Group embedding and duality in semi groups

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It is known that the collection of all continuous additive homomorphisms of a given uniform semi-group in the set of non-negative reals (denoted by C^+) forms a semi-group. If instead of C^+ we take the set C of all the reals then we get a group. In Theorem 1 we show that the group embedding of the semi-group of all continuous additive homomorphisms of a uniform semi-group S with respect to C^+, taken with the symmetric uniform structure is unimorphic to the group of all continuous additive homomorphisms relative to C. In [1] V. S. KRISHNAN has established that the direct sum of the dual semi-groups (dual groups) of a given family of semi-groups (duals taken relative to C^+ and C respectively) taken with the asterisk uniformity is unimorphic with the dual semi-group (dual group) of the direct product of the given family of semi-groups. He has also shown that a similar result holds if the direct sum and direct product are interchanged and appropriate uniformities are taken. Combining the results of Krishnan and Theorem 1 in this paper, we show that the group embedding of the direct sum (direct product) of the dual semi-groups each taken with the associated symmetric uniform structure, endowed with the asterisk uniformity (usual direct product uniformity), is unimorphic with the direct sum (product) of the dual groups taken with the asterisk (usual direct product) uniformity and that this direct sum (direct product) is the same as the dual of the direct product (direct sum) of the given family of semi-groups relative to C and taken with the asterisk (direct product) uniformity.

By a semi-group $(S, +)$ we mean a commutative, associative cancellable binary system with an identity element.

A semi-group $(S, \mathfrak{A})$ is called a *uniform semi-group* if it has a (not necessarily symmetric) uniform structure $\mathfrak{A}=\{U_\alpha\}$ which is compatible with the semi-group operation in the following sense:

$$(x, y)\in U_\alpha,\ U_\alpha\in\mathfrak{A} \text{ if and only if } (x+z, y+z)\in U_\alpha$$

for each z in S.

A continuous additive homomorphism of a uniform semi-group S in C^+ (or in C) is called a *character*.

The collection of all characters with respect to C^+ (resp. to C) forms a semi-group (group) under the operation of addition defined as follows: $(f+g)(x) = f(x)+g(x)$, x in S and f, g being any two characters. We prove this in the following theorem:

Theorem 1. *If D denotes the collection of all characters of the semi-group S relative to C^+ and G is the collection of all characters of S with respect to C, then*

(a) *D is embeddable in a group, and* (b) *the group embedding of D is algebraically isomorphic to G.*

PROOF. Let us first show that D is a semi-group. Trivially the sum of two (and therefore of a finite number of) continuous additive homomorphisms is again a continuous additive homomorphism and therefore D is closed with respect to the addition defined above. This operation in the set of characters is commutative and assocaitive as the same laws are true for the nonnegative reals. The zero character is the function which maps all the elements of S onto the zero of C^+. For the concellation laws we proceed as follows:

Let d_1, d_2 be two elements of D. If $f \neq 0$ is such that $d_1 + f = d_2 + f$, ((i. e.) $(d_1 + f)(x) = (d_2 + f)(x)$ for all x in S), then we have $d_1(x) + f(x) = d_2(x) + f(x)$ and $d_1(x)$, $d_2(x)$, $f(x)$ being reals with $f(x) \neq 0$ for all x we have $d_1(x) = d_2(x)$ for all x in S, i. e. $d_1 = d_2$. Thus D is a semi-group.

Set $H = (D \times D)/E$ where E is the relation defined by $(x, y)E(u, v)$ if and only if $x + v = y + u$. This relation can be shown to be an equivalence relation. Defining addition componentwise we see that H forms a group and that it is the smallest group that contains an isomorphic image of D which proves (a).

