum_{n=1}^{\infty} \frac{n^m}{n!} e^{ns}.$$

If for some $\mathrm{Re}\,(s) = \sigma$ the pth term and the qth term are respectively the maximum terms in the series for $f(s)$ and $f^{(m)}(s)$, then

$$\frac{1}{(p-1)!} e^{(p-1)\sigma} \leqq \frac{1}{p!} e^{p\sigma} > \frac{1}{(p+1)!} e^{(p+1)\sigma}$$

and

$$\frac{(q-1)^m}{(q-1)!} e^{(q-1)\sigma} \leqq \frac{q^m}{q!} e^{q\sigma} > \frac{(q+1)^m}{(q+1)!} e^{(q+1)\sigma}.$$

Hence,

$$(3.1) \qquad p \leqq e^{\sigma} < p+1$$

and

$$(3.2) \qquad \frac{(q-1)^m}{q^{m-1}} \leqq e^{\sigma} < \frac{q^m}{(q+1)^{m-1}}.$$

If e^{σ} has values which satisfy (3. 1) then in order that (3. 2) may also hold for those values, we have,

$$(3.3) \qquad p \leqq \frac{(q-1)^m}{q^{m-1}} \quad \text{and} \quad p+1 \geqq \frac{q^m}{(q+1)^{m-1}}.$$

Taking $m=1$, it can easily be shown that the two inequalities in (3. 3) hold simultaneously only if $q = p+1$. Similarly, taking $m=2$, it can be shown that they hold only if $q = p+2$. Hence, $q-p = m$ when $m=1, 2$. In our notation $p=\nu(\sigma, f)$, $q=\nu(\sigma, f^{(m)})$, $\lambda_n=n$ so that for $m=1, 2$

$$\lim_{\sigma\to\infty} \chi(\sigma, m) = \lim_{\sigma\to\infty} [\lambda_{\nu(\sigma, f^{(m)})} - \lambda_{\nu(\sigma, f)}] = m = m\varrho$$

since $\varrho=1$. To show that (1. 6) holds for $f(s)$ for higher integral values of m also, we proceed as following:

As $\sigma\to\infty$, p, q also tend to infinity. Hence, when σ is very large so is q and the range of the values of e^σ determined by the inequalities in (3. 2) corresponds closely to that determined by

$$q-m \leqq e^\sigma < q+1-m \tag{3. 4}$$

as may be seen by expanding the two expressions in q of (3. 2) in ascending powers of $1/q$ and then neglecting terms of degree higher than 1. The inequalities (3. 1) and (3. 4) then lead to the conclusion that as $\sigma\to\infty$, $q-p\to m = m\varrho$ since $\varrho=1$.

Note. Similar results also hold for entire functions represented by Taylor series [4].

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