For proving (b) it suffices to show that H is isomorphic to G. For this choose from each equivalence class in H an element (f, g) and call it distinguished if for each x in S, $f(x) \neq 0$ implies $g(x) = 0$ and $g(x) \neq 0$ implies $f(x) = 0$. We show that each class can contain only one such distinguished element. If, on the contrary, (f_1, g_1) and (f_2, g_2) are equivalent and distinguished, then we have (i) $f_1 + g_2 = f_2 + g_1$ and (ii) $f_i(x) \neq 0$ implies $g_i(x) = 0$ and $g_i(x) \neq 0$ implies $f_i(x) = 0$, for $i = 1, 2$. Suppose now $f_1(x) \neq 0$, then we have $g_1(x) = 0$ and (i) gives $f_1(x) + g_2(x) = f_2(x)$. Since $f_1(x) \neq 0$, $f_1(x)$, $f_2(x)$ and $g_2(x)$ are all non negative, it follows that $f_2(x) \neq 0$ and this by (ii) gives that $g_2(x) = 0$. Thus whenever $f_1(x) \neq 0$, we have $f_2(x) \neq 0$, and $g_1(x) = g_2(x) = 0$. Again from (ii) and (i) we get that if $g_1(x) \neq 0$ then $g_2(x) \neq 0$ and for these x in S, $f_1(x) = f_2(x) = 0$. Moreover, $f_1(x) = g_1(x) = 0$ implies $f_2(x) = g_2(x) = 0$. Hence we have for all x in S, $f_1(x) = f_2(x)$ and $g_1(x) = g_2(x)$ and this shows that the distinguished element in each class is unique.

If we now associate with each distinguished element (f, g) of H the element $(f - g)$ of G, then we can see easily that this association is one-to-one. It is also onto, since for each g in G, the pair (g^+, g^-) forms a distinguished pair in H where g^+ and g^- are defined as follows: $g^+(x) = g(x)$ whenever $g(x) \in C^+$, x in S, and zero otherwise, and $g^-(x) = -g(x)$ if $g(x) \in C - C^+$, and zero otherwise. If (f_1, h_1) and (f_2, h_2) corresponds to g_1 and g_2 respectively, then the distinguished element in the class determined by $(f_1 + f_2, h_1 + h_2)$ corresponds to $g_1 + g_2$. Thus we see that G and H are algebraically isomorphic. This proves the theorem.

We can make the operations in D and G continuous by suitably prescribing a nuclear base for the zero element of D and G and for this we introduce the notion of a topologizing family of subsets of S.

By a *topologizing family* of subsets of S, we mean a collection of non-null subsets $\{F_\alpha\}$ of S such that (i) each finite subset of S is contained in the family, (ii) any finite union of subsets of the family also belongs to the family. The class of finite or compact or totally bounded subsets of S under its uniformity are examples of topologizing families.

Let us denote the topologizing family of the uniform semi-group S by $\mathfrak{F}=\{F_\alpha\}$. Let $U=\{u_a\}$ and $V=\{v_a\}$ where

$$u_a=\{x\in C^+ | x<a\},$$

and

$$v_a=\{x\in C | -a<x<a\},$$

be the family of neighbourhoods of 0 in C^+ or C. We define the neighbourhoods of 0 in D and G as

$$u_a^\alpha=N(F_\alpha, u_a)=\{f\in D | f(F_\alpha)\subset u_a \quad \text{where} \quad F_\alpha \text{ is in } \mathfrak{F}\},$$

and

$$v_a^\alpha=N(F_\alpha, v_a)=\{g\in G | g(F_\alpha)\subset v_a \quad \text{where} \quad F_\alpha \text{ is in } \mathfrak{F}\}.$$

The group embedding H of D under the uniformity $\mathfrak{U}=\{u_a^\alpha\}$ is only a topological semi-group while G under the uniformity $\mathfrak{V}=\{v_a^\alpha\}$ is a topological group. In order to make H a topological group, we symmetrize the uniform structure $\mathfrak{U}$ and get $\mathfrak{W}=\{w_a^\alpha\}$ as

$$w_a^\alpha=u_a^\alpha\cup(u_a^\alpha)^*$$

where $(u_a^\alpha)^*$ is the collection of all f in H such that the 0 of H is in $f+u_a^\alpha$.

It can be shown without difficulty that

$$u_a^\alpha+(u_a^\alpha)^* = u_a^\alpha\cup(u_a^\alpha)^*.$$

We now prove the following

Theorem 2. *The group G with the uniform structure $\mathfrak{V}$ is unimorphic to H with the uniformity $\mathfrak{W}$.*

PROOF. Let Φ denote the algebraic correspondence which associates with each g in G, a distinguished pair (f, h) in H.

In order to show that Φ is uniformly continuous it is enough if we find a v_a^α when a w_a^α is given such that $\Phi(v_a^\alpha)$ is contained in w_a^α. Now when a w_a^α is given, we have a F_α from $\mathfrak{F}$ and a neighbourhood u_a of 0 in C^+. Let v_a be the symmetric associate of u_a in C. The set of functions that map F_α into v_a will give the v_a^α we are searching for. Because, if $g\in v_a^\alpha$ then $\Phi(g)=(h_1, h_2)$ where h_1, h_2 are in D such that $g(x)=h_1(x)$ if $g(x)\in C^+$, and $-g(x)=h_2(x)$ if $g(x)\in C-C^+$ so that $g(F_\alpha)\subset v_a$ implies that $h_1(F_\alpha)\subset v_a\cap C^+ = u_a$ and $h_2(F_\alpha)\subset v_a\cap C^+$ and this is contained in u_a. Therefore h_1, h_2 are in u_a. This means that $(h_1, 0)+(0, h_2)$ is in $u_a^\alpha+(u_a^\alpha)^*=w_a^\alpha$. Thus $\Phi(w_a^\alpha)\subset w_a^\alpha$.

To show that Φ^{-1} is uniformly continuous, choose an arbitrary surrounding v_a^α. This gives a subset F_α from S and v_a in C. Let $u_a = v_a\cap C^+$. Consider the set of functions of D that take values in u_a or 0 when defined over the set F_α of S. There is at least one such function viz. the zero function on S. If f_1 and f_2 are two such functions such that f_1 is complementary to f_2 then the element g of G whose image is (f_1, f_2) under Φ is such that $g\in v_a^\alpha$. These (f_1, f_2) exhaust w_a^α. Therefore $\Phi^{-1}(w_a^\alpha)\subset v_a^\alpha$. Hence the result.

We need the following concepts for the statement and proof of the next theorems which are extensions of the above theorem to the direct sum and direct product of a family of uniform semi-groups.

DEFINITION. *If x is an element of the uniform semi-group $(S, \mathfrak{A})$ lying in the nucleus u_i, we define the index of x in u_i (denoted by $x|u_i$) to be $(\frac{1}{2})^n$ if $x, 2x, 4x, 8x, \ldots$ $\ldots, 2^n x$ are all in u_i and $2^{n+1}x$ is not in u_i and $x|u_i$ is zero if $2^m x \in u_i$ for all m.*

Given a family of uniform semi-groups $(S_i, \mathfrak{A}_i)$ where i is in I, an indexing set, a *rectangular uniformity* for the direct sum of the semi-groups S_i is defined as the points of the direct sum that lie in the cartesian product of u^i, where u^i is chosen from $\mathfrak{A}_i$ for each $i \in I$. An *asterisk* or **-nucleus* $(\prod_i u^i)^*$ is determined by those points of $(\prod_i u^i)$ such that $\sum_i x_i|u^i < 1$.

We have shown elsewhere ([2] lemma 7) that the group embedding of the direct product of a given family of semi-groups is isomorphic to the direct product of the group embeddings of the individual semi-groups belonging to the same family. With slight modifications it can be shown that the same lemma is valid if we change the direct product into a direct sum.

Theorem 2 above establishes that G_i is isomorphic (in fact unimorphic) to the group embedding H_i of D_i(H_i taken with the associated symmetric uniform structure). Combining this with lemma 1 given below, and the statement made above, we have the following

Theorem 3. *The group completion of the direct sum of duals D_i of S_i relative to C^+ is algebraically isomorphic to the direct sum of the duals G_i of S_i relative to C and this direct sum is also the dual of the direct product of S_i with respect to C. Further, the group completion of the direct product of D_i is algebraically isomorphic to the direct product of G_i and this is the dual of the direct sum of S_i relative to C.*

Lemma 1. (Theorem 6 in [1])' *If D_i is the algebraic dual of the semi-group S_i where i runs over an indexing set I (say), relative to C^+ then $\prod_i D_i$ is the algebraic dual of $\sum_i S_i$ and $\sum_i D_i$ is the algebraic dual of $\prod_i S_i$ with respect to C^+. (Instead of considering the duals with respect to C^+ we can consider them as duals relative to C also).*

In order to prove the above theorem, when we take into consideration the continuity of the operation in the different semi-groups occuring in the theorem, we recall a theorem in [1]. Using the above notations we state the theorem without proof.

Theorem 4. *The semi-group $\sum_i D_i$ with the associated asterisk uniformity is the topological dual of $\prod_i S_i$ with the direct product uniformity, relative to C^+ and $\sum_i G_i$ with the associated asterisk uniformity, is the topological dual of $\prod_i S_i$ taken with the direct product uniformity, when the dual is taken relative to C.*

We remark that whenever we take the group completion of the semi-group, we take only the symmetric uniformity for the group. For proving the topological analogue of Theorem 3, it is enough if we establish a 1—1 correspondence between the rectangular nuclei of the group completion of the direct sum of a family of uniform semi-groups (each taken with the symmetric associate of the given uniformity) and the rectangular nucleus of the direct sum of the group completions of the given family of semi-groups. The proof will be complete if we further show that the index of an element in the group completion, of a given semi-group relative to a fixed nucleus, is equal to the index of the corresponding element in the group relative to the corresponding nucleus under an isomorphic mapping.

We shall now state the following

Theorem 5. *If S_i is an indexed family of uniform semi-groups and $D_i(G_i)$ denotes the dual of S_i relative to $C^+(C)$, then the group embedding of $\sum_i D_i$ (taken with the symmetric associate) having the asterisk uniformity is unimorphic with the direct sum of the groups G_i taken with the asterisk uniformity.*

Proof. We shall now show that there is a $1-1$ correspondence between the rectangular nuclei of the direct sum of groups G and the rectangular nuclei of the direct sum of the group embedding of the dual semi-groups each taken with the symmetric associate of the asymmetric uniformity.

Let $\mathfrak{F}_i$ be a topologizing family of subsets of S_i, and $N(F^i_\alpha, u_a)$ be the asymmetric uniformity for the dual semi-group D_i. Let $M(F^i_\alpha, v_a)$ be the symmetric associate of $N(F^i_\alpha, u_a)$ and the group completion H_i of D_i be endowed with this uniform structure $\mathfrak{M}_i$ (say). Let G_i be the group of characters of S_i relative to C and let $\mathfrak{B}_i = \{V^i_{\alpha,a}\}$ be the uniformity for G_i given by the same topologising family $\mathfrak{F}_i$ of S_i. Then we show that to each $M^i_{\alpha,a}$ there corresponds a $V^i_{\alpha,a}$ and conversely so that the rectangular nucleus formed out of these surroundings are in $1-1$ correspondence.

Let $M^i_{\alpha,a}$ be given in H_i. Then we have a member F^i_α from the topologising family $\mathfrak{F}_i$ of S_i and a neighbourhood u_a of 0 in C^+. Now $v_a = u_a \cup (u_a)^* = u_a + (u_a)^*$ If $h^i \in M^i_{\alpha,a}$ then we can find h^i_1, h^i_2 in D_i such that (h^i_1, h^i_2) forms a distinguished pair. Associate with this distinguished pair an element g^i from G_i. Then from the fact that $h^i \in M^i_{\alpha,a}$ it follows that $h^i_1(F^i_\alpha) \subset u_a$ and $h^i_2(F^i_\alpha) \subset u_a$, so that $(h^i_1, h^i_2)(F^i_\alpha) = (h^i_1, 0)(F^i_\alpha) + (0, h^i_1)(F^i_\alpha) \subset u_a + (u_a)^*$ which implies that $g^i(F^i_\alpha) \subset v_a$. Thus $g^i \in V^i_{\alpha,a}$. As h^i exhauts $M^i_{\alpha,a}$ we have for each $M^i_{\alpha,a}$ a $V^i_{\alpha,a}$.

If now given a $V^i_{\beta,b}$ we have to find a $M^i_{\beta,b}$. When a $V^i_{\beta,b}$ is given we have a F^i_β from the topologising family of S_i and a neighbourhood V_b of 0 in C. Let $g^i \in V^i_{\beta,b}$. From the algebraic correspondence between G_i and H_i it follows that for this g^i there is a distinguished pair (h^i_1, h^i_2) and each component maps the whole of F^i_β into $v_b \cap C^+ = u_b$ so that $h^i_1(F^i_\beta)$ and $h^i_2(F^i_\beta) \subset u_b$. Thus we have a $M^i_{\beta,b}$ from the uniformity $\mathfrak{M}_i$ of H_i.

Thus there is a $1-1$ correspondence between nuclei in G_i with the nuclei in H_i and this is true for each $i \in I$. Therefore it follows that $\prod_i M^i_{\alpha,a}$ corresponds to $\prod_i V^i_{\alpha,a}$. Hence the rectangular uniformities in $\sum_i G_i$ and $\sum_i H_i$ are in $1-1$ correspondence.

For proving the correspondence between the asterisk uniformities it is enough if we show that the index of an element g^i of G_i with respect to some $V^i_{\alpha,a}$ is equal to the index of (h^i_1, h^i_2) of H_i with respect to $M^i_{\alpha,a}$ where (h^i_1, h^i_2) and $M^i_{\alpha,a}$ correspond to g^i and $V^i_{\alpha,a}$ respectively. For then the sum of the indices in both the cases are the same and therefore the neighbourhoods in both the asterisk uniformities correspond to one another.

Let the index of g^i relative to a $V^i_{\alpha,a}$ be $(\frac{1}{2})^n$. Then $2^n \cdot g^i(F^i_\alpha) \subset v_a$ and there exists atleast one element x in F^i_α, such that $2^{n+1} \cdot g^i(x) \notin v_a$. Denote this x by $\bar{x}$. There is no loss of generality in assuming that $g^i(\bar{x})$ is non-negative. Now $g^i(\bar{x}) = h^i_1(\bar{x})$ and $h^i_2(\bar{x}) = 0$ where (h^i_1, h^i_2) corresponds to g^i. Clearly $2^n \cdot g^i(\bar{x}) = 2^n \cdot h^i_1(\bar{x}) \in v_a \cap C^+ = u_a$ and $2^{n+1} \cdot g^i(\bar{x}) \notin v_a$ implies that $2^{n+1} \cdot h^i_1(\bar{x}) \notin u_a$ and $2^m \cdot h^i_2(\bar{x}) \in u_a$ for all m. Therefore $2^n \cdot (h^i_1, h^i_2)(\bar{x}) \in u_a$ while $2^{n+1}(h^i_1 \cdot h^i_2)(\bar{x}) \notin u_a$. Hence $2^n(h^i_1, h^i_2)(F^i_\alpha) \subset u_a$ and $2^{n+1}(h^i_1, h^i_2)(F^i_\alpha) \not\subset u_a$. Thus the index of (h^i_1, h^i_2) relative to $M^i_{\alpha,a}$ is $(\frac{1}{2})^n$.

Hence the asterisk uniformities in $\sum_i G_i$ and $\sum_i H_i$ correspond to each other and thus complete the proof of the theorem.

We now state topological analogue of the above theorem for the direct sum of semi-groups and their duals.

Theorem 6. *If S_i is an indexed family of uniform semi-groups, D_i and G_i denote the duals of S_i relative to C^+ and C respectively, then the dual of $\sum_i S_i$ taken with the asterisk uniformity (it being understood that each S_i has symmetric uniform structure) has $\prod_i D_i$ and $\prod_i G_i$ as the topological duals relative to C^+ and C respectively. The topologies for $\prod_i D_i$ and $\prod_i G_i$ are the direct product topologies. Further the group embedding of $\prod_i D_i$ with the associated symmetric uniform structure is unimorphic with $\prod_i G_i$.*

The first part of the theorem is the content of Theorem 8 of [1] while the second part follows in view of Theorem 1 of this paper and lemma 7 of [2].

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